Canad. Math. Bull. Vol. **23** (2), 1980

FUNCTION THEORETIC INTEGRAL OPERATOR METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS[1]

BY
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1. **Introduction.** It is well known that complex analytic functions and harmonic functions of two real variables are closely related, so that from methods and results in complex function theory one can easily obtain theorems on those harmonic functions. This is the prototype of a relation between complex analysis and a partial differential equation (Laplace's equation in two variables). In the case of more general linear partial differential equations, one can establish similar relations. So far, two types of useful methods are known, as follows.

1.1. **Pseudo-analytic functions.** One can replace the Cauchy-Riemann equations by a system of two more general first-order linear partial differential equations, for instance, by the Hilbert-Carleman system

$$\varphi_x - \psi_y = a_{11}\varphi + a_{12}\psi$$
$$\varphi_y + \psi_x = a_{21}\varphi + a_{22}\psi$$

and then develop the theory of solutions (φ, ψ), taken in complex form as $f(z) = \varphi(x, y) + i\psi(x, y)$, along the lines of classical complex analysis. This idea goes back to Picard and Beltrami. A detailed theory of these *pseudo-analytic functions* was created by L. Bers [7] and (independently) I. N. Vekua. It has found applications in fluid flow and other fields, for instance, in differential geometry in connection with global representations of simply-connected C^3-surfaces in $\mathbb{R}^3$ with Gaussian curvature $K<0$; cf. [14]. This extends classical investigations on minimal surfaces (for related work, see [8, 20]).

1.2. **Integral operators.** One can define integral operators which transform complex analytic functions into solutions of a given partial differential equation. The idea of introducing operators for utilizing methods and results of function theory in the theory of partial differential equations was conceived by S. Bergman [5] and (independently) I. N. Vekua [23]. Bergman also developed general principles as well as many details of a corresponding theory and its applications to fluid dynamics.

(1) This paper is one of a series of survey papers written at the invitation of the editors.
(2) Supported by the N.S.E.R.C. of Canada under grant A9097.
Received by the editors June 5, 1979

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In this paper we shall be concerned exclusively with problems related to integral operators.

It is not difficult to recognize the reason for the success of the integral operator method. Indeed, there is a large number of theorems on specific properties of analytic functions, for instance, on the location and type of singularities, growth, behaviour near the boundary of the domain of holomorphy, coefficient problems of various series developments and so on. And a relation to partial differential equations obtained by an integral operator may then be used as a *principle for translating* those and other results into theorems on general properties of classes of solutions of a given equation. Theorems of this type seem to complement results on partial differential equations obtained by other methods, and we should also note that, in contrast to some other recent methods, our approach is constructive.

The method of integral operators to be considered here may be called a function theoretic approach to partial differential equations. During the past two decades it has become a large field of its own, as can be seen by comparing Bergman's classic [5] with more recent publications, such as [13, 18]. One impetus to the development of the method resulted from the theory of partial differential equations itself. Another motivation came from applications to transition and other flow problems [6, 10], problems in elasticity [9] and applications in connection with the indirect, or inverse, method [21] used for improperly posed and other problems. A particular advantage of these integral operators is that they are useful, on one hand for characterizing classes of solutions in a general way, and on the other hand for solving boundary value problems, either directly or by the inverse method.

An equation being given, one generally has available various integral operators for representing solutions, and the success of deriving theorems on solutions from those in complex analysis will to a large extent depend on the "simplicity" of the operator used. Hence one of the central problems of the whole method and its applications is the construction of suitable classes of kernels. The present paper will be devoted mainly to this problem and its ramifications.

2. **Equations in two variables.** We consider linear equations of the form

$$\Delta w+\alpha(x, y)w_x+\beta(x, y)w_y+\gamma(x, y)w=0. \tag{2.1}$$

Assuming the coefficients α, β, γ to be real-analytic in a domain $\tilde{D}\subset\mathbb{R}^2$, $(0, 0)\in\tilde{D}$, we can continue them analytically to the complex domain by assuming x and y to be complex.

