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ON THE FRACTIONAL POWERS OF A SCHRÖDINGER OPERATOR WITH A HARDY-TYPE POTENTIAL

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Abstract Strong unique continuation properties and a classification of the asymptotic profiles are established for the fractional powers of a Schrödinger operator with a Hardy-type potential, by means of an Almgren monotonicity formula combined with a blow-up analysis.

Keywords: fractional elliptic equations; unique continuation; monotonicity formula; Hardy-type potentials

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1. Introduction

This paper deals with the fractional powers of the operator:

$$L_{\alpha,k}u := -\Delta u - \frac{\alpha}{|x|_k^2}u,$$

on a connected bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ and $0 \in \Omega$ where,

$$|x|_k^2 = \sum_{i=1}^{k} x_i^2 \quad \text{and} \quad \alpha \in \left(-\infty, \left(\frac{k-2}{2}\right)^2\right), \tag{1}$$

for any $k \in \{3, \ldots, N\}$. If $k = N$ we will simply write $|x|$ for $|x|_N$.

The operator $L_{\alpha,k}$ is an elliptic operator with a homogenous potential with a singular set of dimension $N-k$. In view of Hardy–Maz'ja-type inequalities, see § 2, the operator $L_{\alpha,k}$ has a discrete spectrum on $H_0^1(\Omega)$. Hence the fractional powers $L_{\alpha,k}^s$ of $L_{\alpha,k}$ with $s \in (0,1)$ can be defined in a spectral sense in a elementary way, see for example [31]. We will give a more precise definition of $L_{\alpha,k}^s$ in § 2.

In the particular case $\alpha = 0$, the operator $L_{\alpha,k}^s$ reduces to the spectral fractional Laplacian $(-\Delta)^s$ which has been intensely studied, see for example [1, 26] and the references within.

From a mathematical point of view, the interest in Hardy-type potential lies in the criticality with respect to the Laplacian since the have the same order of homogeneity. Furthermore the also is a well-established physical interest in operators with Hardy-type potential thanks to their connections with Coulomb fields, see for example [22] or [24].

In the case of the fractional powers of $L_{\alpha,N}$, a first mathematical interested lies in its connection with Hardy type inequalities. In particular, the fractional powers of $L_{\alpha,N}$ in the whole space $\mathbb{R}^N$ already appear in [20], where generalised and reversed Hardy types inequalities have been obtained using semigroup theory and estimates on the corresponding heat kernel. In this non-compact setting, the operator $L^s_{\alpha,k}$ can be defined by means of the spectral theorem for normal unbounded operator on the Hilbert space $L^2(\mathbb{R}^N)$, see for example [29, Theorem 13.33].

We will focus on the description of the asymptotic behaviour from the singular point 0 for solutions of linear equations involving the operator $L^s_{\alpha,k}$ and prove the validity of a unique continuation principle as a consequence. More precisely, we are interested in the equation:

$$L^s_{\alpha,k} u = gu \quad \text{in } \Omega, \tag{2}$$

where the potential g satisfies:

$$\begin{cases} g \in W^{1,\infty}_{loc}(\Omega \setminus \{0\}), \\ |g(x)| + |x \cdot \nabla g(x)| \leq C_g |x|^{-2s+\varepsilon}, \quad \text{for a.e. } x \in \Omega, \end{cases} \tag{3}$$

for some positive constant $C_g > 0$ and $\varepsilon \in (0, 1)$. We will classify the asymptotic profiles in 0 of solutions of (2) in a suitable weak sense, and obtain a strong unique continuation property from 0, see Theorem 2.11, Theorem 6.9, and Corollary 2.13 for a precise statements of our results. In particular, we will prove that the asymptotic profile of u in 0 is an homogenous function. We will also characterise the possible orders of homogeneity, which have a non-trivial dependence on the singular potential $\alpha|x|_k^{-2}$, see Theorem 2.11. This classification is of particular interest since the presence of an Hardy potential causes an eventual lack of regularity for the weak solutions of (2).

For the restricted fractional Laplacian with a Hardy-type potential, under similar assumptions on the potential g and with a non-linear term, a complete classification of the possible asymptotic profiles and a unique continuation property from 0 have been obtained in [13]. The asymptotic behaviour of the spectral fractional Laplacian with a Hardy-type potential is identical since the equivalent problem obtained with a Caffarelli-Silvestre extension procedure is the same locally. The restricted fractional Laplacian with a Hardy-type potential has been intensively studied in the literature, see for example [2, 4, 12, 14, 18] and the references within.

If $k = N$, it is interesting to compare our results with [13], as far as the minimal order of homogeneity of the asymptotics profiles are concerned, see (25), Theorem 2.11 and [13, Proposition 2.3]. In our cases, it is possible to compute explicitly, while for the restricted fractional Laplacian only a more implicit expression is available.

Similar results in the classical case, that is $s = 1$, in the much more general situation of multiple potentials, including cylindrical and multi-body ones, and with the presence

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of a nonlinear term, have been obtained in [16]. Furthermore in [16], the authors also studied regularity properties of the solutions by means of a Brezis-Kato argument and obtained pointwise estimates.

To study unique continuation properties from 0 for solutions of (2) we start by defining a precise functional setting for (2) by means of Interpolation Theory. Furthermore, our approach is based on an Almgren type monotonicity formula combined with a blow-up argument. Since this approach is local in nature, we need a suitable extension result to localise the problem, see Theorem 2.7 and also [7, 8, 31]. We will also need a Pohozaev type identity to develop a monotonicity formula. The singularity of the Hardy type potential $\alpha|x|_k^{-2}$, the assumptions (3) on g and the singularity or degeneracy of the Muckenhoupt weight y^{1-2s} in the hyperplane $\mathbb{R}^n \times \{0\}$ cause an eventual lack of regularity for solutions to the extended problem. We overcame this issue by means of an approximation procedure based on the Implicit Function Theorem and the ideas contained in [19].

The paper is organised as follows. In § 2 we provide the precise functional setting for (2) and state our main results. In § 3, we prove the extension Theorem 2.7, study an eigenvalue problem on a hemisphere, which will turn out to be correlated to the asymptotic profiles of weak solutions of (2), and discuss some useful inequalities. In § 4 we prove a Pohozaev type identity. In § 5 we develop a monotonicity formula for the extend problem while in § 6 we carry out the blow-up argument and prove our main results. Finally in § 7, we compute the first eigenvalue of the problem studied in 3 while in Appendix 1 we provide some further details on the functional setting for equation (2) which will be introduced in § 2.

2. Functional setting and main results

Since we deal with singular potentials of the form $\alpha|x|_k^{-2}$, Hardy-type inequalities with optimal constants are fundamental to study the positivity of $L_{\alpha,k}$ on $H_0^1(\Omega)$. In the case $k = N$ it is well known that:

$$\int_{\mathbb{R}^N} \frac{\phi^2}{|x|^2}\,dx \le \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |\nabla\phi|^2\,dx, \quad \text{for any } \phi \in C_c^\infty(\mathbb{R}^N),$$

and that $\left(\frac{2}{N-2}\right)^2$ is the optimal constant. A similar result also holds for cylindrical potential, more precisely for any $k \in \{3, \dots, N\}$:

$$\int_{\mathbb{R}^N} \frac{\phi^2}{|x|_k^2}\,dx \le \left(\frac{2}{k-2}\right)^2 \int_{\mathbb{R}^N} |\nabla\phi^2|\,dx, \quad \text{for any } \phi \in C_c^\infty(\mathbb{R}^N), \tag{4}$$

see [27, Subsection 2.1.6, Corollary 3] or [3]. Furthermore, $\left(\frac{2}{k-2}\right)^2$ is the optimal constant as conjectured in [3] and proved in [30].

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Let us consider the eigenvalue problem:

$$\begin{cases} L_{\alpha,k}u = \mu u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We say that μ is an eigenvalue of (5) if there exists $Y \in H_0^1(\Omega) \setminus \{0\}$ such that:

$$\int_\Omega \nabla Y \cdot \nabla v \, dx - \int_\Omega \frac{\alpha}{|x|_k^2} Y v \, dx = \mu \int_\Omega Y v \, dx, \quad \text{for any } v \in H_0^1(\Omega). \tag{6}$$

Thanks to (1) and (4), for any $k \in \{3, \cdots, N\}$ the energy functional:

$$J_{\alpha,k}(u) := \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{\alpha}{|x|_k^2} u^2 \, dx$$

is coercive on $H_0^1(\Omega)$ and so by the Spectral Theorem the set of the eigenvalues of (5) is a non-decreasing, positive, diverging sequence $\{\mu_{\alpha,k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$(we repeat each eigenvalue according to its multiplicity). Furthermore, there exists an orthonormal basis $\{Y_{\alpha,k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ of $L^2(\Omega)$ made of corresponding eigenfunctions. Since the first eigenfunction does not change sign, it is not restrictive to suppose that $Y_{\alpha,k,1}$ is positive.

For any Hilbert space X let $(v_1, v_2)_X$ be the scalar product on X. Furthermore let,

$$v_n := (v, Y_{\alpha,k,n})_{L^2(\Omega)}, \quad \text{for any } v \in L^2(\Omega). \tag{7}$$

Remark 2.1. In view of (4), $\|v\|_{\alpha,k} := (J_{\alpha,k}(v))^{\frac{1}{2}}$ is a norm on $H_0^1(\Omega)$ equivalent to the usual norm $\|v\|_{H_0^1(\Omega)} := \left(\int_\Omega |\nabla v|^2 \, dx\right)^{\frac{1}{2}}$.

The scalar product associated to $\|\cdot\|_{\alpha,k}$ is given by:

$$(v, w)_{\alpha,k} := \int_\Omega \nabla v \cdot \nabla w - \frac{\alpha}{|x|_k^2} v w \, dx.$$

By (6), $\{Y_{\alpha,k,n}/\sqrt{\mu_{\alpha,k,n}}\}_{n\in\mathbb{N}\setminus\{0\}}$ is an orthonormal basis of $H_0^1(\Omega)$ with respect to the norm $\|\cdot\|_{\alpha,k}$ and for any $v, w \in H_0^1(\Omega)$:

$$(v, w)_{\alpha,k} = \sum_{n=1}^{\infty} \mu_{\alpha,k,n} v_n w_n,$$

where v_n and w_n are as in (7).

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Let us consider the functional space:

$$\mathbb{H}^s_{\alpha,k}(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n^2 < +\infty \right\},$$

which is a Hilbert space with respect to the scalar product:

$$(v, w)_{\mathbb{H}^s_{\alpha,k}(\Omega)} := \sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n w_n, \quad \text{for any } v, w \in \mathbb{H}^s_{\alpha,k}(\Omega). \tag{8}$$

For any $j \in \mathbb{N} \setminus \{0\}$, and $v \in L^2(\Omega)$ it is clear that $\sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n} \in L^2(\Omega)$ and that it can be identified with the element of the dual space $(\mathbb{H}^s_{\alpha,k}(\Omega))^*$ acting on $u \in \mathbb{H}^s_{\alpha,k}(\Omega)$ as:

$$\begin{aligned} {}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\left\langle \sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n}, u \right\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} &:= \left(\sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n}, u \right)_{L^2(\Omega)} \\ &= \sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n u_n. \end{aligned}$$

It is easy to see that, if $v \in \mathbb{H}^s_{\alpha,k}(\Omega)$, then the series $\sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n}$ converges in the dual space $(\mathbb{H}^s_{\alpha,k}(\Omega))^*$ to some $F \in (\mathbb{H}^s_{\alpha,k}(\Omega))^*$ such that:

$${}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\langle F, Y_{\alpha,k,n} \rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = \mu^s_{\alpha,k,n} v_n, \quad \text{for any } n \in \mathbb{N} \setminus \{0\}.$$

It follows that, for every $v \in \mathbb{H}^s_{\alpha,k}(\Omega)$, we can define the fractional s-power of the operator $L_{\alpha,k}$ as:

$$L^s_{\alpha,k} v := \sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n} \in (\mathbb{H}^s_{\alpha,k}(\Omega))^*.$$

More precisely, the operator $L^s_{\alpha,k}$ is the Rietz isomorphism between $\mathbb{H}^s_{\alpha,k}(\Omega)$ endowed with the scalar product (8) and its dual space $(\mathbb{H}^s_{\alpha,k}(\Omega))^*$, that is:

$${}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\langle L^s_{\alpha,k} v_1, v_2 \rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = (v_1, v_2)_{\mathbb{H}^s_{\alpha,k}(\Omega)}, \quad \text{for all } v_1, v_2 \in \mathbb{H}^s_{\alpha,k}(\Omega).$$

A similar definition for the spectral fractional Laplacian, that is the operator $L_{0,N}$, was given in [8] and in [10].

We would like to characterise the space $\mathbb{H}^s_{\alpha,k}(\Omega)$ more explicitly. To this end, let $H^s(\Omega)$ be the usual fractional Sobolev space $W^{s,2}(\Omega)$, $H^s_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $H^s(\Omega)$,

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and let,

$$H_{00}^{1/2}(\Omega) := \left\{ u \in H_0^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{u^2(x)}{d(x,\partial\Omega)}\, dx < +\infty \right\},$$

endowed with the norm:

$$\|v\|_{H_{00}^{1/2}(\Omega)} := \|v\|_{H^{1/2}(\Omega)} + \left(\int_\Omega \frac{v^2(x)}{d(x,\partial\Omega)}\, dx \right)^{\frac{1}{2}}, \tag{9}$$

where $d(x,\partial\Omega) := \inf\{|x-y| : y \in \partial\Omega\}$. For any $s \in (0,1)$ let,

$$\mathbb{H}^s(\Omega) := \begin{cases} H_0^s(\Omega), & \text{if } s \in (0,1) \setminus \{\frac{1}{2}\}, \\ H_{00}^{1/2}(\Omega), & \text{if } s = \frac{1}{2}. \end{cases}$$

We also note that $H^s(\Omega) = H_0^s(\Omega)$ if and only if $s \in (0, \frac{1}{2}]$, see [25, Theorem 11.1]. In Appendix 1 we will prove the following Proposition by means of Interpolation Theory.

Proposition 2.2. *For any $k \in \{3, \ldots, N\}$, $s \in (0,1)$ and α as in (1):*

$$\mathbb{H}_{\alpha,k}^s(\Omega) = (L^2(\Omega), H_0^1(\Omega))_{s,2} = \mathbb{H}^s(\Omega),$$

with equivalent norms.

Let for any measurable function $v : \Omega \to \mathbb{R}$,

$$\tilde{v}(x) := \begin{cases} v(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then from [5, Proposition B.1] in the case $s \neq \frac{1}{2}$ and from the proof of [5, Proposition B.1] and (9) if $s = \frac{1}{2}$ we deduce the following result.

Proposition 2.3. *There exists a constant $C_{N,s,\Omega}$ such that:*

$$\|\tilde{v}\|_{H^s(\mathbb{R}^n)} \leq C_{N,s,\Omega} \|v\|_{\mathbb{H}^s(\Omega)}, \tag{10}$$

for any $v \in \mathbb{H}^s(\Omega)$.

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Proposition 2.4. *There exists a constant $K_{N,s,\Omega}$ such that for any $v \in \mathbb{H}^s(\Omega)$:*

$$\int_\Omega \frac{v^2(x)}{|x|^{2s}}\,dx \le K_{N,s,\Omega}\,\|v\|^2_{\mathbb{H}^s(\Omega)}\,. \tag{11}$$

Proof. The following Hardy-type inequality due to Herbst [22]:

$$2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}\int_{\mathbb{R}^N}\frac{v^2(x)}{|x|^{2s}}\,dx \le \int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u, holds for any $v \in H^s(\mathbb{R}^N)$. Then (11) follows from (10). □

By Proposition 2.2, we can define a weak solution to (2) as a function $u \in \mathbb{H}^s(\Omega)$ such that:

$$_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\langle L^s_{\alpha,k}u,\phi\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = \int_\Omega gu\phi\,dx, \quad \text{for any } \phi \in C^\infty_c(\Omega). \tag{12}$$

Thanks to (3), (11) and the Hölder inequality, the right hand side of (12) is well defined, that is it belongs to $(\mathbb{H}^s(\Omega))^*$ as a linear functional of ϕ.

Given the local nature of the Almgren monotonicity formula we need to localise the problem by means of an extension procedure in the spirit of [8] or [7], see also [31, Section 3.1]. Let us set some notation first. Let $z = (x,y) \in \mathbb{R}^N \times [0,+\infty)$ be the total variable in $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times [0,+\infty)$ and let,

$$C := \Omega \times (0,+\infty), \quad \partial C_L := \partial\Omega \times (0,+\infty).$$

For any open set $E \subseteq \mathbb{R}^{N+1}_+$ and any $\phi \in C^\infty(\overline{E})$ we define,

$$\|\phi\|_{H^1(E,y^{1-2s})} := \left(\int_E y^{1-2s}(\phi^2 + |\nabla\phi|^2)\,dz\right)^{\frac{1}{2}}, \tag{13}$$

and $H^1(E,y^{1-2s})$ as the completion of $C^\infty(\overline{E})$ with respect to the norm defined in (13). Thanks to [23, Theorem 11.11, Theorem 11.2, 11.12 Remarks (iii)], for any Lipschitz subset E of $\mathbb{R}^{N+1}_+$, the space $H^1(E,y^{1-2s})$ can be explicitly characterised as:

$$H^1(E,y^{1-2s}) = \left\{V \in W^{1,1}_{\text{loc}}(E) : \int_E y^{1-2s}(V^2 + |\nabla V|^2)\,dz < +\infty\right\}.$$

Proposition 2.5. *For any $\phi \in C^\infty_c(\mathbb{R}^N \times [0,+\infty))$ and any $k \in \{3,\ldots,N\}$:*

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\frac{\phi^2}{|x|^2_k}dz \le \left(\frac{2}{k-2}\right)^2\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla_x\phi|^2\,dz, \tag{14}$$

where ∇_x is the gradient respect to the first N variables.

