A TYPE OF QUASI-FROBENIUS RING

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1. Introduction. In [3], the author proved that a ring R with identity is right noetherian and right injective if and only if R is a direct sum of a finite number of uniform right ideals, which are completely primary in the sense of that paper. In this paper, we shall determine the structure of such rings in the case where the sum of the isomorphic uniform components are two-sided ideals. The ring is found to be a direct sum of total matrix rings over local rings. The local rings are shown to be completely indecomposable, i.e., right and left artinian together with right and left uniform. This result reinforces Morita's suggestion [8, p.121] of a close connection between completely indecomposable systems and injective systems. Not all quasi-Frobenius rings have this decomposition, as shown by an examination of an example of Nakayama. Theorem 2.8 shows this decomposition is unique, while Theorem 2.9 is the converse.

In the third section the essential properties of a local ring (R, M) where M is a principal right ideal, and R is right injective, are investigated. This ring will be completely indecomposable, if M is also a principal left ideal.

The last section lists a few remaining problems.

2. Noetherian and injective X-rings. In this paper the ring R will always have an identity. The definitions and notations of [3] will be used throughout this paper. We begin with a lemma which generalizes 6.3 of [3, p.137].

2.1 LEMMA. Let U and V be nonzero right R modules,

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where U is injective and V is uniform. If α is an R monomorphism of U into V, then α is an isomorphism.

Proof. We have $\alpha U \cong U$ and $\alpha U \subseteq V$. Define $\beta : \alpha U \to U$ by $\beta(\alpha u) = u$. Since U is injective, β extends to $\beta^* : V \to U \to 0$, which is exact. Now $\beta^* \alpha = 1$, the identity mapping on U. Thus $V = \mathrm{Im}\,\alpha \oplus \mathrm{Ker}\,\beta^*$. Since $\mathrm{Im}\,\alpha \neq 0$ and V is uniform, then $\mathrm{Ker}\,\beta^* = 0$. Thus β^* is an isomorphism of V onto U with inverse α. Hence α is an isomorphism.

2.2 DEFINITION. A ring R is called an X-ring provided it satisfies the condition: Let e_i, e_j be distinct primitive idempotents. If $e_iR \not\cong e_jR$, $a \in e_iR$, and $a^r \cap e_jR \neq 0$, then $ae_jR = 0$.

2.3 THEOREM. If R is a right noetherian, right injective X-ring, then R is a direct sum of twosided ideals A_i, where the A_i are right noetherian and right injective over A_i and, in addition, A_i is a total matrix ring over a local ring (D, M)[1], where M is nil and D is right noetherian.

Proof. From 6.4 of [3, p.137], we have that $R = e_iR \oplus \ldots \oplus e_nR$, where e_iR is uniform and injective over R. Let A_1 be the direct sum of these uniform right ideals, which are isomorphic (as R modules) to e_1R, and A_2 be the direct sum of the next set of isomorphic right ideals. We continue until $R = A_1 \oplus A_2 \oplus \ldots \oplus A_r$. To show that A_i is a two-sided ideal, we shall prove that $e_jR\, e_iR = 0$, for e_iR a member of A_i and e_jR a member of A_j, for $i \neq j$.

For arbitrary $a \in R$, we define the mapping $\alpha : e_iR \to e_jR$ by $\alpha(e_i) = e_jae_i$. If $\mathrm{Ker}\,\alpha = 0$, then by 2.1, $e_iR \cong e_jR$ which is impossible. Hence $\mathrm{Ker}\,\alpha \neq 0$, and $e_jae_ia' = 0$ for some non-zero element $e_ia' \in e_iR$. Thus $(e_ja)^r \cap e_iR \neq 0$ for arbitrary

[1] The symbol (D, M) will denote a local ring D with maximal right ideal equal to M.

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$a \in R$. Since R is an X-ring, $(e_j a) e_i R = 0$ for arbitrary $a \in R$. Hence, $e_j R e_i R = 0$.

The fact that R is noetherian and injective if and only if each A_i is right noetherian and right injective over A_i follows as in [3]. From prop. 5 and prop. 6 of [6, p.52], we have that each A_i is a total matrix ring over the ring $D_i = e_i R e_i$, where $e_i R$ is one of the uniform components of A_i.

Now D_i is local with maximal ideal $M_i = \{a \in D_i \mid a^n = 0\}$ by [3, p.137].[1] Q.E.D.

Let $N_i = (M_i)_n$, the $n \times n$ matrices with elements in M_i. Then N_i is the Wedderburn and Jacobson radical of A_i. Thus the Wedderburn and Jacobson radical of R is $N_1 \oplus \dots \oplus N_n$.

Let us now look at the local ring D_i. Now A_i will be right noetherian and right injective over A_i if and only if D_i is right noetherian and right injective over D_i. The injective property can easily be shown by using § 2 of [10]. Thus (D_i, M_i) is a right injective and right noetherian ring, where the not nil elements are units. Hence, D_i/M_i is a division ring and since M_i is nilpotent and finitely generated, one can easily construct a composition series for D_i. Thus D_i is right artinian. (This is the method of 5.5 in [5, p.94]). Since D_i is indecomposable and right injective, it is then right uniform. Applying the important theorem 11.2 of Morita [8, p.122], we have that D_i is completely indecomposable in the following sense.