For convenience we transform (2.1) by using $z=x+iy$, $z^*=x-iy$ and eliminating one of the two first partial derivatives. This yields, say,

$$Lu=u_{zz^*}+b(z, z^*)u_{z^*}+c(z, z^*)u=0, \tag{2.2}$$

https://doi.org/10.4153/CMB-1980-018-0 Published online by Cambridge University Press

where b and c are holomorphic in a domain $\hat{D} \subset \mathbb{C}^2$, $(0, 0) \in \hat{D}$. We exclude the trivial case $c = 0$ throughout this paper. Note that $z^* = \bar{z}$, the conjugate, if and only if x and y are real.

A first principle for obtaining integral operators T of the desired kind is the substitution of an integral into (2.2). For this purpose we consider

$$u(z, z^*) = (Tf)(z, z^*) = \int_{-1}^{1} g(z, z^*, t) f(h_1(z, t)) h_2(t)\, dt, \tag{2.3}$$

where $f \in C^\omega(G_0)$, $G_0 \subset \mathbb{C}$, $0 \in G_0$, and t is real.

It is important that f be a function of a *single* variable, since the theorems on specific properties of analytic functions mentioned above have almost no known counterparts for functions of several complex variables. This motivates $h_1 : \tilde{G} \times J \to G_0$, $J = [-1, 1]$, $\tilde{G}$ a domain in $\mathbb{C}$. And h_2 gives some flexibility with respect to conditions for the kernel g.

S. Bergman [5] suggested the standard choice

$$h_1(z, t) = z(1 - t^2)/2, \qquad h_2(t) = (1 - t^2)^{-1/2}. \tag{2.4}$$

Then the following theorem holds.

THEOREM 2.1. *If g is a holomorphic solution of*

$$Mg = (1 - t^2) g_{z^* t} - t^{-1} g_{z^*} + 2ztLg = 0 \tag{2.5}$$

on $R = G \times G^* \times J$ satisfying

$$(1 - t^2)^{1/2} g_{z^*} \to 0 \quad \text{as} \quad t \to \pm 1 \quad \text{uniformly on} \quad G \times G^* \tag{2.6a}$$

$$g_{z^*}/tz \quad \text{continuous on} \quad R, \tag{2.6b}$$

then for every $f \in C^\omega(G_0)$ the function $u = Tf$ in (2.3) *is a C^ω-solution of* (2.2) *on $\Omega = G \times G^*$. Here G lies in the z-plane and G^* is the same domain in the z^*-plane corresponding to G under the above transformation.*

Conversely, every C^2-solution u of (2.2) on Ω can be represented in the form

$$u = Tf + \hat{T}\hat{f},$$

where $\hat{T}$ is of the form (2.3) with another kernel $\hat{g}$, and $\hat{f}$ depending on z^* (instead of z). This can be proved, for instance, by using classical results on the continuation of solutions of elliptic equations to the complex domain [19] and employing the complex Riemann-Vekua function [23], which is a complex analogue of the Riemann function for hyperbolic equations.

3. **Bergman operators of the first kind.** The choice of integral operators for a given equation (2.2), that is, of kernels g satisfying (2.5), (2.6), depends on the purpose. The development of a corresponding systematic method is an open

problem, but practically useful classes of kernels have been introduced in an ad-hoc fashion and investigated in great detail. Each of them is motivated and suggested by a certain requirement. We shall consider three such classes, starting with operators which have a particularly simple inverse. This property is important, for instance, in connection with the coefficient problem for series representations of solutions u of (2.2), where one needs $f=T^{-1}u$, obtains properties of f from a coefficient theorem in complex analysis, and studies how these properties are transformed in (2.3). An operator satisfying that requirement is defined as follows.