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Proof. Let $\phi \in C_c^\infty(\Omega \times [0,+\infty))$ and $k \in \{3,\dots,N\}$. Then $\phi(\cdot,y) \in C_c^\infty(\Omega)$ for any $y \in [0,\infty)$ and so multiplying by y^{1-2s} and integrating over $(0,\infty)$ we deduce (14) from (4). □

Let,

$$H^1_{0,L}(C,y^{1-2s}) := \left\{V \in H^1(C,y^{1-2s}) : V = 0 \quad \text{on } \partial C_L\right\}. \tag{15}$$

The condition $V=0$ on ∂C_L is meant in a classical trace sense. Indeed the weight y^{1-2s} is smooth, bounded and strictly positive on $\Omega \times [y_1,y_2]$ for any $0 < y_1 < y_2 < +\infty$, and so we can use classical trace theory for the space $H^1(\Omega \times (y_1,y_2))$ for any $0 < y_1 < y_2 < +\infty$.

From [8, Proposition 2.1] and [6, Proposition 2.1, Lemma 2.6] we deduce the following result.

Proposition 2.6. *There exists a linear and continuous trace operator:*

$$\mathrm{Tr} : H^1_{0,L}(C,y^{1-2s}) \to \mathbb{H}^s(\Omega)$$

which is also surjective.

See § 3 for a proof of the following next extension theorem,

Theorem 2.7. *Let $v \in \mathbb{H}^s(\Omega)$, $k \in \{3,\dots,N\}$ and α as in (1). Then there exists a unique function $V \in H^1_{0,L}(C,y^{1-2s})$ such that V weakly solves the problem:*

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla V) = y^{1-2s}\frac{\alpha}{|x|_k^2}V, & \textit{in } C,\\ \mathrm{Tr}(V) = v, & \textit{on } \Omega,\\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial V}{\partial y} = c_{N,s}L^s_{k,\alpha}v, & \textit{on } \Omega, \end{cases} \tag{16}$$

where $c_{N,s} > 0$ is a constant depending only on N and s, in the sense that:

$$\int_C y^{1-2s}\nabla V \cdot \nabla\phi\, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}V\phi\, dz = c_{N,s}\ {}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\left\langle L^s_{\alpha,k}v, \phi(\cdot,0)\right\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)}, \tag{17}$$

for any $\phi \in C_c^\infty(\Omega \times [0,+\infty))$. Furthermore,

$$\int_C y^{1-2s}|\nabla V(x,y)|^2\, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}V^2\, dz = c_{N,s}\, \|v\|^2_{\mathbb{H}^s_{\alpha,k}(\Omega)}, \tag{18}$$

and V is the only solution to the minimisation problem:

$$\inf\left\{\int_C y^{1-2s}\left(|\nabla W|^2 - \frac{\alpha}{|x|_k^2}w^2\right)dz : W \in H^1_{0,L}(C,y^{1-2s}) \textit{and } \mathrm{Tr}(W) = v\right\}. \tag{19}$$

From Theorem 2.7 we deduce the following corollary.

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Corollary 2.8. *Let $u \in \mathbb{H}^s(\Omega)$ be a solution of (12). Then there exists a unique $U \in H^1_{0,L}(C, y^{1-2s})$ such that:*

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = y^{1-2s}\frac{\alpha}{|x|_k^2}U, & \textit{in } C, \\ \operatorname{Tr}(U) = u, & \textit{on } \Omega, \\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y} = c_{N,s}gu, & \textit{on } \Omega, \end{cases} \tag{20}$$

where $c_{N,s} > 0$ is the constant depending only on N and s defined in Theorem 2.7, in the sense that:

$$\int_C y^{1-2s}\nabla U \cdot \nabla\phi\, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}U\phi\, dz = c_{N,s}\int_\Omega gu\phi(\cdot,0)\, dx, \tag{21}$$

for any $\phi \in C_c^\infty(\Omega \times [0,+\infty))$. Furthermore,

$$\int_C y^{1-2s}|\nabla U(x,y)|^2\, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}U^2\, dz = c_{N,s}\,\|u\|^2_{\mathbb{H}^s_{\alpha,k}(\Omega)} = c_{N,s}\int_\Omega gu^2\, dx.$$

Let for, any $r > 0$,

$$B_r^+ := \{z \in \mathbb{R}^{N+1}_+ : |z| < r\}, \quad S_r^+ := \{z \in \mathbb{R}^{N+1}_+ : |z| = r\},$$
$$B_r' := \{z = (x,y) \in \mathbb{R}^{N+1} : |x| < r, y = 0\}.$$

Let $\theta = \frac{z}{|z|}$ for any $z \in \mathbb{R}^{N+1}$ and $\theta' = (\theta_1, \ldots, \theta_N)$.

The asymptotic profile of a solution U of (21) in 0 will turn out to be related to the following eigenvalue problem:

$$\begin{cases} -\operatorname{div}_{\mathbb{S}}(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}}Z) - \theta_{N+1}^{1-2s}\frac{\alpha}{|\theta|_k^2}Z = \gamma\theta_{N+1}^{1-2s}Z, & \text{in } \mathbb{S}^+, \\ -\lim\limits_{\theta_{N+1}\to 0^+}\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}}Z\cdot\nu = 0, & \text{on } \mathbb{S}', \end{cases} \tag{22}$$

where ν is the outer normal vector to $\mathbb{S}^+$ on $\mathbb{S}'$, that is $\nu = -(0, \ldots, 0, 1)$ and

$$\mathbb{S} := \{\theta \in \mathbb{R}^{N+1} : |\theta|^2 = 1\},$$
$$\mathbb{S}^+ := \{\theta = (\theta', \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} > 0\},$$
$$\mathbb{S}' := \{\theta = (\theta', \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} = 0\}.$$

We refer to § 3.1 for a variational formulation of (22). By classical spectral theory, see § 3.1 for further details, the eigenvalues of (22) are a non-decreasing and diverging sequence $\{\gamma_{\alpha,k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ (we repeat each eigenvalue according to its multiplicity).

We have the following estimate on $\gamma_{\alpha,k,1}$:

$$\gamma_{\alpha,k,1} > -\left(\frac{N-2s}{2}\right)^2,$$

for any $k \in \{3,\dots,N\}$ and α as in (1), see Proposition 3.4. We can actually compute $\gamma_{\alpha,k,1}$ in terms of the first eigenvalue $\eta_{\alpha,k,1}$ of the problem:

$$-\Delta_{\mathbb{S}'}\Psi - \frac{\alpha}{|\theta'|_k^2}\Psi = \eta\Psi \quad \text{in } \mathbb{S}', \tag{23}$$

as

$$\gamma_{\alpha,k,1} = 2(1-s)\left[\sqrt{\left(\frac{N-2}{2}\right)^2 + \eta_{\alpha,k,1}} - \frac{N-2}{2}\right] + \eta_{\alpha,k,1}, \tag{24}$$

see § 7. In particular, if $k = N$ then $\eta_{\alpha,k,1} = -\alpha$ and so,

$$\gamma_{\alpha,N,1} = 2(1-s)\left[\sqrt{\left(\frac{N-2}{2}\right)^2 - \alpha} - \frac{N-2}{2}\right] - \alpha. \tag{25}$$

If $k = N$, we are able to obtain an explicit expression of $\gamma_{\alpha,N,1}$ for any $\alpha \in \left(-\infty, \frac{N-2}{2}\right)$. For the restricted fractional Laplacian with a Hardy-type potential it is also possible to obtain a formula for the first eigenvalue of the corresponding problem on a hemisphere although with a more implicit expression, see [13, Proposition 2.3].

Theorem 2.9. *Let U be a non-trivial solution of (21) and suppose that g satisfies (3). Then there exist an eigenvalue $\gamma_{\alpha,k,n}$ of (22) and a correspondent eigenfunction Z such that:*

$$\lambda^{\frac{N-2s}{2}-\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,n}}}U(\lambda z) \to |z|^{-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,n}}}Z(z/|z|) \quad \text{as } \lambda \to 0^+$$

strongly in $H^1(B_1^+, y^{1-2s})$.

Remark 2.10. Let $r > 0$. Thanks to [25] there exists a linear and continuous trace operator:

$$\mathrm{Tr}_{B_r'} : H^1(B_r^+, y^{1-2s}) \to H^s(B_r').$$

If $\overline{B_r'} \subset \Omega$, then any function $V \in H^1(B_r^+, y^{1-2s})$ can be extended to an element $\tilde{V}$ of $H^1_{0,L}(\mathcal{C}, y^{1-2s})$ (see (15) and [9]) and $\mathrm{Tr}_{B_r'}(V) = \mathrm{Tr}(\tilde{V})_{|B_r'}$. Therefore with a slight abuse we will simply use Tr instead of $\mathrm{Tr}_{B_r'}$ to indicate the operator $\mathrm{Tr}_{B_r'}$.

From Remark 2.10 and the previous theorem we obtain the following.

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Theorem 2.11. *Let u be a non-trivial solution solution of (12) and suppose that g satisfies (3). Then there exist an eigenvalue $\gamma_{\alpha,k,n}$ of (22) and a correspondent eigenfunction Z such that:*

$$\lambda^{\frac{N-2s}{2}-\sqrt{(\frac{N-2s}{2})^2+\gamma_{\alpha,k,n}}}u(\lambda x) \to |x|^{-\frac{N-2s}{2}+\sqrt{(\frac{N-2s}{2})^2+\gamma_{\alpha,k,n}}}\mathrm{Tr}(Z(|))(x) \quad as \lambda \to 0^+$$

strongly in $H^s(B_1')$.

We will also prove a more precise and complete version of Theorem 2.9 and Theorem 2.11 in § 6, computing the coordinates of the eigenfunction Z respect to a basis of the eigenspace corresponding to $\gamma_{\alpha,k,n}$. Furthermore, we can deduce the following strong unique continuation properties as corollaries of Theorem 2.9 and Theorem 2.11 respectively.

Corollary 2.12. *Let U be a solution of (21) and suppose that g satisfies (3). If*

$$U(z) = o(|z|^n) = o(|(x,y)|^n) as\ x \to 0,\ y \to 0^+, \quad for\ any\ n \in \mathbb{N}, \tag{26}$$

then $U \equiv 0$ on $\Omega \times (0,\infty)$.

Corollary 2.13. *Let u be a solution of (12) and suppose that g satisfies (3). If*

$$u(x) = o(|x|^n) as\ x \to 0, \quad for\ any\ n \in \mathbb{N},$$

then $u \equiv 0$ on Ω.

Remark 2.14. We have considered equation (2) with assumption (3) on the potential g for the sake of simplicity. With simple modifications to our arguments it is also possible to obtain the same results for a potential $g \in W^{\frac{N}{2s}+\varepsilon}(\Omega)$ for some $\varepsilon \in (0,1)$, see [19, Proposition 2.3] for the corresponding Pohozaev identity. Furthermore, we can obtain analogous results for the more general equation:

$$L^s_{k,\alpha}u = \frac{\lambda}{|x|^{2s}}u + gu,$$

with $\lambda \in \left(-\infty, 2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}\right)$ with the same approach, where Γ is the usual Γ-function.

3. Preliminaries

We start this section by proving Theorem 2.7.

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Proof of Theorem 2.7. We follow the proof of [8, Proposition 2.1]. Let $v \in \mathbb{H}^s(\Omega)$ and consider:

$$V(x,y) := \sum_{n=1}^{\infty} v_n Y_{\alpha,k,n}(x) h_n(y), \quad \text{where } v_n = \int_\Omega v Y_{\alpha,k,n}\, dx, \tag{27}$$

and $h_n : (0,+\infty) \to \mathbb{R}$ is a solution to the problem:

$$\begin{cases} h_n'' + \frac{1-2s}{y} h_n' - \mu_{\alpha,k,n} h_n = 1, & \text{on } (0,+\infty), \\ h_n(0) = 1, \\ \lim_{y\to\infty} h_n(y) = 0. \end{cases} \tag{28}$$

From the proof of [8, Proposition 2.1], (28) admits a unique solution h_n for any $n \in \mathbb{N} \setminus \{0\}$ and:

$$-\lim_{y\to 0^+} y^{1-2s} h_n'(y) = c_{N,s} \mu_{\alpha,k,n}^s, \tag{29}$$

for some positive constant $c_{N,s} > 0$ depending only on N and s. Furthermore for any $y \in [0+,\infty)$ by (27) and Remark 2.1.

$$\begin{aligned} \int_\Omega \left| \frac{\partial V}{\partial y}(x,y) \right|^2 dx + \int_\Omega |\nabla_x V(x,y)|^2\, dx - \int_\Omega \frac{\alpha}{|x|_k^2} V^2(x,y)\, dx \\ = \sum_{n=1}^{\infty} v_n^2 (h_n'(y))^2 + \mu_{\alpha,k,n} v_n^2 h_n(y)^2. \end{aligned} \tag{30}$$

Proceeding exactly as in [8, Proposition 2.1] we can show that (18) holds. Hence, in view of (14), $V \in H^1(C, y^{1-2s})$ and $\sum_{n=1}^{j} v_n Y_{\alpha,k,n}(x) h_n(y) \to V$ in $H^1(C, y^{1-2s})$ as $j \to \infty$. In conclusion $V \in H^1_{0,L}(C, y^{1-2s})$ since $\sum_{n=1}^{j} v_n Y_{\alpha,k,n}(x) h_n(y) \in H^1_{0,L}(C, y^{1-2s})$ for any $j \in \mathbb{N}, j \geq 1$.

In contrast to [8, Proposition 2.1], V might not be smooth for $y > 0$ since the functions $Y_{\alpha,k,n}$ might not be smooth on Ω. Then we prove that V satisfies (16) in the weak sense given by (17). Let $\phi \in C_c^\infty(\Omega \times [0,+\infty))$. Then,

$$\phi(x,y) = \sum_{n=1}^{\infty} \phi_n(y) Y_{\alpha,k,n}(x), \quad \text{where } \phi_n(y) := \int_\Omega \phi(x,y) Y_{\alpha,k,n}(x)\, dx,$$

and similarly to (30):

$$\int_\Omega |\nabla \phi(x,y)|^2\, dx - \int_\Omega \frac{\alpha}{|x|_k^2} \phi^2(x,y)\, dx = \sum_{n=1}^{\infty} (\phi_n'(y))^2 + \mu_{\alpha,k,n} \phi_n(y)^2. \tag{31}$$

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Then by (27) and Remark 2.1

$$\int_\Omega \nabla V(x,y)\cdot\nabla\phi(x,y)\,dx - \int_\Omega \frac{\alpha}{|x|_k^2}V(x,y)\phi(x,y)\,dx$$
$$= \sum_{n=1}^{\infty} v_n h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} v_n h_n(y)\phi_n(y). \tag{32}$$

Furthermore, for any $j\in\mathbb{N}$, by Hölder's inequality:

$$\left|\int_0^{+\infty} y^{1-2s}\left(\sum_{n=j}^{\infty} v_n h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} v_n h_n(y)\phi_n(y)\right)dy\right|$$
$$\leq \frac{1}{2}\int_0^{+\infty} y^{1-2s}\left(\sum_{n=j}^{\infty} v_n^2 (h_n'(y))^2 + \mu_{\alpha,k,n} v_n^2 h_n(y)^2\right)dy$$
$$+\frac{1}{2}\int_0^{+\infty} y^{1-2s}\left(\sum_{n=j}^{\infty} (\phi_n'(y))^2 + \mu_{\alpha,k,n}\phi_n(y)^2\right)dy.$$

By (30), (31) and the Monotone Convergence Theorem we conclude that:

$$\lim_{j\to\infty}\int_0^{\infty} y^{1-2s}\left(\sum_{n=j}^{\infty} v_n h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} v_n h_n(y)\phi_n(y)\right)dy = 0.$$

Hence we may change the order of summation and integration in (32) obtaining:

$$\int_C y^{1-2s}\left(\nabla V\cdot\nabla\phi - \frac{\alpha}{|x|_k^2}V\phi\right)dz$$
$$= \sum_{n=1}^{\infty} v_n \int_0^{+\infty} y^{1-2s}(h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} h_n(y)\phi_n(y))\,dy.$$

An integration by parts, in view of (28) and (29), yields:

$$\int_0^{+\infty} y^{1-2s}(h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} h_n(y)\phi_n(y))\,dy = c_{N,s}\mu_{\alpha,k,n}^s\phi_n(0).$$

It follows that:

$$\int_C y^{1-2s}\nabla V\cdot\nabla\phi\,dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}V\phi\,dz = c_{N,s}\sum_{n=1}^{\infty}\mu_{\alpha,k,n}^s v_n\phi_n(0),$$

and so we have proved (17). If V_1 and V_2 solve (16) then by (1), (17) and (14) we deduce that:

$$\int_C y^{1-2s}|\nabla(V_1 - V_2)|^2\,dz = 0, \quad \text{and} \quad \operatorname{Tr}(V_1 - V_2) = 0$$

thus $V_1 = V_2$. Finally V solves the minimising problem (19) in view of (17) and a density argument. □

By [13] and [28, Theorem 19.7] we have the following result.

Proposition 3.1. *For any $r>0$ there exists a linear and continuous trace operator:*

$$\operatorname{Tr}_{S_r^+} : H^1(B_r^+, y^{1-2s}) \to L^2(B_r^+, y^{1-2s}),$$

which is also compact.

For the sake of simplicity, we will write V instead of $\operatorname{Tr}_{S_r^+}(V)$ on S_r^+.

Remark 3.2. For any $r>0$ and any $V, W \in H^1(B_r^+, y^{1-2s})$, thanks to the Coarea Formula,

$$\int_{B_r^+} \left| y^{1-2s}\nabla U \cdot \frac{z}{|z|} W \right| dz = \int_0^r \left(\int_{S_\rho^+} \left| y^{1-2s}\nabla U \cdot \frac{z}{\rho} W \right| dS \right) d\rho,$$

hence the function $f(\rho) := \int_{S_\rho^+} \left| y^{1-2s}\nabla U \cdot \frac{z}{\rho} W \right| dS$ is a well-defined element of $L^1(0,r)$. In particular a.e. $\rho \in (0,r)$ is a Lebesgue point of f.

Reasoning as in [13, Lemma 3.1] or[19, Proposition 3.7] we can prove the following.

Proposition 3.3. *Let U be a solution of (21). For a.e. $r>0$ such that $\overline{B_r'} \subset \Omega$ and any $W \in H^1(B_r^+, y^{1-2s})$:*

$$\begin{aligned}\int_{B_r^+} y^{1-2s} &\left(\nabla U \cdot \nabla W - \frac{\alpha}{|x|_k^2} UW \right) dz \\ &= \frac{1}{r} \int_{S_r^+} y^{1-2s} \nabla U \cdot z\, W\, dS + c_{N,s} \int_{B_r'} g \operatorname{Tr}(U)\operatorname{Tr}(W)\, dx. \end{aligned} \tag{33}$$

3.1. An eigenvalue problem on $\mathbb{S}^+$

In this section, we provide a variational formulation of problem (22). To this end we consider the space:

$$L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) := \{\Psi : \mathbb{S}^+ \to \mathbb{R}\,\text{measurable:}\ \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}\Psi^2\, dS < +\infty\},$$

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and the space $H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ defined as the completion of $C^\infty(\overline{\mathbb{S}^+})$ with respect to the norm:

$$\|\phi\|_{H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})} := \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} (\phi^2 + |\nabla_{\mathbb{S}} \phi|^2) \, dS \right)^{1/2},$$

where $\nabla_{\mathbb{S}}$ is the Riemannian gradient respect to the standard metric on $\mathbb{S}$.

Proposition 3.4. *For any $k \in \{3, \ldots, N\}$:*

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\Psi^2}{|\theta|_k^2} \, dS \leq \left(\frac{N-2s}{2}\right)^2 \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\Psi|^2 \, dS + \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \Psi|^2 \, dS, \quad (34)$$

for any $\Psi \in H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$.

Proof. Let $\phi \in C^\infty(\overline{\mathbb{S}^+})$, $f \in C_c^\infty((0, +\infty))$ with $f \neq 0$, and $V(z) := V(r\theta) = \phi(\theta) f(r)$. From (14) we obtain, passing in polar coordinates,

$$\begin{aligned}
&\left(\frac{k-2}{2}\right)^2 \left(\int_0^\infty r^{N-1-2s} f^2(r) \, dr \right) \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\phi^2}{|\theta|_k^2} \, dS \right) \\
&\qquad \leq \left(\int_0^\infty r^{N+1-2s} |f'(r)|^2 \, dr \right) \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \phi^2 \, dS \right) \\
&\qquad + \left(\int_0^\infty r^{N-1-2s} f^2(r) \, dr \right) \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \phi|^2 \, dS \right),
\end{aligned}$$

and so, thanks to the optimality of the classical Hardy constant, see [21, Theorem 330],

$$\begin{aligned}
&\left(\frac{k-2}{2}\right)^2 \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\phi^2}{|\theta|_k^2} \, dS \right) \\
&\leq \inf_{f \in C_c^\infty((0,+\infty)), f \neq 0} \frac{\int_0^\infty r^{N+1-2s} |f'(r)|^2 \, dr}{\int_0^\infty r^{N-1-2s} f(r)^2 \, dr} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \phi^2 \, dS \right) \\
&+ \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \phi|^2 \, dS \\
&= \left(\frac{N-2s}{2}\right)^2 \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\phi|^2 \, dS + \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \phi|^2 \, dS.
\end{aligned}$$

In conclusion (34) follows by density. □

For any $k \in \{3, \ldots, N\}$ and α as in (1), we say that γ is an eigenvalue of (22) if there exists a function $Z \in H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \setminus \{0\}$ such that:

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}} Z \cdot \nabla_{\mathbb{S}} \Psi \, dS - \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\alpha}{|\theta|_k^2} Z \Psi \, dS = \gamma \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} Z \Psi \, dS, \quad (35)$$

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for any $\Psi \in H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$. By (1), (34), the Spectral Theorem, and the compactness of the embedding $H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \hookrightarrow L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ (see [28, Theorem 19.7]) the eigenvalues of (22) are a non-decreasing and diverging sequence $\{\gamma_{\alpha,k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ (we repeat each eigenvalue according to its multiplicity). Let, for future reference,

$$V_{\alpha,k,n} \text{be the eigenspace of problem (22) associated to the eigenvalue } \gamma_{\alpha,k,n}, \tag{36}$$

$$M_{\alpha,k,n} \text{be the dimension of } V_{\alpha,k,n}, \tag{37}$$

$$\begin{aligned}&\{Z_{\alpha,k,n,i} : i \in \{1,\dots,M_{\alpha,k,n}\}\} \text{be a } L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \text{ orthonormal basis of } V_{\alpha,k,n}\\ &\text{of eigenfunctions of problem (22).}\end{aligned} \tag{38}$$

Finally $\{Z_{\alpha,k,n}\}_{n\in\mathbb{N}\setminus\{0\}} := \bigcup_{n=1}^{\infty}\{Z_{\alpha,k,n,i} : i \in \{1,\dots,M_{\alpha,k,n}\}\}$ is an orthonormal basis of $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$.

Remark 3.5. It is worth noticing that $Z_{\alpha,k,n}$ cannot vanish identically on $\mathbb{S}'$. We argue by contradiction. In view of [13, Lemma 2.1], we can show with a direct computation that $V(z) := |z|^{-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,n}}} Z_{\alpha,k,n}(z/|z|)$ solves $\operatorname{div}(y^{1-2s}\nabla V) - y^{1-2s}\frac{\alpha}{|x|_k^2}V = 0$ on $\mathbb{R}^{N+1}_+$ and satisfies both zero Dirichlet and zero Neumann condition on $\mathbb{R}^N \times \{0\}$. Let

$$\Sigma_k := \{z \in \mathbb{R}^{N+1} : |x|_k = 0\}. \tag{39}$$

Note that Σ_k has Lebesgue measure 0 and that V is a solution to an elliptic equitation with a Muckenhoupt weight and bounded coefficients away from Σ_k. Then by the unique continuation principles proved in [32], we conclude that $V \equiv 0$. Hence $Z_{\alpha,k,n} \equiv 0$ which is a contradiction.

3.2. Inequalities in $H^1(B_r^+, y^{1-2s})$

In this subsection, we prove some useful inequalities.

Proposition 3.6. *For any $r>0$, any $k \in \{0,\dots,N\}$, and any $V \in H^1(B_r^+, y^{1-2s})$:*

$$\left(\frac{k-2}{2}\right)^2 \int_{B_r^+} y^{1-2s}\frac{V^2}{|x|_k^2}\,dz \le \int_{B_r^+} y^{1-2s}|\nabla V|^2\,dz + \frac{N-2s}{2r}\int_{S_r^+} y^{1-2s}V^2\,dz. \tag{40}$$

Proof. By density it is enough to prove (40) for any $\phi \in C^\infty(\overline{B_r^+})$. Passing in polar coordinates, by (34) and [13, Lemma 2.4], we have that:

$$\left(\frac{k-2}{2}\right)^2 \int_{B_r^+} y^{1-2s}\frac{V^2}{|x|_k^2}\,dz = \left(\frac{k-2}{2}\right)^2 \int_0^r \rho^{N-1-2s}\left(\int_{\mathbb{S}^+}\frac{V^2(\rho\theta)}{|\theta|_k^2}\,dS\right)d\rho$$

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$$
\begin{aligned}
&\leq \int_0^r \rho^{N-1-2s}\left(\left(\frac{N-2s}{2}\right)^2\int_{\mathbb{S}+}\theta_{N+1}^{1-2s}|V^2(\rho\theta)|^2\,dS+\int_{\mathbb{S}+}\theta_{N+1}^{1-2s}|\nabla_{\mathbb{S}}V(\rho\theta)|^2\,dS\right)d\rho\\
&=\left(\frac{N-2s}{2}\right)^2\int_{B_r^+}y^{1-2s}\frac{V^2}{|z|^2}\,dz+\int_0^r\rho^{N-1-2s}\left(\int_{\mathbb{S}+}\theta_{N+1}^{1-2s}|\nabla_{\mathbb{S}}V(\rho\theta)|^2\,dS\right)d\rho\\
&\leq \frac{N-2s}{2r}\int_{S_r^+}y^{1-2s}V^2\,dS\\
&+\int_0^r\rho^{N+1-2s}\left(\int_{\mathbb{S}+}\theta_{N+1}^{1-2s}\left(\frac{1}{\rho^2}|\nabla_{\mathbb{S}}V(\rho\theta)|^2+\left|\frac{\partial V}{\partial\rho}(\rho\theta)\right|^2\right)dS\right)d\rho\\
&=\frac{N-2s}{2r}\int_{S_r^+}y^{1-2s}V^2\,dS+\int_{B_r^+}y^{1-2s}|\nabla V|^2\,dz,
\end{aligned}
$$

hence we have proved (40). □

Proposition 3.7. *Let $r>0$ and suppose that $h : B'_r :\to \mathbb{R}$ is a measurable function such that:*

$$
|h(x)|\leq C_h|x|^{-2s+\varepsilon},\quad \text{for a.e. } x\in B'_r, \tag{41}
$$

for some positive constant C_h and some $\varepsilon\in(0,1)$. Then for any $k\in\{3,\dots,N\}$, any α as in (1) and any $V\in H^1(B_r^+,y^{1-2s})$

$$
\begin{aligned}
&\int_{B'_r}|h|\,\mathrm{Tr}(V)^2\,dx\\
&\quad\leq k_{N,s,h}r^{\varepsilon}\left(\int_{B_r^+}y^{1-2s}|\nabla V|^2\,dz-\int_{B_r^+}y^{1-2s}\frac{\alpha}{|x|_k^2}V^2\,dz+\frac{N-2s}{2r}\int_{S_r^+}y^{1-2s}V^2\,dz\right),
\end{aligned} \tag{42}
$$

where $k_{N,s,h}$ is a positive constant depending only on N, s, C_h.

Proof. The claim follows from (41), [13, Lemma 2.5], and (40). □

In view of (1) there exists $r_0>0$ such that:

$$
\overline{B_{r_0}^+}\subset C\quad\text{and}\quad \alpha\left(\frac{2}{k-2}\right)^2+c_{N,s}k_{N,s,g}r_0^{\varepsilon}<1, \tag{43}
$$

where $k_{N,s,g}$ is as in Proposition 3.7, $c_{N,s}$ as in Theorem 2.7 and g as in (3).

Proposition 3.8. *Let $k\in\{3,\dots,N\}$, α as (1), g as in (3), $c_{N,s}$ as in Theorem 2.7 and r_0 as in (43). Then for any $V\in H^1(B_r^+,y^{1-2s})$ and any $r\in(0,r_0]$*

$$
\int_{B_r^+}y^{1-2s}\left(|\nabla W|^2-\frac{\alpha}{|x|_k^2}W^2\right)dz
$$

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$$
\begin{aligned}
& - c_{N,s} \int_{B_r'} g \,\mathrm{Tr}(W)^2 \, dx + \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} W^2 dS \\
& \geq \left(1 - \alpha \left(\frac{2}{k-2}\right)^2 + c_{N,s} k_{N,s,g} r_0^{\varepsilon}\right) \left(\int_{B_r^+} y^{1-2s} |\nabla W|^2 \, dz \right. \\
& \left. + \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} W^2 dS \right). \qquad (44)
\end{aligned}
$$

Proof. The claim follows from Proposition 3.7, (3) and (40). □

4. Approximated problems and a Pohozaev-type Identity

In order to obtain a Pohozaev type identity for a weak solution of (20), we approximate it with a family of solutions to more regular problems. Then we obtain a Pohozaev-type identity for such solutions and pass to the limit.

Let for any $r > 0$

$$
H^1_{0,S_r^+}(B_r^+, y^{1-2s}) := \overline{\{\phi \in C^\infty(\overline{B_r^+}) : \phi = 0 \,\text{on}\, S_r^+\}}^{\|\cdot\|_{H^1(B_r^+, y^{1-2s})}}. \qquad (45)
$$

Remark 4.1. Let r_0 be as in (43). By (44) and the Poincaré inequality, for any $r \in (0, r_0)$,

$$
\|W\|_{g,\alpha,k,0} := \left(\int_{B_r^+} y^{1-2s} \left(|\nabla W|^2 - \frac{\alpha}{|x|_k^2} W^2 \right) dz - c_{N,s} \int_{B_r'} g \,\mathrm{Tr}(W)^2 \, dx \right)^{\frac{1}{2}},
$$

defines a norm on $H^1_{0,S_r^+}(B_r^+, y^{1-2s})$ equivalent to (13). Furthermore

$$
\begin{aligned}
\|W\|_{g,\alpha,k} := & \left(\int_{B_r^+} y^{1-2s} \left(|\nabla W|^2 - \frac{\alpha}{|x|_k^2} W^2 \right) dz - c_{N,s} \int_{B_r'} g \,\mathrm{Tr}(W)^2 \, dx \right. \\
& \left. + \int_{S_r^+} y^{1-2s} W^2 \, dz \right)^{\frac{1}{2}}
\end{aligned}
$$

defines a norm on $H^1(B_r^+, y^{1-2s})$ equivalent to (13).

Theorem 4.2. *Let U be a weak solutions of (20), and r_0 as in (43). Then there exists $\tilde{\lambda} > 0$ such that for any $\lambda \in (0, \tilde{\lambda})$ the problem:*

$$
\begin{cases}
-\,\mathrm{div}(y^{1-2s} \nabla V) = y^{1-2s} \frac{\alpha}{|x|_k^2 + \lambda^2} V, & \text{in } B_{r_0}^+, \\
V = U, & \text{on } S_{r_0}^+, \\
-\lim_{y \to 0^+} y^{1-2s} \frac{\partial V}{\partial y} = c_{N,s} g \,\mathrm{Tr}(V), & \text{on } B_{r_0}',
\end{cases}
$$

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where $c_{N,s}>0$ is as in Theorem 2.7, admits a weak solution $U_\lambda\in H^1(B^+_{r_0},y^{1-2s})$, i.e.:

$$\int_{B^+_{r_0}} y^{1-2s}\nabla U_\lambda\cdot\nabla W\,dz-\int_{B^+_{r_0}} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda W\,dz=c_{N,s}\int_{B'_{r_0}} g\,\mathrm{Tr}(V)\,\mathrm{Tr}(W)\,dx, \tag{46}$$

for any $W\in H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s})$, and $U_\lambda=U$ on $S^+_{r_0}$. Furthermore,

$$U_\lambda\to U\,\textit{strongly in } H^1(B^+_{r_0},y^{1-2s})\quad \textit{as } \lambda\to 0^+.$$

Proof. Let us consider the map $\Phi:\mathbb{R}\times H^1_{0,S^+_r}(B^+_r,y^{1-2s})\to(H^1_{0,S^+_r}(B^+_r,y^{1-2s}))^*$ defined as

$$\begin{aligned}\Phi(\lambda,V)(W):=&\int_{B^+_{r_0}} y^{1-2s}\nabla V\cdot\nabla W\,dz-\int_{B^+_{r_0}} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}VW\,dz\\&-c_{N,s}\int_{B'_{r_0}} g\,\mathrm{Tr}(V)\,\mathrm{Tr}(W)\,dx+\int_{B^+_{r_0}} y^{1-2s}\left(\frac{\alpha}{|x|_k^2+\lambda^2}-\frac{\alpha}{|x|_k^2}\right)UW\,dz,\end{aligned}$$

for any $W\in H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s})$. It is clear that Φ is well defined and that Φ is continuous in $(0,0)$ in view of Hölder's inequality, Proposition 3.7, (3), and (40). Furthermore $\Phi(0,0)=0$.

Let us prove that $\Phi_V(0,0)\in\mathcal{L}(H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s}),(H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s})^*)$ is an isomorphism, where Φ_V is the partial derivative with respect to V of Φ. For any $W_1,W_2\in H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s})$:

$${}_{(H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s}))^*}\langle\Phi_V(0,0)(W_1),W_2\rangle_{H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s})}=(W_1,W_2)_{g,\alpha,k,0}\,.$$

Hence, by Remark 4.1, $\Phi_V(0,0)$ is the Rietz isomorphism associated to the norm $\|\cdot\|_{g,\alpha,k,0}$.

We are now in position to apply the Implicit Function Theorem to Φ in the point $(0,0)$ and conclude that there exist $\tilde\lambda>0$, $\rho>0$ and a function:

$$f:(-\tilde\lambda,\tilde\lambda)\to B_\rho(0), \tag{47}$$

continuous in 0, such that $\Phi(\lambda,V)=0$ if and only if $V=f(\lambda)$ for any $\lambda\in(-\tilde\lambda,\tilde\lambda)$ and $V\in B_\rho(0)$. The set $B_\rho(0)$ in (47) is defined as $B_\rho(0)=\{V\in H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s}):\|V\|_{H^1(B^+_{r_0},y^{1-2s})}<\rho\}$.