Definition 3.1. A *Bergman operator of the first kind* is an integral operator T defined by (2.3), (2.4) which has a kernel satisfying (2.5), (2.6) and, for all z, z^*, t considered,

$$g(z,0,t)=1, \qquad g(0,z^*,t)=1. \tag{3.1}$$

Such a kernel can be obtained by substituting

$$g(z,z^*,t)=1+\sum_{j=1}^{\infty} z^j t^{2j} \int_0^{z^*} q_j(z,\zeta)\, d\zeta$$

into (2.5), determining the q_j's from a recursive system of nonlinear second-order partial differential equations and proving convergence by Cauchy's classical majorant method. This system turns out to be rather manageable in most cases of practical interest. The inverse T^{-1} is simple; indeed, $f=T^{-1}u$ can immediately be obtained from

$$u(z,0)=\int_{-1}^{1} f\left(\frac{z}{2}(1-t^2)\right)(1-t^2)^{-1/2}\, dt.$$

Note that for T^{-1} to exist, we may have to define an equivalence relation on $\{f \mid f\in C^\omega(G_0)\}$, G_0 a given domain, $0\in G_0$, but can then readily characterize a minimal representative of each class in terms of the coefficients of Maclaurin series.

4. **Exponential kernels.** Operators of the first kind yield local solutions, in general. This fact suggests the study of "kernels of finite form" for obtaining global solutions. An important class of such kernels and operators is defined as follows.

Definition 4.1. L in (2.2) is said to be of *class E*, written $L\in E$, if solutions $u\neq 0$ of (2.2) can be obtained by using an *operator T of class E*, that is, an operator defined by (2.3), (2.4) with a kernel of the form

$$g=e^q, \qquad q(z,z^*,t)=\sum_{\mu=0}^{m} q_\mu(z,z^*)t^\mu \qquad (m\in\mathbb{N}). \tag{4.1}$$

Clearly, the assumption of a special form of g imposes conditions on b and c

in (2.2). This fact entails the basic problem of finding out whether a special class of kernels is suitable for a sufficiently large class of operators L. For class E this holds true, and necessary and sufficient conditions for $L \in E$ are known; see [1], which also discusses kernels of the form $g = pe^q$, where p and q are polynomials in t with coefficients depending on z^* and z, respectively.

A remarkable property of class E operators is as follows.

THEOREM 4.2. *If $L \in E$ and $f(z) = z^n$, $n \in \mathbb{N}$, then $u = Tf$ satisfies an ordinary linear differential equation whose order is at most $m+1$ and is independent of n.*

This theorem enables us to apply the Fuchs-Frobenius theory of ordinary differential equations for investigating singularities and other properties of particular solutions of certain partial differential equations, for instance, of the Helmholtz equation, for which $L \in E$.

As an equation for which a kernel $g = pe^q$ with p and q as characterized above is suitable, we mention

$$\psi(z^*)^2 u_{zz^*} + \varphi(z)(\psi(z^*)u)_{z^*} = 0$$

where φ and ψ are analytic functions of the respective variable, holomorphic in a neighborhood of the origin.

5. **Differential operators.** E. Peschl and his school [4, 22] have recently developed a function theory of solutions of the equation

$$u_{zz^*} + m(m+1)(1+zz^*)^{-2}u = 0 \qquad (m \in \mathbb{N}) \tag{5.1}$$

in analogy to complex analysis. This equation has also been considered by M. Eichler, I. N. Vekua and many others. It is important since it occurs in the separation of the wave equation $\Delta_3 w = w_{tt}$ in spherical coordinates under the application of a stereographic projection to the equation involving the angular variables. The main tool in that recent approach is a differential operator D for representing solutions of (5.1). It is interesting that D is a special case of differential operators which can be derived from certain integral operators. The latter are defined as follows.

DEFINITION 5.1. L in (2.2) is said to be of *class P*, written $L \in P$, if solutions $u \neq 0$ of (2.2) can be obtained by using an *operator T of class P*, that is, an operator defined by (2.3), (2.4) with a kernel of the form

$$g(z, z^*, t) = \sum_{\mu=0}^{m} p_{2\mu}(z, z^*)t^{2\mu} \qquad (m \in \mathbb{N}). \tag{5.2}$$

Note that in (5.2) odd powers of t have been omitted without loss of generality. The determination of *explicit* necessary and sufficient conditions for $L \in P$ is an open problem. Implicit conditions are as follows.