It follows that $U_\lambda:=U-f(\lambda)$ solves (46) for any $\lambda\in(0,\tilde\lambda)$ since U is a solution of (33). Furthermore, $U_\lambda\to U$ strongly in $H^1(B^+_{r_0},y^{1-2s})$ as $\lambda\to0^+$ since f is continuous in 0 and $f(0)=0$. $\square$

Remark 4.3. Let U_λ be a solution of (46). Then, reasoning in the same way of Proposition 3.3, we can prove that for a.e. $r\in(0,r_0)$, a.e. $\rho\in(0,r)$ and any

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$W \in H^1(B_r^+ \setminus B_\rho^+, y^{1-2s})$

$$\int_{B_r^+\setminus B_\rho^+} y^{1-2s}\left(\nabla U_\lambda\cdot\nabla W - \frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda W\right)dz$$
$$= \frac{1}{r}\int_{S_r^+} y^{1-2s}\nabla U_\lambda\cdot z\, W\, dS - \frac{1}{\rho}\int_{S_\rho^+} y^{1-2s}\nabla U_\lambda\cdot z\, W\, dS$$
$$+ c_{N,s}\int_{B_r'\setminus B_\rho'} g\,\mathrm{Tr}(U_\lambda)\,\mathrm{Tr}(W)\,dx. \tag{48}$$

Let ν be the outer normal vector to B_r^+ on S_r^+, that is $\nu(z) = \frac{z}{|z|}$.

Proposition 4.4. *For any $\lambda \in (0, \tilde\lambda)$, let U_λ be a solution of (46). Then for a.e. $r \in (0, r_0)$*

$$\frac{r}{2}\int_{S_r^+} y^{1-2s}|\nabla U_\lambda|^2\,dS - r\int_{S_r^+} y^{1-2s}|\nabla U_\lambda\cdot\nu|^2\,dS$$
$$+\frac{c_{N,s}}{2}\int_{B_r'}(Ng + x\cdot\nabla g)|\,\mathrm{Tr}(U_\lambda)|^2\,dx - \frac{c_{N,s}r}{2}\int_{S_r'} g|\,\mathrm{Tr}(U_\lambda)|^2\,dS$$
$$=\frac{N-2s}{2}\int_{B_r^+} y^{1-2s}|\nabla U_\lambda|^2\,dz + \int_{B_r^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z\,dz. \tag{49}$$

Proof. We proceed in the spirit of [19, Proposition 2.3], since $(|x|_k^2+\lambda^2)^{-1}U_\lambda \in L^2(B_r^+, y^{1-2s})$ and $g \in W^{1,\infty}_{loc}(\Omega\setminus\{0\})$. Then by [19, Theorem 2.1, Proposition 3.6] and the proof of [19, Proposition 2.2], for any $r \in (0, r_0)$ and $\rho \in (0, r)$,

$$\nabla_x U_\lambda \in H^1(B_r^+\setminus B_\rho^+, y^{1-2s}), \quad \text{and} \quad y^{1-2s}\frac{\partial U_\lambda}{\partial y} \in H^1(B_r^+\setminus B_\rho^+, y^{2s-1}), \tag{50}$$

$$\mathrm{Tr}(U_\lambda) \in H^{1+s}(B_r'\setminus B_\rho'), \quad \text{and} \quad \mathrm{Tr}(\nabla_x U_\lambda) = \nabla\,\mathrm{Tr}(U_\lambda),$$
$$\nabla U_\lambda\cdot z \in H^1(B_r^+\setminus B_\rho^+, y^{1-2s}), \quad \text{and} \quad \mathrm{Tr}(\nabla U_\lambda\cdot z) = \mathrm{Tr}(\nabla U_\lambda)\cdot x, \tag{51}$$

where $H^{1+s}(B_r'\setminus B_\rho') := \{w \in H^1(B_r'\setminus B_\rho') : \frac{\partial w}{\partial x_i} \in W^{s,2}(B_r'\setminus B_\rho')\text{ for any } i = 1,\dots,N\}$. We also have, in view of (46), the following identity:

$$\mathrm{div}(y^{1-2s}|\nabla U_\lambda|^2 z - 2y^{1-2s}\nabla U_\lambda\cdot z\nabla U_\lambda) = (N-2s)|\nabla U_\lambda|^2 + 2\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z, \tag{52}$$

in a distributional sense in $B_r^+\setminus B_\rho^+$. Furthermore, thanks to (50),

$$\mathrm{div}(y^{1-2s}\nabla U_\lambda\cdot z\nabla U_\lambda) = -y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z + y^{1-2s}\nabla U_\lambda\cdot\nabla(\nabla U_\lambda\cdot z) \in L^1(B_r^+\setminus B_\rho^+), \tag{53}$$

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and so by (52):

$$\operatorname{div}(y^{1-2s}|\nabla U_\lambda|^2 z) \in L^1(B_r^+ \setminus B_\rho^+).$$

Let, for any $\delta \in (0, r)$,

$$B_{r,\delta}^+ := \{(x,y) \in B_r^+ : y > \delta\} \quad \text{and} \quad S_{r,\delta}^+ := \{(x,y) \in S_r^+ : y > \delta\}. \tag{54}$$

Integrating by part on $B_r^+ \setminus B_\rho^+$ we obtain, for any $\delta \in (0, \rho)$,

$$\begin{aligned}\int_{B_{r,\delta}^+ \setminus B_{\rho,\delta}^+} \operatorname{div}(y^{1-2s}|\nabla U_\lambda|^2 z)\, dz = r \int_{S_{r,\delta}^+} y^{1-2s}|\nabla U_\lambda|^2\, dS - \rho \int_{S_{\rho,\delta}^+} y^{1-2s}|\nabla U_\lambda|^2\, dS \\ - \delta^{2-2s} \int_{B'_{\sqrt{r^2-\delta^2}} \setminus B'_{\sqrt{\rho^2-\delta^2}}} |\nabla U_\lambda|^2(x,\delta)\, dx.\end{aligned} \tag{55}$$

We claim that there exists a sequence $\delta_n \to 0^+$ such that:

$$\lim_{n\to\infty} \delta^{2-2s} \int_{B'_{\sqrt{r^2-\delta_n^2}} \setminus B'_{\sqrt{\rho^2-\delta_n^2}}} |\nabla U_\lambda|^2(x,\delta)\, dx = 0, \tag{56}$$

arguing by contradiction. If the claim does not hold than there exist a constant $C > 0$ and $\delta_0 \in (0, \rho)$ such that $B'_r \times (0, \delta_0) \subseteq B_{r_0}^+$ and:

$$\delta^{1-2s} \int_{B'_{\sqrt{r^2-\delta^2}} \setminus B'_{\sqrt{\rho^2-\delta^2}}} |\nabla U_\lambda|^2(x,\delta)\, dx \geq \frac{C}{\delta}, \quad \text{for any } \delta \in (0, \delta_0). \tag{57}$$

Then integrating (57) over $(0, \delta_0)$ we obtain:

$$\int_0^{\delta_0} \left(\delta^{1-2s} \int_{B'_r} |\nabla U_\lambda|^2(x,\delta)\, dx \right) d\delta \geq \int_0^{\delta_0} \frac{C}{\delta} d\delta = +\infty,$$

which is a contradiction in view of the Fubini-Tonelli Theorem. Then we can pass to the limit as $\delta = \delta_n$ in (55) and conclude that, thanks to the Dominate Convergence Theorem and the Monotone Convergence Theorem,

$$\int_{B_r^+ \setminus B_\rho^+} \operatorname{div}(y^{1-2s}|\nabla U_\lambda|^2 z)\, dz = r \int_{S_r^+} y^{1-2s}|\nabla U_\lambda|^2\, dS - \rho \int_{S_\rho^+} y^{1-2s}|\nabla U_\lambda|^2\, dS, \tag{58}$$

for a.e $r \in (0, r_0)$ and a.e. $\rho \in (0, r)$. Testing (48) with $\nabla U \cdot z$ we obtain, in view of (53) and Remark 4.3,

$$
\begin{aligned}
&\int_{B_r^+\setminus B_\rho^+} \operatorname{div}(y^{1-2s}\nabla U_\lambda\cdot z\nabla U_\lambda)\,dz\\
&\quad=\int_{B_r^+\setminus B_\rho^+} y^{1-2s}\nabla U_\lambda\cdot\nabla(\nabla U_\lambda\cdot z)\,dz-\int_{B_r^+\setminus B_\rho^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z\,dz\\
&\quad=\frac{1}{r}\int_{S_r^+} y^{1-2s}|\nabla U_\lambda\cdot z|^2\,dS-\frac{1}{\rho}\int_{S_\rho^+} y^{1-2s}|\nabla U_\lambda\cdot z|^2\,dS\\
&\qquad+c_{N,s}\int_{B_r'\setminus B_\rho'} g\operatorname{Tr}(U_\lambda)\,\nabla_x\operatorname{Tr}(U_\lambda)\cdot x\,dx. \qquad (59)
\end{aligned}
$$

We note that $g\operatorname{Tr}(U_\lambda)^2x\in W^{1,1}(B_r'\setminus B_\rho',\mathbb{R}^N)$ by (3) and (51) hence integrating by part we obtain

$$
\begin{aligned}
\int_{B_r'\setminus B_\rho'} g\,\operatorname{Tr}(U_\lambda)\nabla_x\operatorname{Tr}(U_\lambda)\cdot x\,dx&=-\frac{1}{2}\int_{B_r'\setminus B_\rho'}(Ng+x\cdot\nabla g)\operatorname{Tr}(U_\lambda)^2\,dx\\
&\quad+\frac{r}{2}\int_{S_r'} g|\operatorname{Tr}(U_\lambda)|^2dS'-\frac{\rho}{2}\int_{S_\rho'} g|\operatorname{Tr}(U_\lambda)|^2dS'. \qquad (60)
\end{aligned}
$$

Arguing as in the proof of (56), we see that there exists a sequence $\rho_n\to 0^+$ such that:

$$
\begin{aligned}
\lim_{n\to\infty}\rho_n\int_{S_{\rho_n}^+} y^{1-2s}|\nabla U_\lambda|^2\,dS&=\lim_{n\to\infty}\rho_n\int_{S_{\rho_n}^+} y^{1-2s}\left|\nabla U_\lambda\cdot\frac{z}{|z|}\right|^2\,dS\\
&=\lim_{n\to\infty}\rho_n\int_{S_{\rho_n}'} g|\operatorname{Tr}(U_\lambda)|^2dS'=0.
\end{aligned}
$$

Then by the Dominated Convergence Theorem, we can pass to the limit as $\rho=\rho_n$ and $n\to\infty$ in (58), (59), (60) and conclude that (49) holds in view of (52). □

Proposition 4.5. *Let U be a solution of (21). Then, for a.e. $r\in(0,r_0)$*

$$
\begin{aligned}
&\frac{r}{2}\int_{S_r^+} y^{1-2s}\left(|\nabla U|^2-\frac{\alpha}{|x|_k^2}U^2\right)\,dS-r\int_{S_r^+} y^{1-2s}|\nabla U\cdot\nu|^2\,dS\\
&+\frac{c_{N,s}}{2}\int_{B_r'}(Ng+x\cdot\nabla g)|\operatorname{Tr}(U)|^2\,dx-\frac{c_{N,s}}{2}r\int_{S_r'} g|\operatorname{Tr}(U)|^2\,dS'\\
&=\frac{N-2s}{2}\int_{B_r^+} y^{1-2s}\left(|\nabla U|^2-\frac{\alpha}{|x|_k^2}U^2\right)\,dz. \qquad (61)
\end{aligned}
$$

Proof. Let $r\in(0,r_0)$ and $B_{r,\delta}^+$, $S_{r,\delta}^+$ be as in (54) for any $\delta\in(0,r)$. Then, by (1),

$$
\begin{aligned}
&\operatorname{div}\left(y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2\,z\right)\\
&\quad=y^{1-2s}\left(2\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z+(N+2-2s)\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2-2\frac{\alpha|x|_k^2}{(|x|_k^2+\lambda^2)^2}U_\lambda^2\right) \qquad (62)
\end{aligned}
$$

and $y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2 z \in W^{1,1}(B_{r,\delta}^+, \mathbb{R}^{N+1})$. Integrating (62) by part in $B_{r,\delta}^+$ we obtain

$$\begin{aligned} r\int_{S_{r,\delta}^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2\, dS - \delta^{2-2s}\int_{B'_{\sqrt{r^2-\delta^2}}} \frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2(x,\delta)\, dx \\ = \int_{B_{r,\delta}^+} y^{1-2s}\left(2\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z + (N+2-2s)\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2\right. \\ \left. -2\frac{\alpha|x|_k^2}{(|x|_k^2+\lambda^2)^2}U_\lambda^2\right)\, dz. \end{aligned} \tag{63}$$

We claim that there exists a sequence $\delta_n \to 0^+$ as $n\to\infty$ such that:

$$\lim_{n\to\infty}\delta_n^{2-2s}\int_{B'_{\sqrt{r^2-\delta_n^2}}}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2(x,\delta_n)\, dx = 0, \tag{64}$$

arguing by contradiction. If (64) does not hold, then there exists a constant $C>0$ and $\delta_0\in(0,r)$ such that $(0,\delta_0)\times B'_r \subseteq B_{r_0}^+$, and

$$\delta^{1-2s}\int_{B'_{\sqrt{r^2-\delta^2}}}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2(x,\delta)\, dx \geq \frac{C}{\delta}$$

for any $\delta\in(0,\delta_0)$. Integrating over $(0,\delta_0)$ we obtain:

$$+\infty > \int_0^{\delta_0}\delta^{1-2s}\left(\int_{B'_r}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2(x,\delta)\, dx\right) d\delta \geq \int_0^{\delta_0}\frac{C}{\delta}\, d\delta,$$

a contradiction in view of the Fubini-Tonelli Theorem. Passing to the limit for $\delta=\delta_n$ as $n\to\infty$ in (63) we conclude that

$$\begin{aligned} \int_{B_r^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z\, dz = \frac{r}{2}\int_{S_r^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2\, dS \\ -\frac{1}{2}\int_{B_r^+} y^{1-2s}\left((N+2-2s)\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2\right. \\ \left. -2\frac{\alpha|x|_k^2}{(|x|_k^2+\lambda^2)^2}U_\lambda^2\right)\, dz. \end{aligned} \tag{65}$$

Now we pass to the limit as $\lambda\to 0^+$, eventually along a suitable sequence $\lambda_n\to 0^+$, in each term of (49) taking into account (65). We recall that, by Theorem 4.2, $U_\lambda \to U$

strongly in $H^1(B_r^+, y^{1-2s})$ for any $r \in (0, r_0]$. It is clear that for any $r \in (0, r_0)$:

$$\lim_{\lambda \to 0^+} \int_{B_r^+} y^{1-2s} |\nabla U_\lambda|^2 \, dz = \int_{B_r^+} y^{1-2s} |\nabla U|^2 \, dz.$$

Furthermore, there exists a sequence $\lambda_n \to 0$ as $n \to \infty$ and $G \in L^2(B_{r_0}^+, y^{1-2s}|x|_k^{-2})$ such that:

$$\begin{aligned}
&(N+2-2s)\frac{\alpha}{|x|_k^2+\lambda_n^2}U_{\lambda_n}^2 - 2\frac{\alpha|x|_k^2}{(|x|_k^2+\lambda_n^2)^2}U_{\lambda_n}^2 \to (N-2s)\frac{\alpha}{|x|_k^2}U^2, \quad \text{for a.e. } z \in B_{r_0}^+,\\
&\frac{\alpha}{|x|_k^2+\lambda_n^2}U_{\lambda_n} - \frac{\alpha}{|x|_k^2}U \to 0, \quad \text{for a.e. } z \in B_{r_0}^+, \\
&|U_{\lambda_n}| \le |G|, \quad \text{for a.e. } z \in B_{r_0}^+ \text{and any } n \in \mathbb{N}.
\end{aligned} \tag{66}$$

Then by the Dominated Convergence Theorem, we conclude that for any $r \in (0, r_0)$

$$\begin{aligned}
\lim_{n\to\infty} \int_{B_r^+} y^{1-2s} \left((N+2-2s)\frac{\alpha}{|x|_k^2+\lambda_n^2}U_{\lambda_n}^2 - 2\frac{\alpha|x|_k^2}{(|x|_k^2+\lambda_n^2)^2}U_{\lambda_n}^2 \right) dz \\
= (N-2s)\int_{B_r^+} y^{1-2s}\frac{\alpha}{|x|_k^2}U^2 \, dz,
\end{aligned}$$

and

$$\lim_{n\to\infty} \int_{B_r^+} y^{1-2s} \left| \frac{\alpha}{|x|_k^2+\lambda_n^2}U_{\lambda_n}^2 - \frac{\alpha}{|x|_k^2}U^2 \right| dz = 0. \tag{67}$$

By (3), (42), (40) and Proposition 3.1:

$$\lim_{\lambda\to 0^+} \int_{B_r'} |Ng + \nabla g \cdot x| \, |\operatorname{Tr}(U_\lambda) - \operatorname{Tr}(U)|^2 \, dx = 0, \tag{68}$$

hence, for any $r \in (0, r_0)$,

$$\lim_{\lambda\to 0^+} \int_{B_r'} (Ng + x\cdot\nabla g)|\operatorname{Tr}(U_\lambda)|^2 \, dx = \int_{B_r'} (Ng + \nabla g \cdot x)|\operatorname{Tr}(U)|^2 \, dx.$$

By Fatou's Lemma and the Coarea Formula,

$$\int_0^{r_0} \left(\liminf_{\lambda\to 0^+} \int_{S_r^+} y^{1-2s}|\nabla U_\lambda - \nabla U|^2 \, dS \right) dr \le \liminf_{\lambda\to 0^+} \int_{B_{r_0}^+} y^{1-2s}|\nabla U_\lambda - \nabla U|^2 \, dS = 0,$$

and so

$$\liminf_{\lambda\to 0^+} \int_{S_r^+} y^{1-2s}|\nabla U_\lambda|^2 \, dS = \int_{S_r^+} y^{1-2s}|\nabla U|^2 \, dS,$$

for a.e. $r \in (0, r_0)$. Similarly, for a.e. $r \in (0, r_0)$:

$$\liminf_{\lambda \to 0^+} \int_{S_r^+} y^{1-2s} |\nabla U_\lambda \cdot \nu|^2 \, dS = \int_{S_r^+} y^{1-2s} |\nabla U \cdot \nu|^2 \, dS,$$

and, by (68) and Fatou's Lemma,

$$\liminf_{\lambda \to 0^+} \int_{S_r'} g |\operatorname{Tr}(U_\lambda)|^2 \, d'S = \int_{S_r'} g |\operatorname{Tr}(U)|^2 \, dS'.$$

Furthermore passing to the limit for $\lambda = \lambda_n$ as $n \to \infty$ and λ_n is as in (66), we obtain:

$$\lim_{n \to \infty} \int_{S_r^+} y^{1-2s} \frac{\alpha}{|x|_k^2 + \lambda_n^2} U_{\lambda_n}^2 \, dS = \int_{S_r^+} y^{1-2s} \frac{\alpha}{|x|_k^2} U^2 \, dS,$$

for a.e. $r \in (0, r_0)$, thanks to Fatou's Lemma and (67). In conclusion (61) holds. □

5. The monotonicity formula

Let U be a non-trivial solution of (21), let r_0 be as in (43). For any $r \in (0, r_0]$ we define the height and energy functions respectively as:

$$H(r) := \frac{1}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} U^2 \, dS, \tag{69}$$

$$D(r) := \frac{1}{r^{N-2s}} \left(\int_{B_r^+} y^{1-2s} \left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2} U^2 \right) dz - c_{N,s} \int_{B_r'} g |\operatorname{Tr}(U)|^2 \, dx \right). \tag{70}$$

The proof of the next Proposition is very similar to [11, Lemma 3.1] and we omit it. We also recall that ν is the outer normal vector to B_r^+ on S_r^+, that is $\nu(z) = \frac{z}{|z|}$.