LEMMA 5.2. *Let $r_0(z)$ be arbitrary analytic, $r_{2,z^*}=-2r_0c$ and*

$$(2\mu+1)r_{2\mu+2,z^*}=-2Lr_{2\mu} \qquad \mu=1,2,\cdots.$$

Then $L\in P$ if and only if there is an $m\in\mathbb{N}$ such that $r_{2m+2}=0$.

The conversion of a class P operator to a differential operator (and vice versa) can be accomplished as follows (cf. [15]).

THEOREM 5.3. *If $L\in P$, then*

$$(Tf)(z,z^*)=(D\tilde{f})(z,z^*)=\sum_{\mu=0}^{m}\frac{(2\mu)!}{2^{2\mu}\mu!}z^{-\mu}p_{2\mu}(z,z^*)\tilde{f}^{(m-\mu)}(z) \tag{5.3}$$

where

$$f(z)=\sum_{n=0}^{\infty}a_nz^n, \qquad \tilde{f}(z)=\sum_{n=0}^{\infty}\frac{n!B(\frac{1}{2},n+\frac{1}{2})}{2^n(n+m)!}a_nz^{n+m}$$

and B is the beta function.

For the special equation (5.1) we obtain from Theorems 2.1 and 5.3 the representation

$$u(z,z^*)=(D\tilde{f})(z,z^*)=\sum_{\mu=0}^{m}\frac{(2m-\mu)!}{(m-\mu)!\mu!}\left(\frac{-z^*}{1+zz^*}\right)^{m-\mu}\tilde{f}^{(\mu)}(z). \tag{5.4}$$

Here D is the aforementioned differential operator. Converting it back to an integral operator, we arrive at (2.3) with the kernel

$$g(z,z^*,t)=\sum_{\mu=0}^{m}\binom{m+\mu}{2\mu}\left(\frac{-4zz^*}{1+zz^*}\right)^{\mu}$$

and can now establish many results of that function theory for (5.1) simply by using the theory of integral operators. Incidentally, the case $m=1$ of (5.1) plays a role in connection with minimal surfaces (cf. [20], p. 105), and its general solution now follows from (5.4) without employing the Weierstrass representation of those surfaces.

6. **An application. Tricomi equation.** A part of Bergman's earlier work on integral operators was motivated by problems in compressible fluid flow. He improved Chaplygin's pioneer work, which failed for flows past obstacles in cases requiring analytic continuation; cf. the references given in [6]. We shall consider a different approach involving the *Tricomi equation*

$$\sigma\psi_{\theta\theta}+\psi_{\sigma\sigma}=0. \tag{6.1}$$

Here ψ is the stream function of the flow, which is assumed stationary, two-dimensional, nonviscous and compressible. θ is the angle between the velocity vector $v=(v_1,v_2)$ and the v_1-axis in the *hodograph plane* (the v_1v_2-plane), and σ is defined by $d\sigma/dq=-\rho/q$, where $q=|v|$ is the speed and ρ is the density. (6.1) is probably the best known equation of mixed type. It occurs in *transition problems*, that is, flows which are subsonic in a portion of the region of flow, sonic on a curve (or several curves), and supersonic in the remaining portion of the region. An example of a transition problem is a nozzle in the case of subsonic flow of high speed; then a supersonic zone may develop in an area near the minimal cross section of the nozzle. Another problem is an airfoil travelling at a high subsonic speed. In this case, a supersonic zone may develop on the upper portion of the wing where the flow is fastest. For a recent method of computer design of such airfoils, see [3].

Equation (6.1) is an approximation to the famous Chaplygin equation

$$K(\sigma)\psi_{\theta\theta}+\psi_{\sigma\sigma}=0 \tag{6.2}$$

in regions such that

$$K(\sigma)=\rho^{-2}(1-M^2)\approx l\sigma \qquad (|\sigma| \text{ small}),$$

where $M=q/a$ is the Mach number, $a=(dp/d\rho)^{1/2}$ the speed of sound, and p the pressure. We may assume $l=1$, which can be accomplished by a suitable linear transformation of the independent variable. We mention that the transition from the xy-plane of the flow to the hodograph plane corresponds to the transition from a nonlinear system

$$\rho\varphi_x=\psi_y, \qquad \rho\varphi_y=-\psi_x$$

to the linear equation (6.2). Here, φ is the potential function of the (irrotational) flow and we have denoted the stream function again by ψ, for simplicity. Indeed, using q, θ, we first obtain

$$\varphi_q=\frac{M^2-1}{\rho q}\psi_\theta, \qquad \varphi_\theta=\frac{q}{\rho}\psi_q.$$