Proposition 5.1. *We have that $H \in W^{1,1}_{loc}((0, r_0])$ and:*

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} \frac{\partial U}{\partial \nu} U \, dS = \frac{2}{r} D(r), \tag{71}$$

in a distributional sense and for a.e. $r \in (0, r_0)$.

Proposition 5.2. *Let H be as in (69). Then $H(r) > 0$ for any $r \in (0, r_0]$.*

Proof. Assume by contradiction that there exists $r \in (0, r_0]$ such that $H(r) = 0$. From (33) and Remark 4.1 we deduce that $U \equiv 0$ on B_r^+. Let Σ_k be as in (39). The function U is a solution of an elliptic equation with bounded coefficients away from Σ_k and $\mathbb{R}^N \times \{0\}$. Then the claim follows from classical unique continuation principles, see for example [34]. □

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Proposition 5.3. *The function D defined in (70) belongs to $W^{1,1}_{loc}((0,r_0])$ and:*

$$D'(r) = \frac{2}{r^{N+1-2s}}\left(r\int_{S_r^+} y^{1-2s}|\nabla U\cdot\nu|^2\,dS - c_{N,s}\int_{B'_r}\left(sg+\frac{1}{2}x\cdot\nabla g\right)|\operatorname{Tr}(U)|^2\,dx\right), \tag{72}$$

in a distributional sense and for a.e. $r\in(0,r_0)$.

Proof. By the Coarea Formula

$$D'(r) = (2s-N)r^{-N+2s-1}\left(\int_{B_r^+} y^{1-2s}\left(|\nabla U|^2-\frac{\alpha}{|x|_k^2}U^2\right)dz - c_{N,s}\int_{B'_r} g|\operatorname{Tr}(U)|^2\,dx\right)$$
$$+r^{-N+2s}\left(\int_{S_r^+} y^{1-2s}\left(|\nabla U|^2-\frac{\alpha}{|x|_k^2}U^2\right)dS - c_{N,s}\int_{S'_r} g|\operatorname{Tr}(U)|^2\,dS'\right),$$

and so (72) follows from (61). Furthermore $D\in W^{1,1}_{loc}((0,r_0])$ by (72), (70) and the Coarea Formula. □

Let us define, for any $r\in(0,r_0]$, the frequency function $\mathcal{N}$ as:

$$\mathcal{N}(r) := \frac{D(r)}{H(r)}. \tag{73}$$

In view of Proposition 5.2 the definition of $\mathcal{N}$ is well-posed.

Proposition 5.4. *We have that $\mathcal{N}\in W^{1,1}_{loc}((0,r_0])$ and for any $r\in(0,r_0]$:*

$$\mathcal{N}(r) > -\frac{N-2s}{2}. \tag{74}$$

Furthermore,

$$\mathcal{N}'(r) = v_1(r)+v_2(r), \tag{75}$$

in a distributional sense and for a.e. $r\in(0,r_0)$, where

$$v_1(r) := \frac{2r\left(\left(\int_{S_r^+} y^{1-2s}U^2\,dS\right)\left(\int_{S_r^+} y^{1-2s}\left|\frac{\partial U}{\partial\nu}\right|^2 dS\right)-\left(\int_{S_r^+} y^{1-2s}U\frac{\partial U}{\partial\nu}\,dS\right)^2\right)}{\left(\int_{S_r^+} y^{1-2s}U^2\,dS\right)^2},$$

and

$$v_2(r) := -c_{N,s}\frac{\int_{B'_r}(2sg+x\cdot\nabla g)|\operatorname{Tr}(U)|^2\,dx}{\int_{S_r^+} y^{1-2s}U^2\,dS}. \tag{76}$$

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Finally,

$$v_1(r) \geq 0, \quad \textit{for any } r \in (0, r_0]. \tag{77}$$

Proof. Since $1/H, D \in W^{1,1}_{loc}((0, r_0])$ it follows that $\mathcal{N} \in W^{1,1}_{loc}((0, r_0])$. We can deduce (74) directly from (44) and (73).

Furthermore by (71):

$$\frac{d}{dr}\mathcal{N}'(r) = \frac{D'(r)H(r) - D(r)H'(r)}{H^2(r)} = \frac{D'(r)H(r) - \frac{r}{2}(H'(r))^2}{H^2(r)},$$

and so (75) follows from (69), (70) and (72). Finally (77) is a consequence of the Cauchy-Schwartz inequality in $L^2(S_r^+, y^{1-2s})$ between the vectors U and $\frac{\partial U}{\partial \nu}$. $\square$

Proposition 5.5. *There exists a constant* $C > 0$ *such that:*

$$|v_2(r)| \leq Cr^{-1+\varepsilon}\left(\mathcal{N}(r) + \frac{N-2s}{2}\right), \quad \textit{for any } r \in (0, r_0]. \tag{78}$$

Proof. The claim follows from (3), (42), (44) and (76). $\square$

Proposition 5.6. *There exists a constant* $C_1 > 0$ *such that:*

$$\mathcal{N}(r) \leq C_1, \quad \textit{for any } r \in (0, r_0]. \tag{79}$$

Proof. Thanks to Proposition 5.4, for a.e. $r \in (0, r_0)$:

$$\left(\mathcal{N} + \frac{N-2s}{2}\right)'(r) \geq v_2(r) \geq -Cr^{-1+\varepsilon}\left(\mathcal{N}(r) + \frac{N-2s}{2}\right).$$

Hence an integration over (r, r_0) yields:

$$\mathcal{N}(r) \leq -\frac{N-2s}{2} + \left(\mathcal{N}(r_0) + \frac{N-2s}{2}\right)e^{\frac{C}{\varepsilon}r_0^{\varepsilon}}$$

for any $r \in (0, r_0)$. $\square$

Proposition 5.7. *The limit,*

$$\gamma := \lim_{r \to 0^+} \mathcal{N}(r), \tag{80}$$

exists and it is finite.

Proof. Since $\mathcal{N} \in W^{1,1}_{loc}((0, r_0])$ by Proposition 5.4, for any $r \in (0, r_0)$:

$$\mathcal{N}(r) = \mathcal{N}(r_0) - \int_r^{r_0} \mathcal{N}'(r)\,dr = \mathcal{N}(r_0) - \int_r^{r_0} v_1(r)\,dr - \int_r^{r_0} v_2(r)\,dr. \tag{81}$$

Since $v_1 \geq 0$ by (77) and $v_2 \in L^1(0, r_0)$ by (78) and (79), we can pass to the limit as $r \to 0^+$ in (81) and conclude that the limit (80) exists. From (74) and (79) it is finite. □

The proofs of the Proposition 5.8 and 5.9 are standard and we omit them, see for example [11, Lemma 3.7, Lemma 4.6], [15, Lemma 5.6, Lemma 6.4] or [16, Lemma 5.9, Lemma 6.6].

Proposition 5.8. *Let γ be as in (80). Then there exists a constant $K > 0$ such that:*

$$H(r) \leq K r^{2\gamma}, \quad \text{for any } r \in (0, r_0). \tag{82}$$

Furthermore for any $\sigma > 0$ there exist a constant K_σ such that:

$$H(r) \geq K_\sigma r^{2\gamma+\sigma}, \quad \text{for any } r \in (0, r_0). \tag{83}$$

Proposition 5.9. *Let γ be as in (80). Then there exists the limit:*

$$\lim_{r \to 0^+} r^{-2\gamma} H(r), \tag{84}$$

and it is finite.

6. The blow-up analysis

Let U be a non-trivial solution of (21) and let r_0 be as in (43). For any $\lambda \in (0, r_0]$ let,

$$V^\lambda(z) := \frac{U(\lambda z)}{\sqrt{H(\lambda)}}. \tag{85}$$

By a change of variables, it is clear that V^λ weakly solves:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla V^\lambda) = y^{1-2s}\frac{\alpha}{|x|_k^2} V^\lambda, & \text{in } B^+_{r_0/\lambda}, \\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial V^\lambda}{\partial y} = c_{N,s}\lambda^{2s} g(\lambda\cdot)\operatorname{Tr}(V^\lambda), & \text{on } B'_{r_0/\lambda}, \end{cases}$$

in the sense that:

$$\int_{B^+_{r_0/\lambda}} y^{1-2s}\nabla V^\lambda \cdot \nabla W\,dz - \int_{B'_{r_0/\lambda}} y^{1-2s}\frac{\alpha}{|x|_k^2} V^\lambda W\,dz$$
$$= c_{N,s}\lambda^{2s} \int_{B^+_{r_0/\lambda}} g(\lambda\cdot)\operatorname{Tr}(V^\lambda)\operatorname{Tr}(W)\,dx,$$

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for any $W \in H^1_{0,S^+_{r_0/\lambda}}(B^+_{r_0/\lambda}, y^{1-2s})$ (see (45)). Furthermore by (69) and a change of variables:

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |V^\lambda(\theta)|^2 dS = 1, \quad \text{for any } \lambda \in (0, r_0]. \tag{86}$$

Since the frequency function $\mathcal{N}$ is bounded on $[0, r_0]$ (see (74) and (79)) we can prove the following proposition.

Proposition 6.1. *The family of functions $\{V^\lambda\}_{\lambda \in (0,r_0]}$ is bounded in $H^1(B_1^+, y^{1-2s})$.*

Proof. For any $\lambda \in (0, r_0)$, thanks to (44), (85) and a change of variables,

$$\begin{aligned}
\mathcal{N}(\lambda) &= \frac{\lambda^{2s-N}}{H(\lambda)} \left(\int_{B^+_\lambda} y^{1-2s} \left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2} U^2 \right) dz - c_{N,s} \int_{B'_\lambda} g |\operatorname{Tr}(U)|^2 \, dx \right) \\
&\geq \left(1 - \alpha \left(\frac{2}{k-2} \right)^2 + c_{N,s} k_{N,s,g} r_0^\varepsilon \right) \frac{\lambda^{2s-N}}{H(\lambda)} \left(\int_{B^+_\lambda} y^{1-2s} |\nabla U|^2 \, dz \right) - \frac{N-2s}{2} \\
&= \left(1 - \alpha \left(\frac{2}{k-2} \right)^2 + c_{N,s} k_{N,s,g} r_0^\varepsilon \right) \left(\int_{B^+_1} y^{1-2s} |\nabla V^\lambda|^2 \, dz \right) - \frac{N-2s}{2}.
\end{aligned}$$

Hence the claim follows from (79), (86) and (40). □

Now we establish the following doubling property.

Proposition 6.2. *There exists a constant $C_3 > 0$ such that:*

$$\frac{1}{C_3} H(\lambda) \leq H(R\lambda) \leq C_3 H(\lambda), \tag{87}$$

$$\int_{B^+_R} y^{1-2s} |V^\lambda|^2 \, dz \leq C_3 2^{N+2-2s} \int_{B^+_1} y^{1-2s} |V^{R\lambda}|^2 \, dz, \tag{88}$$

$$\int_{B^+_R} y^{1-2s} |\nabla V^\lambda|^2 \, dz \leq C_3 2^{N-2s} \int_{B^+_1} y^{1-2s} |\nabla V^{R\lambda}| \, dz, \tag{89}$$

for any $\lambda \in (0, r_0)$ and any $R \in [1, 2]$.

Proof. By (71), (74), and (79):

$$-\frac{N-2s}{r} \leq \frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} \leq \frac{2C_1}{r} \quad \text{for a.e. } r \in (0, r_0).$$

An integration over $(\lambda, R\lambda)$ with $R \in (1, 2]$ yields:

$$R^{2s-N} \leq \frac{H(R\lambda)}{H(\lambda)} \leq R^{2C_1},$$

thus (87) holds for $R \in (1,2]$ while if $R=1$ it is obvious.

Furthermore for any $\lambda \in (0, r_0)$, by (87) and a change of variables,

$$\int_{B_R^+} y^{1-2s}|V^\lambda|^2\,dz = \frac{\lambda^{-N-2+2s}}{H(\lambda)} \int_{B_{R\lambda}^+} y^{1-2s}|U|^2\,dz \le C_3 \frac{\lambda^{-N-2+2s}}{H(\lambda R)} \int_{B_{R\lambda}^+} y^{1-2s}|U|^2\,dz$$
$$= C_3 R^{N+2-2s} \int_{B_1^+} y^{1-2s}|V^{\lambda R}|^2\,dz \le C_3 2^{N+2-2s} \int_{B_1^+} y^{1-2s}|V^{\lambda R}|^2\,dz,$$

for any $R \in [1,2]$. Hence, we have proved (88) and (89) follows from (87) in the same way. □

In view of the Coarea Formula, there exists a subset $\mathcal{M} \subset (0, r_0)$ of Lebesgue measure 0 such that $|\nabla U| \in L^2(S_r^+, y^{1-2s})$ and (33) holds for any $r \in (0, r_0) \setminus \mathcal{M}$.