Introducing the above σ yields a system which immediately implies (6.2), namely

$$\varphi_\sigma=K(\sigma)\psi_\theta, \qquad \varphi_\theta=-\psi_\sigma.$$

If we set

$$z=\tfrac{2}{3}\sigma^{3/2}+i\theta, \qquad z^*=\tfrac{2}{3}\sigma^{3/2}-i\theta, \qquad u=\sigma^{1/4}\psi,$$

then (6.1) becomes

$$L_0u=u_{zz^*}+k(z+z^*)^{-2}u=0, \qquad k=5/36. \tag{6.3}$$

We see that this is a special case of (2.2), so that T defined in Sec. 2 is applicable. A better choice is the following.

LEMMA 6.1. *Let B be defined by*

$$u(z,z^*)=(Bf)(z,z^*)=\int_{z_0}^{z} g(z,z^*,t)f(t)\,dt-f(z) \tag{6.4}$$

where $f\in C^\omega(G_1)$, $G_1\subset\mathbb{C}$ *is a domain,* $z_0\in G_1$, $g\in C^\omega(G)$, $G=G_1\times G_2\times G_3$,

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and G_2 and G_3 correspond to G_1 in the z^* and t planes, respectively. Assume that for $(z, z^*, t) \in G$ we have

$$(6.5) \qquad \text{(a)}\ L_0 g = 0, \qquad \text{(b)}\ g_{z^*}(z, z^*, z) = k(z+z^*)^{-2}.$$

Then $L_0 u = L_0 Bf = 0$ on $G_1 \times G_2$.

This becomes even simpler if we set

$$\tau = h(t) = (2t - z + z^*)(z+z^*)^{-1}, \qquad g(z, z^*, t) = 2(z+z^*)^{-1}\, dP/d\tau.$$

Then (6.5a) reduces to the Legendre equation for P with parameter $-\frac{1}{6}$, and (6.5b) gives $dP(1)/d\tau = -k/2$. Hence we obtain the following result (cf. also [16]).

THEOREM 6.2. *Let*

$$(6.6) \qquad u(z, z^*) = \int_1^{h(z_0)} P_{-1/6}(\tau) \frac{d}{d\tau} f(\tau(z+z^*)/2 + (z-z^*)/2)\, d\tau$$

where $P_{-1/6}$ is the Legendre function of the first kind of order $-\frac{1}{6}$. Then for every $f \in C^\omega(G_1)$ with $f(z_0) = 0$ we have $L_0 u = 0$ in $G_1 \times G_2$.

This theorem can be used in a number of ways. First of all, by inserting various functions f (even simple ones, such as powers, polynomials, exponential and logarithmic functions) one can obtain particular solutions of the Tricomi equation, including those derived by Chaplygin, Falkowitsch, Guderley, Tamada and Tomotika from various sources and by different methods. Furthermore, (6.6) suggests a study of general properties of classes of solutions corresponding to certain classes of analytic functions, such as meromorphic or algebraic functions. Finally, one may employ (6.6) in connection with the indirect or inverse method and its application to improperly posed problems (cf. [21]).

7. **Solutions in domains with corners.** As long as we use integral operators as a principle for translating methods and results of complex analysis, it is clear that we should define the operators on spaces of analytic functions. On the other hand, the above and similar representations by means of the complex Riemann function (cf. [23]) make sense also under much weaker assumptions. This is of interest, for instance, in connection with boundary value problems in domains with corners. One can start, say, from $C^{m+\alpha}$-solutions ($m \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$) in a simply-connected bounded plane domain with corners, and ask for differentiability properties of an associated function f of such a solution u (that is, $u = Tf$) in Bergman or Riemann–Vekua representations. Recent results by S. C. Eisenstat [11] based on asymptotic expansions developed by H. Lewy and his school (cf. [17]) show that, roughly, Hölder continuity of the mth derivatives of u in the closure of the domain (assumed without interior or

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exterior cusps) implies the same for f in the Riemann–Vekua representation. For another approach to boundary value problems in domains with corners, see [2].