Proposition 6.3. *There exist $M>0$ and $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there exists $R_\lambda \in [1,2]$ such that $R_\lambda \lambda \notin \mathcal{M}$, and*

$$\int_{S_{R_\lambda}^+} y^{1-2s}|\nabla V^\lambda|^2\,dS \le M \int_{B_{R_\lambda}^+} y^{1-2s}(|\nabla V^\lambda|^2 + |V^\lambda|^2)\,dz. \tag{90}$$

Proof. By Proposition 6.1 $\{V^\lambda\}_{\lambda \in \left(0, \frac{r_0}{2}\right)}$ is bounded in $H^1(B_2^+, y^{1-2s})$. Hence

$$\limsup_{\lambda \to 0^+} \int_{B_2^+} y^{1-2s}(|\nabla V^\lambda|^2 + |V^\lambda|^2)\,dz < +\infty. \tag{91}$$

Let, for any $\lambda \in \left(0, \frac{r_0}{2}\right)$,

$$f_\lambda(R) := \int_{B_R^+} y^{1-2s}(|\nabla V^\lambda|^2 + |V^\lambda|^2)\,dz.$$

The function f is absolutely continuous on $[1,2]$ and, thanks to the Coarea Formula, its distributional derivative is given by:

$$f_\lambda'(R) = \int_{S_R^+} y^{1-2s}(|\nabla V^\lambda|^2 + |V^\lambda|^2)\,dS \quad \text{for a.e. } R \in [1,2].$$

We argue by contradiction supposing that for any $M>0$ there exists $\lambda_n \to 0^+$ such that:

$$\int_{S_R^+} y^{1-2s}(|\nabla V^{\lambda_n}|^2 + |V^{\lambda_n}|^2)\,dS > M \int_{B_R^+} y^{1-2s}(|\nabla V^{\lambda_n}|^2 + |V^{\lambda_n}|^2)\,dz,$$

for any $n \in \mathbb{N}$ and any $R \in [1,2] \setminus \frac{1}{\lambda_n}\mathcal{M}$, hence for a.e. $R \in [1,2]$. Therefore,

$$f_{\lambda_n}'(R) > M f_{\lambda_n}(R), \quad \text{for a.e. } R \in [1,2] \text{ and any } n \in \mathbb{N}.$$

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An integration over $[1,2]$ yields $f_{\lambda_n}(2) > e^M f_{\lambda_n}(1)$ for any $n \in \mathbb{N}$. Hence

$$\liminf_{n\to\infty} f_{\lambda_n}(1) \le \limsup_{n\to\infty} f_{\lambda_n}(1) \le e^{-M} \limsup_{n\to\infty} f_{\lambda_n}(2),$$

and so

$$\liminf_{\lambda\to 0^+} f_\lambda(1) \le e^{-M} \limsup_{\lambda\to 0^+} f_\lambda(2),$$

for any $M>0$. It follows that $\liminf_{\lambda\to 0^+} f_\lambda(1) = 0$ by (91). We conclude that there exists a sequence $\lambda_n \to 0^+$ as $n\to\infty$ and $V \in H^1(B_1^+, y^{1-2s})$ such that:

$$\lim_{n\to\infty} \int_{B_1^+} y^{1-2s}(|\nabla V^{\lambda_n}|^2 + |V^{\lambda_n}|^2)\, dz = 0,$$

and $V_{\lambda_n} \rightharpoonup V$ weakly in $H^1(B_1^+, y^{1-2s})$, taking into account Proposition 6.1. By Proposition 3.1, (86) and the lower semicontinuity of norms, we obtain,

$$\int_{B_1^+} y^{1-2s}(|\nabla V|^2 + |V|^2)\, dz = 0 \quad \text{and} \quad \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^2\, dS = 1,$$

which is a contradiction. □

Proposition 6.4. *Let R_λ be as in Proposition 6.3. Then there exists a constant $\overline{M} > 0$ such that:*

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla V^{R_\lambda \lambda}|^2 dS \le \overline{M} \quad \textit{for any } \lambda \in \left(0, \min\left\{\lambda_0, \frac{r_0}{2}\right\}\right). \tag{92}$$

Proof. By a change of variables, the fact that $R_\lambda \in [1,2]$ and (85):

$$\begin{aligned}\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla V^{R_\lambda \lambda}|^2 dS &= R_\lambda^{-N+1+2s} \frac{H(\lambda)}{H(R_\lambda \lambda)} \int_{S^+_{R_\lambda}} y^{1-2s} |\nabla V^\lambda|^2 dS \\ &\le 2C_3 M \int_{B^+_{R_\lambda}} y^{1-2s}(|\nabla V^\lambda|^2 + |V^\lambda|^2)\, dz \\ &\le 2^{N+3-2s} C_3^2 M \int_{B_1^+} y^{1-2s}(|\nabla V^{R_\lambda \lambda}|^2 + |V^{R_\lambda\lambda}|^2)\, dz \le \overline{M} < +\infty,\end{aligned}$$

for some $\overline{M} > 0$, in view of Proposition 6.1, (87), (88), (89), and (90). □

Proposition 6.5. *Let U be a non-trivial solution of (21) and γ be as in (80). Then* *(i) there exists $n \in \mathbb{N} \setminus \{0\}$ such that:*

$$\gamma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}, \tag{93}$$

where $\gamma_{\alpha,k,n}$ is an eigenvalue of problem (22),

(ii) for any sequence $\lambda_p \to 0^+$ as $p \to \infty$ there exists a subsequence $\lambda_{p_q} \to 0^+$ as $q \to \infty$ and a eigenfunction Z of problem (22), corresponding to the eigenvalue $\gamma_{\alpha,k,n}$, such that $\|Z\|_{L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})} = 1$ and

$$\frac{U(\lambda_{p_q} z)}{\sqrt{H(\lambda_{p_q})}} \to |z|^\gamma Z\left(\frac{z}{|z|}\right) \quad \textit{strongly in } H^1(B_1^+, y^{1-2s}) \quad \textit{as } q \to \infty.$$

Proof. Let V^λ be as in (85) and R_λ as in Proposition 6.3. The family $\{V^{R_\lambda \lambda}\}_{\lambda \in \left(0, \min\left\{\lambda_0, \frac{r_0}{2}\right\}\right)}$ is bounded in $H^1(B_1^+, y^{1-2s})$, thanks to Proposition 6.1. Let $\lambda_p \to 0^+$ as $p \to \infty$. Then there exists a subsequence $\lambda_{p_q} \to 0^+$ as $q \to \infty$ and $V \in H^1(B_1^+, y^{1-2s})$ such that $V^{R_{\lambda_{p_q}} \lambda_{p_q}} \rightharpoonup V$ weakly in $H^1(B_1^+, y^{1-2s})$ as $q \to \infty$. By Proposition 3.1 the trace operator $\mathrm{Tr}_{S_1^+}$ is compact and so

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |V|^2 \, dS = 1, \tag{94}$$

in view of (86). Hence V is non-trivial. We claim that:

$$V^{R_{\lambda_{p_q}} \lambda_{p_q}} \rightharpoonup V \quad \text{strongly in } H^1(B_1^+, y^{1-2s}) \text{as } q \to \infty. \tag{95}$$

For q sufficiently large $B_1^+ \subseteq B^+_{r_0/(R_{\lambda_{p_q}} \lambda_{p_q})}$ and since $R_{\lambda_{p_q}} \lambda_{p_q} \notin \mathcal{M}$, where $\mathcal{M}$ is as in Proposition 6.3, we have that

$$\begin{aligned}
&\int_{B_1^+} y^{1-2s} \left(\nabla V^{R_{\lambda_{p_q}} \lambda_{p_q}} \cdot \nabla W - \frac{\alpha}{|x|_k^2} V^{R_{\lambda_{p_q}} \lambda_{p_q}} W \right) dz \\
&= \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\partial V^{R_{\lambda_{p_q}} \lambda_{p_q}}}{\partial \nu} W \, dS + c_{N,s} (R_{\lambda_{p_q}} \lambda_{p_q})^{2s} \int_{B_1'} g(R_{\lambda_{p_q}} \lambda_{p_q} \cdot) \, \mathrm{Tr}(V^{R_{\lambda_{p_q}} \lambda_{p_q}}) \, \mathrm{Tr}(W) \, dx,
\end{aligned} \tag{96}$$

for any $W \in H^1(B_1^+, y^{1-2s})$, thanks to (33) and a change of variables. We will pass to the limit as $q \to \infty$ in (96). To this end we observe that, for any $W \in H^1(B_1^+, y^{1-2s})$,

$$\left| \lambda^{2s} \int_{B_1'} g(\lambda \cdot) \, \mathrm{Tr}(V^\lambda) \, \mathrm{Tr}(W) \, dx \right| = \left| \frac{\lambda^{2s-N}}{H(\lambda)} \int_{B_\lambda'} g(x) \, \mathrm{Tr}(U)(x) \, \mathrm{Tr}(W)(\lambda x) \, dx \right|$$

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$$
\begin{aligned}
&\le k_{N,s,g}\frac{\lambda^{2s+\varepsilon-N}}{H(\lambda)}\left|\int_{B_\lambda^+} y^{1-2s}|\nabla U|^2\,dz-\int_{B_\lambda^+} y^{1-2s}\frac{\alpha}{|x|_k^2}|U|^2\,dz\right.\\
&\qquad\left.+\frac{N-2s}{2\lambda}\int_{S_\lambda^+} y^{1-2s}|U|^2\,dS\right|^{\frac12}\\
&\times\left|\int_{B_\lambda^+} y^{1-2s}|\nabla W(\lambda\cdot)|^2\,dz-\int_{B_\lambda^+} y^{1-2s}\frac{\alpha}{|x|_k^2}W(\lambda\cdot)^2\,dz\right.\\
&\qquad\left.+\frac{N-2s}{2\lambda}\int_{S_\lambda^+} y^{1-2s}|W(\lambda\cdot)|^2\,dS\right|^{\frac12}\\
&=k_{N,s,g}\lambda^{\varepsilon}\left|\int_{B_1^+} y^{1-2s}|\nabla V^\lambda|^2\,dz-\int_{B_1^+} y^{1-2s}\frac{\alpha}{|x|_k^2}|V^\lambda|^2\,dz+\frac{N-2s}{2}\right|^{\frac12}\\
&\times\left|\int_{B_1^+} y^{1-2s}|\nabla W|^2\,dz-\int_{B_1^+} y^{1-2s}\frac{\alpha}{|x|_k^2}W^2\,dz+\frac{N-2s}{2}\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}|W|^2\,dS\right|^{\frac12}
\end{aligned}
$$

by a change of variables, the Hölder inequality, (3), (42), (85) and (86). We conclude that:

$$
\lim_{\lambda\to0^+}\left|\lambda^{2s}\int_{B_1'} g(\lambda\cdot)\,\mathrm{Tr}(V^\lambda)\,\mathrm{Tr}(W)\,dx\right|=0, \tag{97}
$$

by Proposition 6.1 and (40). Thanks to (92), there exists a function $f\in L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})$ such that:

$$
\frac{\partial V^{R_{\lambda_{pq}}\lambda_{pq}}}{\partial\nu}\rightharpoonup f\quad\text{weakly in } L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})\text{as } q\to\infty, \tag{98}
$$

up to a subsequence. Hence,

$$
\lim_{q\to\infty}\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}\frac{\partial V^{R_{\lambda_{pq}}\lambda_{pq}}}{\partial\nu}W\,dS=\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}fW\,dS
$$

for any $W\in H^1(B_1^+,y^{1-2s})$. Furthermore,

$$
\begin{aligned}
&\lim_{q\to\infty}\int_{B_1^+} y^{1-2s}\left(\nabla V^{R_{\lambda_{pq}}\lambda_{pq}}\cdot\nabla W-\frac{\alpha}{|x|_k^2}V^{R_{\lambda_{pq}}\lambda_{pq}}W\right)\,dz\\
&\quad=\int_{B_1^+} y^{1-2s}\left(\nabla V\cdot\nabla W-\frac{\alpha}{|x|_k^2}VW\right)\,dz
\end{aligned}
$$

by Remark 4.1. It follows that:

$$\int_{B_1^+} y^{1-2s} \left(\nabla V \cdot \nabla W - \frac{\alpha}{|x|_k^2} VW \right) dz = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} f W \, dS,$$

for any $W \in H^1(B_1^+, y^{1-2s})$, that is V is a weak solution of the problem:

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla V) = \frac{\alpha}{|x|_k^2} V, & \text{in } B_1^+, \\ -\lim_{y \to 0^+} y^{1-2s} \frac{\partial V}{\partial y} = 0, & \text{on } B_1'. \end{cases} \tag{99}$$

Furthermore testing (96) with $V^{R_{\lambda_{pq}} \lambda_{pq}}$,

$$\lim_{q \to \infty} \int_{B_1^+} y^{1-2s} \left(\left| \nabla V^{R_{\lambda_{pq}} \lambda_{pq}} \right|^2 - \frac{\alpha}{|x|_k^2} \left| V^{R_{\lambda_{pq}} \lambda_{pq}} \right|^2 \right) dz$$

$$= \lim_{q \to \infty} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\partial V^{R_{\lambda_{pq}} \lambda_{pq}}}{\partial \nu} V^{R_{\lambda_{pq}} \lambda_{pq}} \, dS = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} f W \, dS,$$

thanks to (98) and the compactness of the trace operator $\operatorname{Tr}_{S_1^+}$, see Proposition 3.1. Hence from Remark 4.1 and (86) we deduce (95). Let for any $r \in (0,1]$

$$D_q(r) = \frac{1}{r^{N-2s}} \Big(\int_{B_r^+} y^{1-2s} \left(|\nabla V^{R_{\lambda_{pq}} \lambda_{pq}}|^2 - \frac{\alpha}{|x|_k^2} |V^{R_{\lambda_{pq}} \lambda_{pq}}|^2 \right) dz$$
$$- c_{N,s} (R_{\lambda_{pq}} \lambda_{pq})^{2s} \int_{B_r'} g(R_{\lambda_{pq}} \lambda_{pq} \cdot) |\operatorname{Tr}(V^{R_{\lambda_{pq}} \lambda_{pq}})|^2 \, dx \Big),$$

and

$$H_q(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} |V^{R_{\lambda_{pq}} \lambda_{pq}}|^2 \, dS.$$

For any $r \in (0,1]$ we also define:

$$D_V(r) = \frac{1}{r^{N-2s}} \int_{B_r^+} y^{1-2s} \left(|\nabla V|^2 - \frac{\alpha}{|x|_k^2} |V|^2 \right) dz$$

and

$$H_V(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} |V|^2 \, dS. \tag{100}$$

Thanks to a scaling argument it is easy to see that:

$$\mathcal{N}_q(r) := \frac{D_q(r)}{H_q(r)} = \frac{D(R_{\lambda_{pq}} \lambda_{pq} r)}{H(R_{\lambda_{pq}} \lambda_{pq} r)} = \mathcal{N}(R_{\lambda_{pq}} \lambda_{pq} r), \quad \text{for any } r \in (0,1].$$

By (95), (97) and Remark 4.1, it follows that:

$$H_q(r) \to H_V(r) \quad \text{and} \quad D_q(r) \to D_V(r), \quad \text{as } q \to \infty, \text{for any } r \in (0,1].$$

Furthermore $H_V(r) > 0$ for any $r \in (0,1]$ by Proposition 5.2 in the case $g \equiv 0$ and $\Omega = B_2'$. In particular the function:

$$\mathcal{N} : (0,1] \to \mathbb{R}, \quad \mathcal{N}_V(r) := \frac{D_V(r)}{H_V(r)},$$

is well defined and $\mathcal{N}_V \in W^{1,1}_{loc}((0,1])$ by Proposition 5.4 in the case $g \equiv 0$ and $\Omega = B_2'$. In view of (100), (80):

$$\mathcal{N}_V(r) = \lim_{q\to\infty} \mathcal{N}(R_{\lambda_{p_q}}\lambda_{p_q} r) = \gamma \quad \text{for any } r \in (0,1]. \tag{101}$$

Hence $\mathcal{N}_V(r)$ is constant in $[0,1]$ and so:

$$\mathcal{N}_V'(r) \equiv 0, \text{for any } r \in (0,1].$$

By Proposition 5.4 it follows that:

$$\left(\int_{S_r^+} y^{1-2s} V^2\, dS\right)\left(\int_{S_r^+} y^{1-2s} \left|\frac{\partial V}{\partial \nu}\right|^2 dS\right) - \left(\int_{S_r^+} y^{1-2s} V \frac{\partial V}{\partial \nu}\, dS\right)^2 = 0,$$

for a.e. $r \in (0,1)$, that is, equality holds in the Cauchy-Schwartz inequality for the vectors V and $\frac{\partial V}{\partial \nu}$ in $L^2(S_r^+, y^{1-2s})$ for a.e. $r \in (0,1)$.

Therefore, there exists a function $\eta(r)$ defined a.e. in $(0,1)$ such that:

$$\frac{\partial V}{\partial \nu}(r\theta) = \eta(r) V(r\theta), \quad \text{for a.e. r } \in (0,1) \text{ and a.e. } \theta \in \mathbb{S}^+.$$

Multiplying by $V(r\theta)$ and integrating over $\mathbb{S}^+$,

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\partial V}{\partial \nu}(r\theta) V(r\theta)\, dS = \eta(r) \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |V(r\theta)|^2\, dS, \text{for a.e. r } \in (0,1),$$

and so $\eta(r) = \frac{H_V'(r)}{2H_V(r)} = \frac{\gamma}{r}$ for a.e. $r \in (0,1)$ by (71), (71) and (101). Since V is smooth away from Σ_k by classical elliptic regularity theory (see (39)), an integration over $(r,1)$ yields:

$$V(r\theta) = r^\gamma V(1\theta) = r^\gamma Z(\theta), \quad \text{for any } r \in (0,1] \quad \text{and a.e. } \theta \in \mathbb{S}^+, \tag{102}$$

where $Z = V_{|\mathbb{S}^+}$ and $\|Z\|_{L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})} = 1$ by (94). In view of [13, Lemma 1.1], (102) and (99) the function Z is an eigenfunction of problem (22) and the correspondent eigenvalue

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$\gamma_{\alpha,k,n}$ satisfies the relationship $\gamma(N-2s+\gamma)=\gamma_{\alpha,k,n}$, that is:

$$\gamma = -\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,n}} \text{ or } \gamma = -\frac{N-2s}{2}-\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,n}}$$

Since $r^\gamma Z(\theta)\in H^1(B_1^+,y^{1-2s})$ by (102) then $r^{2\gamma-2}Z^2(\theta)\in L^1(B_1^+,y^{1-2s})$ by (40) and so we conclude that (93) must hold.

Consider now the sequence $\{V^{\lambda_{pq}}\}_{q\in\mathbb{N}}$. Up to a further subsequence, $V^{\lambda_{pq}}\rightharpoonup\tilde V$ weakly in $H^1(B_1^+,y^{1-2s})$ as $q\to\infty$, for some $\tilde V\in H^1(B_1^+,y^{1-2s})$ and $R_{\lambda_{pq}}\to\tilde R$, for some $\tilde R\in[1,2]$ as $q\to\infty$. The strong convergence of $\{V^{R_{\lambda_{pq}}\lambda_{pq}}\}_{q\in\mathbb{N}}$ to V in $H^1(B_1^+,y^{1-2s})$ implies that, up to a further subsequence, both $V^{R_{\lambda_{pq}}\lambda_{pq}}$ and $\left|\nabla V^{R_{\lambda_{pq}}\lambda_{pq}}\right|$ are dominated a.e. by a $L^2(B_1^+,y^{1-2s})$ function, uniformly with respect to $q\in\mathbb{N}$. Up to a further subsequence, we may also assume that the limit:

$$\ell=\lim_{q\to\infty}\frac{H(R_{\lambda_{pq}}\lambda_{pq})}{H(\lambda_{pq})},$$

exists, it is finite and strictly positive, taking into account (87). Then from the Dominated Convergence Theorem and a change of variables we deduce that:

$$\begin{aligned}
&\lim_{q\to\infty}\int_{B_1^+}y^{1-2s}V^{\lambda_{pq}}(z)\phi(z)\,dz=\lim_{q\to\infty}R_{\lambda_{pq}}^{N+2-2s}\int_{B^+_{1/R_{\lambda_{pq}}}}y^{1-2s}V^{\lambda_{pq}}(R_{\lambda_{pq}}z)\phi(R_{\lambda_{pq}}z)\,dz\\
&=\lim_{q\to\infty}R_{\lambda_{pq}}^{N+2-2s}\sqrt{\frac{H(R_{\lambda_{pq}}\lambda_{pq})}{H(\lambda_{pq})}}\int_{B_1^+}y^{1-2s}\chi_{B^+_{1/R_{\lambda_{pq}}}}(z)V^{R_{\lambda_{pq}}\lambda_{pq}}(z)\phi(R_{\lambda_{pq}}z)\,dz\\
&=\tilde R^{N+2-2s}\sqrt{\ell}\int_{B^+_{1/\tilde R}}y^{1-2s}V(z)\phi(\tilde Rz)\,dz=\sqrt{\ell}\int_{B_1^+}y^{1-2s}V(z/\tilde R)\phi(z)\,dz,
\end{aligned}$$

for any $\phi\in C^\infty(\overline{B_1^+})$. By density we conclude that $V^{\lambda_{pq}}\rightharpoonup\sqrt{\ell}V(\cdot/\tilde R)$ weakly in $L^2(B_1^+,y^{1-2s})$ as $q\to\infty$. Since $V^{\lambda_{pq}}\rightharpoonup\tilde V$ weakly in $H^1(B_1^+,y^{1-2s})$ as $q\to\infty$ we conclude that $\tilde V=\sqrt{\ell}V(\cdot/\tilde R)$ and so $V^{\lambda_{pq}}\rightharpoonup\sqrt{\ell}V(\cdot/\tilde R)$ weakly in $H^1(B_1^+,y^{1-2s})$ as $q\to\infty$. Furthermore

$$\begin{aligned}
&\lim_{q\to\infty}\int_{B_1^+}y^{1-2s}|\nabla V^{\lambda_{pq}}(z)|^2\,dz=\lim_{q\to\infty}R_{\lambda_{pq}}^{N+2-2s}\int_{B^+_{1/R_{\lambda_{pq}}}}y^{1-2s}|\nabla V^{\lambda_{pq}}(R_{\lambda_{pq}}z)|^2\,dz\\
&=\lim_{q\to\infty}R_{\lambda_{pq}}^{N-2s}\frac{H(R_{\lambda_{pq}}\lambda_{pq})}{H(\lambda_{pq})}\int_{B_1^+}y^{1-2s}\chi_{B^+_{1/R_{\lambda_{pq}}}}(z)|\nabla V^{R_{\lambda_{pq}}\lambda_{pq}}(z)|^2\,dz\\
&=\tilde R^{N-2s}\ell\int_{B^+_{1/\tilde R}}y^{1-2s}|\nabla V|^2dz=\int_{B_1^+}y^{1-2s}|\sqrt{\ell}\nabla V(\cdot/\tilde R)|^2\,dz,
\end{aligned}$$

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by the Dominated Convergence Theorem and a change of variables. Hence $V^{\lambda_{p_q}} \to \sqrt{\ell}V(\cdot/\tilde{R})$ strongly in $H^1(B_1^+, y^{1-2s})$ as $q \to \infty$.

Thanks to (102), V is a homogeneous function of degree γ and so $\tilde{V} = \sqrt{\ell}\tilde{R}^{-\gamma}V$. Moreover, since $V^{\lambda_{p_q}} \to \tilde{V}$ strongly in $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ as $q \to \infty$ by Proposition 3.1,

$$1 = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}|\tilde{V}(\theta)|^2 dS = \sqrt{\ell}\tilde{R}^{-\gamma} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}|V(\theta)|^2 dS = \sqrt{\ell}\tilde{R}^{-\gamma}$$

in view of (86) and (94). We conclude that $\tilde{V} = V$ thus completing the proof. □

Now, we show that the limit (84) is strictly positive, by means of a Fourier analysis with respect to the $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$-orthonormal basis $\{Z_{\alpha,k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ of eigenfunctions of problem (22), see § 3.1. To this end let us define for any $k \in \{3, \dots, N\}$, α as in (1), and $n \in \mathbb{N} \setminus \{0\}$:

$$\varphi_{n,i}(\lambda) := \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} U(\lambda\theta) Z_{\alpha,k,n,i}(\theta)\, dS, \quad \text{for any } \lambda \in (0, r_0],\ i \in 1, \dots, M_{\alpha,k,n}, \tag{103}$$

see (37) for the definition of $M_{\alpha,k,n}$, and

$$\Upsilon_{n,i}(\lambda) := c_{N,s} \int_{B'_\lambda} g\, \mathrm{Tr}(U)\, \mathrm{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right) dx, \tag{104}$$

for any $\lambda \in (0, r_0]$, $i \in 1, \dots, M_{\alpha,k,n}$. Thanks to Proposition 5.7 and Proposition 6.5 there exists $n_0 \in \mathbb{N} \setminus \{0\}$ such that:

$$\gamma = \lim_{r\to 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n_0}}. \tag{105}$$

For any $i \in \{1, \dots, M_{\alpha,k,n_0}\}$ we need to compute the asymptotics of $\varphi_{n_0,i}(\lambda)$ as $\lambda \to 0^+$.

Proposition 6.6. *Let n_0 be as in (105). Then for any $i \in \{1, \dots, M_{\alpha,k,n_0}\}$ and any $r \in (0, r_0]$:*

$$\begin{aligned}\varphi_{n_0,i}(\lambda) = \lambda^\gamma \Bigg(&\frac{\varphi_{n_0,i}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma} \int_0^r \rho^{-1+\rho} \Upsilon_{n_0,i}(\rho) d\rho \\ &+ \frac{N-2s+\gamma}{N-2s+2\gamma} \int_\lambda^r \rho^{-N-1+2s-\gamma} \Upsilon_{n_0,i}(\rho)\, d\rho \Bigg) + O(\lambda^{\gamma+\varepsilon}) \quad \text{as } \lambda \to 0^+.\end{aligned} \tag{106}$$

Proof. Let $n \in \mathbb{N}$ and $i \in \{1, \dots, M_{\alpha,k,n}\}$. Let $f \in C_c^\infty(0, r_0)$. Then testing (21) with the function $|z|^{N+1-2s} f(|z|) Z_{\alpha,k,n,i}(z/|z|)$ and passing in polar coordinates, by (35),

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we obtain:

$$-\varphi_{n,i}''(\lambda) - \frac{N+1-2s}{\lambda}\varphi_{n,i}'(\lambda) + \frac{\gamma_{\alpha,k,n}}{\lambda^2}\varphi_{n,i}(\lambda) = \zeta_{n,i}(\lambda) \quad \text{in } (0, r_0),$$

in a distributional sense, where the distribution $\zeta_{n,i} \in \mathcal{D}'(0,r_0)$ is define as:

$$_{\mathcal{D}'(0,r_0)}\langle \zeta_{n,i}, f\rangle_{\mathcal{D}(0,r_0)} = \int_0^{r_0} \frac{f(\lambda)}{\lambda^{2-2s}} \left(\int_{\mathbb{S}'} g(\lambda\cdot)\,\mathrm{Tr}(U)(\lambda\cdot)\,\mathrm{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right) dS' \right) d\lambda, \tag{107}$$

for any $f \in C_c^\infty(0, r_0)$. In particular $\zeta_{n,i}$ belongs to $L^1_{loc}((0, r_0])$ by the Coarea Formula and a change of variables. If $\Upsilon_{n,i}$ is as in (104), a direct computation shows that:

$$\Upsilon_{n,i}'(\lambda) = \lambda^{N+1-2s}\zeta_{n,i}(\lambda) \quad \text{in } \mathcal{D}'(0, r_0),$$

hence

$$-\left(\lambda^{N+1-2s+2\sigma_n}\left(\lambda^{-\sigma_n}\varphi_{n,i}(\lambda)\right)'\right)' = \lambda^{\sigma_n}\Upsilon_{n,i}'(\lambda) \quad \text{in } \mathcal{D}'(0, r_0), \tag{108}$$

where

$$\sigma_n := -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}.}. \tag{109}$$

From (108) and (107), we deduce that $\lambda \to \lambda^{N+1-2s+2\sigma_n}\left(\lambda^{-\sigma_n}\varphi_{n,i}'(\lambda)\right)$ belongs to $W^{1,1}_{loc}((0, r_0])$ hence an integration over (λ, r) yields:

$$\begin{aligned}\left(\lambda^{-\sigma_n}\varphi_{n,i}(\lambda)\right)' &= -\lambda^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\lambda) \\ &\quad - \lambda^{-N-1+2s-2\sigma_n}\sigma_n\left(C(r) + \int_\lambda^r \rho^{\sigma_n-1}\Upsilon_{n,i}(\rho)\,d\rho\right),\end{aligned} \tag{110}$$

for any $r \in (0, r_0]$, for some real number $C(r)$ depending on r, α, k, n and i. Since in view of (110) $\lambda \to \lambda^{-\sigma_n}\varphi_{n,i}(\lambda)$ belongs to $W^{1,1}_{loc}((0, r_0])$, a further integration yields:

$$\begin{aligned}\varphi_{n,i}(\lambda) &= \lambda^{\sigma_n}\Bigg(r^{-\sigma_n}\varphi_{n,i}(r) + \int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho \\ &\quad + \sigma_n\int_\lambda^r \rho^{-N-1+2s-2\sigma_n}\left(C(r) + \int_\rho^r t^{\sigma_n-1}\Upsilon_{n,i}(t)\,dt\right)d\rho\Bigg) \\ &= \lambda^{\sigma_n}\Bigg(r^{-\sigma_n}\varphi_{n,i}(r) + \int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho + \frac{\sigma_n C(r) r^{-N+2s-2\sigma_n}}{-N+2s-2\sigma_n} \\ &\quad - \frac{\sigma_n C(r)\lambda^{-N+2s-2\sigma_n}}{-N+2s-2\sigma_n} - \frac{\sigma_n\lambda^{-N+2s-2\sigma_n}}{-N+2s-2\sigma_n}\int_\lambda^r t^{\sigma_n-1}\Upsilon_{n,i}(t)\,dt\end{aligned}$$

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$$+\frac{\sigma_n}{-N+2s-2\sigma_n}\int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho\Bigg)$$
$$=\lambda^{\sigma_n}\left(\frac{\varphi_{n,i}(r)}{r^{\sigma_n}}-\frac{\sigma_n C(r) r^{-N+2s-2\sigma_n}}{N-2s+2\sigma_n}+\frac{N-2s+\sigma_n}{N-2s+2\sigma_n}\int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho\right)$$
$$+\frac{\sigma_n\lambda^{-N+2s-\sigma_n}}{N-2s+2\sigma_n}\left(C(r)+\int_\lambda^r t^{\sigma_n-1}\Upsilon_{n,i}(t)\,dt\right), \tag{111}$$

for any $\lambda\in(0,r_0]$.

Let n_0 be as in (105) and $i\in\{1,\dots,M_{\alpha,k,n_0}\}$. By (105) and (109), $\gamma=\sigma_{n_0}$ and

$$\lambda^{-N-1+2s-\gamma}\left|\Upsilon_{n_0,i}(\lambda)\right| < c_{N,s}\lambda^{-N-1+2s-\gamma}\int_{B'_\lambda}|g||\operatorname{Tr}(U)\left|\operatorname{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right)\right|dx$$
$$\le \lambda^{-N-1+2s-\gamma}\left(\int_{B'_\lambda}|g||\operatorname{Tr}(U)|^2dx\right)^{\frac12}\left(\int_{B'_\lambda}|g|\left|\operatorname{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right)\right|^2dx\right)^{\frac12}$$
$$\le k_{N,s,g}\lambda^{-N-1+2s-\gamma+\varepsilon}\left(\int_{B^+_\lambda}y^{1-2s}|\nabla U|^2\,dz-\int_{B^+_\lambda}y^{1-2s}\frac{\alpha}{|x|_k^2}U^2\,dz\right.$$
$$\left.+\frac{N-2s}{2\lambda}\int_{S^+_\lambda}y^{1-2s}U^2\,dz\right)^{\frac12}$$
$$\times\left(\int_{B^+_\lambda}y^{1-2s}|\nabla Z_{\alpha,k,n,i}(z/|z|)|^2\,dz-\int_{B^+_\lambda}y^{1-2s}\frac{\alpha}{|x|_k^2}|Z_{\alpha,k,n,i}(z/|z|)|^2\,dz\right.$$
$$\left.+\frac{N-2s}{2\lambda}\int_{S^+_\lambda}y^{1-2s}|Z_{\alpha,k,n,i}(z/|z|)|^2\,dz\right)^{\frac12}$$
$$=k_{N,s,g}\lambda^{-1-\gamma+\varepsilon}\sqrt{H(\lambda)}\left(\int_{B^+_1}y^{1-2s}|\nabla V^\lambda|^2\,dz-\int_{B^+_1}y^{1-2s}\frac{\alpha}{|x|_k^2}|V^\lambda|^2\,dz+\frac{N-2s}{2}\right)^{\frac12}$$
$$\times\left(\int_{B^+_1}y^{1-2s}|\nabla Z_{\alpha,k,n,i}(z/|z|)|^2\,dz-\int_{B^+_1}y^{1-2s}\frac{\alpha}{|x|_k^2}|Z_{\alpha,k,n,i}(z/|z|)|^2\,dz+\frac{N-2s}{2}\right)^{\frac12}$$
$$\le \text{const}\ \lambda^{-1+\varepsilon}$$

for any $\lambda\in(0,r_0]$, by Holder inequality, a change of variables, (3), (42), (82), (85), (86), (104). Hence

$$\left|\Upsilon_{n_0,i}(\lambda)\right|\le \text{const}\ \lambda^{N-2s+\gamma+\varepsilon}\quad \text{for any }\lambda\in(0,r_0]. \tag{112}$$

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Now we show that for any $r \in (0, r_0]$:

$$C(r) + \int_0^r \lambda^{-1+\gamma} \Upsilon_{n_0,i}(\lambda) d\lambda = 0. \tag{113}$$

From (112) it is clear that $\int_0^{r_0} \lambda^{-1+\gamma} \Upsilon_{n_0,i}(\lambda) d\lambda < +\infty$. We argue by contradiction. Since $\sigma_{n_0} = \gamma > -\frac{N-2s}{2}$ by (105) and (109), then from (111) we deduce that:

$$\varphi_{n_0,i}(\lambda) \sim \frac{\gamma \lambda^{-N+2s-\gamma}}{N-2s+2\gamma} \left(C(r) + \int_\lambda^r t^{-1+\gamma} \Upsilon_{n_0,i}(t)\, dt \right) \quad \text{as } \lambda \to 0^+$$

and so by (105):

$$\int_0^{r_0} \lambda^{N-1-2s} |\varphi_{n_0,i}(\lambda)|^2 \, d\lambda = +\infty. \tag{114}$$

On the other hand by Hölder inequality, a change of variables, (103) and [13, Lemma 2.4]:

$$\begin{aligned} &\int_0^{r_0} \lambda^{N-1-2s} |\varphi_{n_0,i}(\lambda)|^2 \, d\lambda \le \int_0^{r_0} \lambda^{N-1-2s} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |U(\lambda\theta)|^2 \, dS \right) d\lambda \\ &= \int_{B_{r_0}^+} y^{1-2s} \frac{U^2}{|z|^2} \, dz < +\infty, \end{aligned}$$

which contradicts (114). It follows that:

$$\begin{aligned} &\lambda^{-N+2s-\gamma} \left| C(r) + \int_\lambda^r \lambda^{-1+\gamma} \Upsilon_{n_0,i}(\lambda) d\lambda \right| = \lambda^{-N+2s-\gamma} \left| \int_0^\lambda \lambda^{-1+\gamma} \Upsilon_{n_0,i}(\lambda) d\lambda \right| \\ &= O(\lambda^{\gamma+\varepsilon}), \end{aligned} \tag{115}$$

in view of (112). In conclusion (106) follows from (111), (113), and (115). □

Proposition 6.7. *Let U be a non-trivial solution of (21) and γ be as in (80). Then,*

$$\lim_{r \to 0^+} r^{-2\gamma} H(r) > 0.$$

Proof. From (103), since $\{Z_{\alpha,k,n}\}_{n \in \mathbb{N} \setminus \{0\}}$ is a orthonormal basis of $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$, see § 3.1, we have that:

$$H(\lambda) = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |U(\lambda\theta)|^2 \, dS = \sum_{n=1}^{\infty} \sum_{i=1}^{M_{\alpha,k,n}} |\varphi_{n,i}(\lambda)|^2, \tag{116}$$

by (69) and a change of variables. We argue by contradiction supposing that:

$$\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda) = 0.$$

Let n_0 be as in (105). By (116) for any $i \in \{1, \dots, M_{\alpha,k,n_0}\}$,

$$\lim_{\lambda \to 0^+} \lambda^{-2\gamma} |\varphi_{n_0,i}(\lambda)|^2 = 0.$$

By (106), for any $i \in \{1, \dots, M_{\alpha,k,n_0}\}$ and any $r \in (0, r_0]$

$$\frac{\varphi_{n,i}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma} \int_0^r \rho^{-1+\rho} \Upsilon_{n_0,i}(\rho) d\rho \\ + \frac{N-2s+\gamma}{N-2s+2\gamma} \int_0^r \rho^{-N-1+2s-\gamma} \Upsilon_{n,i}(\rho)\, d\rho = 0. \tag{117}$$

Hence by (106), (112) and (117):

$$\varphi_{n,i}(\lambda) = -\lambda^\gamma \frac{N-2s+\gamma}{N-2s+2\gamma} \int_0^\lambda \rho^{-N-1+2s-\gamma} \Upsilon_{n,i}(\rho)\, d\rho + O(\lambda^{\gamma+\varepsilon}) = O(\lambda^{\gamma+\varepsilon}),$$

as $\lambda \to 0^+$ for any $i \in \{1, \dots, M_{\alpha,k,n_0}\}$. In view of (69) and (85), it follows that:

$$\sqrt{H(\lambda)} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^\lambda Z\, dS = O(\lambda^{\gamma+\varepsilon}) \quad \text{as } \lambda \to 0^+,$$

for any $Z \in V_{n_0}$, see (36). Then, in view of (83) with $\sigma = \frac{\varepsilon}{2}$,

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^\lambda Z\, dS = O(\lambda^{\frac{\varepsilon}{2}}) \quad \text{as } \lambda \to 0^+, \tag{118}$$

for any $Z \in V_{n_0}$. On the other hand by Proposition 6.5 and Proposition 3.1, there exist $Z_0 \in V_{n_0}$ with $\|Z_0\|_{L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})} = 1$ and a sequence $\lambda_q \to 0^+$ as $q \to \infty$ such that:

$$V^{\lambda_q} \to Z_0 \quad \text{strongly in } L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \text{ as } q \to \infty. \tag{119}$$

Since $Z_0 \in V_{n_0}$, from the Parseval identity, (118), and (119) we deduce that $Z_0 \equiv 0$ which contradicts the fact that $\|Z_0\|_{L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})} = 1$. $\square$

We are now in position to state and prove our main results which are a more precise version of Theorem 2.9 and Theorem 2.11 respectively.