References

1. A. Azzam and E. Kreyszig, *Construction of kernels of integral operators for linear partial differential equations*, in press.

2. A. Azzam and E. Kreyszig, *Regularity properties of solutions of elliptic equations near corners*, in press.

3. F. Bauer, P. Garabedian, D. Korn, and A. Jameson, *Supercritical Wing Sections II*, Springer, Berlin, 1975.

4. K. W. Bauer, *Differentialoperatoren bei partiellen Differentialgleichungen*, Gesellsch. Math. u. Datenverarb. Bonn, Bericht **77** (1973) 7–17.

5. S. Bergman, *Integral Operators in the Theory of Linear Partial Differential Equations*, Springer, Berlin, 1969.

6. S. Bergman, On the mathematical theory of flow patterns of compressible fluids, in [18] (below), 1–9.

7. L. Bers, *Function Theoretic Aspects of the Theory of Elliptic Partial Differential Equations*, in R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. II, Interscience/Wiley, New York, 1962.

8. E. Bombieri, *Variational problems and elliptic equations*, Proc. Int. Congr. Math. Vancouver, 1974. (Vol. 1, pp. 53–63.)

9. D. L. Clements and C. Rogers, *On the Bergman operator method and anti-plane contact problems involving an inhomogeneous half-space*, SIAM J. Appl. Math. **34,** (1978), 764–773.

10. D. L. Colton, *Partial differential Equations in the complex Domain*, Pitman, London, 1976.

11. S. C. Eisenstat, *On the rate of convergence of the Bergman–Vekua method for the numerical solution of elliptic boundary value problems*, SIAM J. Numer. Anal. **11** (1974), 654–680.

12. R. P. Gilbert, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York, 1969.

13. R. P. Gilbert, *Constructive Methods for Elliptic Equations*, Springer, Berlin, 1974.

14. G. Kneis, *Darstellung negativ gekrümmter Flächen durch pseudoholomorphe Funktionen*, in G. Anger (ed.), Elliptische Differentialgleichungen, vol. 1, Akademie–Verlag, Berlin, 1970.

15. M. Kracht and E. Kreyszig, *Bergman-Operatoren mit Polynomen als Erzeugenden, Manuscripta Math.* **1** (1969), 369–376.

16. E. Lanckau, *Konstruktive Methoden zur Lösung von elliptischen Differentialgleichungen mittels Bergman–Operatoren*, in G. Anger (ed.), Elliptische Differentialgleichungen, vol. 1, Akademie–Verlag, Berlin, 1970.

17. R. S. Lehman, *Developments at an analytic corner of solutions of elliptic partial differential equations*, J. Math. Mech. **8** (1959), 727–760.

18. V. E. Meister, N. Weck, and W. L. Wendland (eds.), *Function Theoretic Methods for Partial Differential Equations, Proceedings of the International Symposium Held at Darmstadt, Germany*, April 12–15, 1976, Springer, Berlin, 1976.

19. C. Miranda, *Partial Differential Equations of Elliptic Type*, 2nd ed., Springer, Berlin, 1970.

20. J. C. C. Nitsche, *Vorlesungen über Minimalflächen*, Springer, Berlin, 1975.

21. L. E. Payne, *Some general remarks on improperly posed problems for partial differential equations*, in R. J. Knops, Symposium on Non-Well-Posed Problems and Logarithmic Convexity Held in Heriot-Watt University, Edinburgh/Scotland March 22–24, 1972, Springer, Berlin, 1973.

22. S. Ruscheweyh, *Geometrische Eigenschaften der Lösungen der Differentialgleichung* $(1-z\bar{z})^2 w_{z\bar{z}} - n(n+1)w = 0$, J. Reine Angew. Math. **270** (1974), 143–157.

23. I. N. Vekua, *New Methods for Solving Elliptic Equations*, Wiley, New York, 1967.

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