Theorem 6.8. *Let U be a solution of (21) and suppose that g satisfies (3). Then there exists $n \in \mathbb{N} \setminus \{0\}$ such that:*

$$\gamma = \lim_{r \to 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}. \tag{120}$$

Furthermore, let $M_{\alpha,k,n}$ and $\{Z_{\alpha,k,n,i}\}_{i \in \{1,\dots,M_{\alpha,k,n}\}}$ be as in (37) and (38) respectively. Then for any $i \in \{0, \dots M_{\alpha,k,n}\}$ there exists $\beta_i \in \mathbb{R}$ such that $(\beta_1, \dots, \beta_{M_{\alpha,k,n}}) \neq$

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$(0,\dots,0)$ *and*

$$\frac{U(\lambda z)}{\lambda^\gamma} \to |z|^\gamma \sum_{i=1}^{M_{\alpha,k,n}} \beta_i Z_{\alpha,k,n,i}(z/|z|) \quad \textit{strongly in } H^1(B_1^+, y^{1-2s}) \textit{ as } \lambda \to 0^+, \tag{121}$$

where

$$\beta_i := \frac{\varphi_{n,i}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma}\int_0^r \rho^{-1+\rho}\Upsilon_{n,i}(\rho)d\rho \\ + \frac{N-2s+\gamma}{N-2s+2\gamma}\int_0^r \rho^{-N-1+2s-\gamma}\Upsilon_{n,i}(\rho)\,d\rho, \quad \textit{for any } r \in (0,r_0], \tag{122}$$

with $\varphi_{n,i}$ and $\Upsilon_{n,i}$ given by (103) and (104) respectively.

Proof. In view of (80) and Proposition 6.5 we know that (120) holds for some $n \in \mathbb{N}\setminus\{0\}$. Furthermore for any sequence of strictly positive numbers $\lambda_p \to 0^+$ as $p \to \infty$ there exist a subsequence $\lambda_{p_q} \to 0^+$ as $q \to \infty$ and real numbers $\beta_1,\dots,\beta_{M_{\alpha,k,n}}$ such that:

$$\frac{U(\lambda z)}{\lambda^\gamma} \to |z|^\gamma \sum_{i=1}^{M_{\alpha,k,n}} \beta_i Z_{\alpha,k,n,i}(z/|z|) \quad \text{strongly in } H^1(B_1^+, y^{1-2s}) \text{ as } q \to \infty^+, \tag{123}$$

taking into account Proposition 6.5 and (38). We claim that for any $i \in \{1,\dots,M_{\alpha,k,n}\}$ the number β_i does not depend neither on the sequence $\lambda_p \to 0^+$ nor on its subsequence $\lambda_{p_q} \to 0^+$. In view of (38), (103), (123) and Proposition 3.1

$$\lim_{q\to\infty} \lambda_{p_q}^{-\gamma}\varphi_{n,j}(\lambda_{p_q}) = \lim_{q\to\infty}\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}\lambda_{p_q}^{-\gamma} U(\lambda_{p_q}\theta) Z_{\alpha,k,n,j}(\theta)\,dS \\ = \sum_{i=1}^{M_{\alpha,k,n}} \beta_i \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} Z_{\alpha,k,n,i} Z_{\alpha,k,n,j}\,dS = \beta_j,$$

for any $j \in \{1,\dots,M_{\alpha,k,n}\}$. On the other hand for any $r \in (0, r_0]$

$$\lim_{q\to\infty} \lambda_{p_q}^{-\gamma}\varphi_{n,j}(\lambda_{p_q}) = \frac{\varphi_{n,j}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma}\int_0^r \rho^{-1+\rho}\Upsilon_{n,j}(\rho)d\rho \\ + \frac{N-2s+\gamma}{N-2s+2\gamma}\int_0^r \rho^{-N-1+2s-\gamma}\Upsilon_{n,j}(\rho)\,d\rho,$$

by (106). Hence

$$\beta_j = \frac{\varphi_{n,j}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma}\int_0^r \rho^{-1+\rho}\Upsilon_{n,j}(\rho)d\rho$$

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$$+\frac{N-2s+\gamma}{N-2s+2\gamma}\int_0^r \rho^{-N-1+2s-\gamma}\Upsilon_{n,j}(\rho)\,d\rho, \tag{124}$$

for any $j \in \{1, \dots, M_{\alpha,k,n}\}$ and in particular β_j does not depend neither on the sequence $\lambda_p \to 0^+$ nor on its subsequence $\lambda_{pq} \to 0^+$. Then by (124) and the Urysohn Subsequence Principle we conclude that (121) holds, thus completing the proof. □

From Theorem 6.8, Proposition 2.6 and Remark 2.10, we can easily deduce the following theorem.

Theorem 6.9. *Let u be a solution of (12) and suppose that g satisfies (3). Let γ, $n \in \mathbb{N} \setminus \{0\}$, $M_{\alpha,k,n}$ and $\{Z_{\alpha,k,n,i}\}_{i\in\{1,\dots,M_{\alpha,k,n}\}}$ be as in Theorem 6.8. Then*

$$\frac{u(\lambda x)}{\lambda^\gamma} \to |x|^\gamma \sum_{i=1}^{M_{\alpha,k,n}} \beta_i \operatorname{Tr}(Z_{\alpha,k,n,i}((\cdot/|\cdot|))(x) \quad \textit{strongly in } H^s(B_1')\textit{as } \lambda \to 0^+,$$

where β_i is as in (122) for any $i \in \{1, \cdots, M_{\alpha,k,n}\}$.

Proof of Corollary 2.12 and Corollary 2.13. We start by proving Corollary 2.12. Let U be a solution of (21) such that (26) holds and assume by contradiction that $U \not\equiv 0$ on $\Omega \times (0,\infty)$. Let γ be as in Theorem 6.8. Then there exists a sequence $\lambda_q \to 0^+$ such that:

$$\lim_{q\to\infty} \lambda_q^{-\gamma} U(\lambda_q z) = 0, \quad \text{for a.e } z \in B_1^+.$$

On the other hand by Theorem 2.9 there exists an eigenfunction Z of (22) such that:

$$\lim_{q\to\infty} \lambda_q^{-\gamma} U(\lambda_q z) = |z|^\gamma Z(z/|z|), \quad \text{for a.e. } z \in B_1^+,$$

up to a further subsequence, which is a contradiction. Arguing in the same way, we can deduce Corollary 2.13 from Theorem 2.11, taking into account Remark 3.5. □

7. Computation of the first eigenvalue on a hemisphere

Proposition 7.1. *Equation (24) holds for any $k \in \{3, \dots, N\}$. If $k = N$ then (25) holds.*

Proof. Let $Y_{\alpha,k,1}$ be the first eigenfunction of (5) defined in § 2. In particular $Y_{\alpha,k,1}$ is positive. By [17, Theorem 1.1] there exists an eigenfunction Ψ of problem (23), corresponding to the first eigenvalue $\eta_{\alpha,k,1}$, such that:

$$\lambda^{\frac{N-2}{2}-\sqrt{\left(\frac{N-2}{2}\right)^2+\eta_{\alpha,k,1}}} Y_{\alpha,k,1}(\lambda x) \to |x|^{-\frac{N-2}{2}+\sqrt{\left(\frac{N-2}{2}\right)^2+\eta_{\alpha,k,1}}}\,\Psi\left(\frac{x}{|x|}\right), \tag{125}$$

strongly in $H^1(B_1')$ as $\lambda \to 0^+$, since $Y_{\alpha,k,1}$ is positive. Furthermore for any $\phi \in C_c^\infty(\Omega)$:

$$_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\langle L^s_{\alpha,k} Y_{\alpha,k,1}, \phi\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = (Y_{\alpha,k,1}, \phi)_{\mathbb{H}^s_{\alpha,k}(\Omega)} = \mu^s_{\alpha,k,1}\int_\Omega Y_{\alpha,k,1}\phi\,dx,$$

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in view of (8), that is $Y_{\alpha,k,1}$ is weak solution of $L^s_{\alpha,k} Y_{\alpha,k,1} = \mu^s_{\alpha,k,1} Y_{\alpha,k,1}$ in the sense given by (12). Let U be the extension of $Y_{\alpha,k,1}$ provided by Theorem 2.7. Since $Y_{\alpha,k,1}$ is positive then $|U|$ is the only solution to the minimisation problem (19) and so we conclude that U is positive. Then, in view of by Theorem 6.8 and Theorem 6.9,

$$\lambda^{\frac{N-2s}{2}-\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,1}}} Y_{\alpha,k,1}(\lambda x) \to |x|^{-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,1}}} \beta_1 \operatorname{Tr}(Z_{\alpha,k,1}((\cdot/|\cdot|))(x), \tag{126}$$

strongly in $H^s(B_1')$ as $\lambda \to 0^+$. Putting together (125) and (126) we obtain:

$$-\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,1}} = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \eta_{\alpha,k,1}}$$

thus (24) follows from a direct computation. Finally, if $k = N$, problem (23) reduces to:

$$-\Delta_{\mathbb{S}'}\Psi - \alpha\Psi = \eta\Psi \quad \text{in } \mathbb{S}',$$

which admits $-\alpha$ as first eigenvalue, hence we have proved (25) in view of (24). □

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Appendix 1. A proof of Proposition 2.2

In this section, we provide, for the sake of completeness, a detailed proof of Proposition 2.2 starting with a preliminary lemma. Let us consider, for any positive sequence $\{q_n\}_{n\in\mathbb{N}}$, the weighted $\ell^2(\mathbb{N})$-space defined as:

$$\ell^2(\mathbb{N},\{q_n\}) := \left\{ \{a_n\}_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} q_n a_n^2 < +\infty \right\}$$

endowed with the norm:

$$\|\{a_n\}\|_{\ell^2(\mathbb{N},\{q_n\})} := \left(\sum_{n=0}^{\infty} q_n a_n^2 \right)^{\frac{1}{2}}.$$

Lemma A.1. *Let $\ell^2(\mathbb{N},\{q_n\})$ and $\ell^2(\mathbb{N},\{p_n\})$ be weighted $\ell^2(\mathbb{N})$-spaces. Then,*

$$(\ell^2(\mathbb{N},\{q_n\}), \ell^2(\mathbb{N},\{p_n\}))_{s,2} = \ell^2(\mathbb{N},\{q_n^{1-s} p_n^s\}), \tag{127}$$

with equivalent norms.

Proof. We follow the proof of [33, Lemma 23.1]. Let us consider a variant of the standard K function defined as:

$$K_2(t,a) := \inf_{b+c=a} \left\{ \left(\|b\|^2_{\ell^2(\mathbb{N},\{q_n\})} + t^2 \|c\|^2_{\ell^2(\mathbb{N},\{p_n\})} \right)^{\frac{1}{2}} : b \in \ell^2(\mathbb{N},\{q_n\}), c \in \ell^2(\mathbb{N},\{p_n\}) \right\},$$

for any $t \geq 0$ and any sequence $a \in \ell^2(\mathbb{N},\{q_n\}) + \ell^2(\mathbb{N},\{p_n\})$. If $K(t,a)$ is the standard K-function it is clear that $K_2(t,a) \leq K(t,a) \leq \sqrt{2} K_2(t,a)$ for any $t \geq 0$ and any sequence $a \in \ell^2(\mathbb{N},\{q_n\}) + \ell^2(\mathbb{N},\{p_n\})$. It follows that we can use K_2 to define a norm on $(\ell^2(\mathbb{N},\{q_n\}), \ell^2(\mathbb{N},\{p_n\}))_{s,2}$ equivalent to the standard one.

We can compute $K_2(a,t)$ explicitly. Indeed, fixed $a \in \ell^2(\mathbb{N},\{q_n\}) + \ell^2(\mathbb{N},\{p_n\})$ and $t \geq 0$, we can, for any $n \in \mathbb{N}$, minimise the value of $b_n^2 q_n + t^2 (a_n - b_n)^2 p_n$ as a function

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of b_n choosing:

$$b_n := \frac{t^2 p_n}{q_n + t^2 p_n} a_n.$$

With this optimal choice it follows that:

$$c_n = a_n - b_n = \frac{q_n}{q_n + t^2 p_n} a_n,$$

and so we obtain:

$$K_2(t,a)^2 = \sum_{n=0}^{\infty} \frac{t^2 p_n q_n}{q_n + t^2 p_n} a_n^2.$$

Then by the Monotone Convergence Theorem and the change of variables $t = \tau\sqrt{\frac{q_n}{p_n}}$:

$$\int_0^\infty K_2(t,a)^2 t^{-1-2s}\, dt = \sum_{n=0}^{\infty} a_n^2 \int_0^\infty \frac{t^{1-2s} p_n q_n}{q_n + t^2 p_n}\, dt = \left(\int_0^\infty \frac{\tau^{1-2s}}{1+\tau^2}\, d\tau\right) \sum_{n=0}^{\infty} a_n^2 q_n^{1-s} p_n^s.$$

Since for any $s \in (0,1)$:

$$\int_0^\infty \frac{\tau^{1-2s}}{1+\tau^2}\, d\tau < +\infty,$$

we conclude that (127) holds. □

Proof of Proposition 2.2. Let us start by proving that for any $k \in \{3, \dots, N\}$ and α as in (1)

$$\mathbb{H}^1_{\alpha,k}(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \mu_{\alpha,k,n} v_n^2 < +\infty \right\} = H^1_0(\Omega), \tag{128}$$

with equivalent norms. If $u \in H^1_0(\Omega)$ then, in view of Remark 2.1,

$$u = \sum_{n=1}^{\infty} \left(u, \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}}\right)_{\alpha,k} \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}},$$

and so by the Parseval's identity, (6), (7) and Remark 2.1:

$$+\infty > \|u\|^2_{\alpha,k} = \sum_{n=1}^{\infty} \mu_{\alpha,k,n} u_n^2. \tag{129}$$

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On the other hand if $u \in \mathbb{H}^1_{\alpha,k}(\Omega)$ let, in view of (6),

$$u^{(j)} := \sum_{n=1}^{j} \left(u, \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}}\right)_{\alpha,k} \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}} = \sum_{n=1}^{j} u_n Y_{\alpha,k,n}.$$

For any $j \in \mathbb{N} \setminus \{0\}$ it is clear that $u^{(j)} \in H^1_0(\Omega)$ and if $j > i$:

$$\left\| u^{(j)} - u^{(i)} \right\|^2_{\alpha,k} = \sum_{n=i}^{j} \mu_{\alpha,k,n} u_n^2. \tag{130}$$

It follows that $\{u^{(j)}\}_{j\in\mathbb{N}\setminus\{0\}}$ converges to u in $H^1_0(\Omega)$ by Remark 2.1, and (130). In conclusion $u \in H^1_0(\Omega)$. From Remark 2.1 and (129) we deduce that the norms on $H^1_0(\Omega)$ and $\mathbb{H}^1_{\alpha,k}(\Omega)$ are equivalent.

For any $s \in (0,1]$, since $L^2(\Omega)$ and $\mathbb{H}^s_{\alpha,k}(\Omega)$ are isomorphic to $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{N}, \{\mu^s_{\alpha,k,n}\})$ respectively, from Lemma A.1 and (128) it follows that:

$$\mathbb{H}^s_{\alpha,k}(\Omega) = (L^2(\Omega), \mathbb{H}^1_{\alpha,k}(\Omega))_{s,2} = (L^2(\Omega), H^1_0(\Omega))_{s,2} = \begin{cases} H^s_0(\Omega), & \text{if } s \in (0,1) \setminus \{\frac{1}{2}\}, \\ H^{1/2}_{00}(\Omega), & \text{if } s = \frac{1}{2}, \end{cases}$$

with equivalent norms. The last equality is a classical interpolation result, see for example [25]. □

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