2.4 DEFINITION[2]. A ring R is termed completely in-

[1] That each A_i is a total matrix ring over a local ring also follows from Lambek [7, p.285].

[2] See [2] for the definition for modules.

decomposable provided (1) R is right and left noetherian and artinian (2) R is right and left uniform.

We shall now write two theorems which characterize the local rings of the type D_i.

2.5 THEOREM. Let (R, M) be a local right Noetherian ring and M be the nil ideal; then R is completely indecomposable if and only if R is right injective.

Proof. Since R is local, it is indecomposable. The proof now follows directly from theorem 11.2 of [8, p.122] and the preceding discussion.

The following also follows from [8, p.122].

2.6 THEOREM. Let R be right noetherian and right artinian. Then R is completely indecomposable if and only if (1) the ring (R, M) is a local ring, where M is the nil radical. (2) R is right uniform and right injective.

Since D_i is completely indecomposable, it is quasi-Frobenius. By problem 2 on [1, p.402], then A_i is quasi-Frobenius. We have shown

2.7 DECOMPOSITION THEOREM. If R is a right noetherian, right injective X-ring, then R is quasi-Frobenius and a direct sum of a finite number of two-sided ideals each of which is a total matrix ring over a completely indecomposable ring.

For 2.7 we have the following strong uniqueness theorem.

2.8. UNIQUENESS THEOREM. Let

$$R = \oplus \sum_{i=1}^{n} A_i = \oplus \sum_{i=1}^{m} A_i'$$

be two decompositions of R as in 2.7. Then $m = n$ and each A_i is an A_j'.

Proof. Since A_i is a total matrix ring over a local ring with nil maximal ideal, then the only proper two sided ideals are nil. Hence, the A_i are indecomposable as two-sided ideals. The result now follows from 55.2 of [1, p. 378].

For 2.7 we have the following converse which applies when R is quasi-Frobenius.

2.9 CONVERSE THEOREM. Let R be a ring which is right noetherian and artinian. If R is the direct sum of two-sided ideals A_i each of which is the direct sum of the isomorphic indecomposable components, then R is an X-ring.

Proof. If e is a primitive idempotent in A_i, then eR is indecomposable [1, p. 369]. By the Krull-Schmidt Theorem [1, p. 83], we have that eR is isomorphic to each of the components of A_i. Thus, if e and f are primitive idemponents and $eR \not\cong fR$, then they must be in different two sided ideals. Therefore $eRfR = 0$ and R is an X-ring.

Thus if one constructs R as a direct sum of total matrix rings over completely indecomposable rings, we have a quasi-Frobenius X-ring. Note that in this case,[1] $E_{ii}R$ has only one idempotent since R has only one idempotent. Thus E_{ii} is a primitive idempotent.[2]

Not every quasi-Frobenius ring will be an X-ring. Consider the example of Nakayama [9, p. 624]. Let R be the set of all matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} & c_1 & 0 & 0 & 0 \\ a_{21} & a_{22} & c_2 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & d_1 & d_2 \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{21} & a_{22} \end{bmatrix}$$

1 See next example for notation E_{ii}.

2 This also follows from 54.9 of [1, p. 372].

where the elements are in a field of characteristic 0. Let E_{ij} denote a matrix with 1 in the (i, j) position and zero elsewhere. If $e_i = E_{11} + E_{55}$, $e_2 = E_{22} + E_{66}$, and $e_3 = E_{33} + E_{44}$. Then $R = e_1R \oplus e_2R \oplus e_3R$ is the decomposition of [3, p.137]. Now $e_1R \cong e_2R \not\cong e_3R$. (Use Prop. 4 of [6, p.51] to establish this). However, neither $e_1R \oplus e_2R$ nor e_3R are two-sided ideals.

3. Local rings where M is principal. In this section we shall discuss properties of a local ring (R, M), where $M = pR$, a principal right ideal.

3.1 LEMMA. Let (R, M) be a local ring. For nonzero $a, b \in M$ we have $aR = bR$ if and only if $a = bu$, where u is a unit of R.

Proof. If $aR = bR$, then $a = br$ and $b = as$. Thus $a = asr$. If r or s is in N, then the relation $a(1-sr) = 0$ implies $a = 0$. Thus r and s are units.

3.2 DEFINITION. A non unit p of a ring R is termed irreducible, provided $p = ab$ implies either a or b is a unit.

3.3 PROPOSITION. If (R, M) is a local ring and $M = pR$, then p is irreducible.

Proof. Let $p = ab$. If b is not a unit, then by 3.1, $pR \subset aR$. This implies $aR = R$ and a is a unit.

3.4 LEMMA. Let (R, M) be a local ring. If $0 \neq a = q^k u$, where u is a unit and $q \in M$, then this factorization is essentially unique.

Proof. If $a = q^k u = q^m v$ for $k > m$, then $q^m(1-q^{k-m}w) = 0$. This implies $a = 0$.

3.15 THEOREM. If (R, M) is a local ring and $M = pR$, then every $a \in M$ has an essentially unique factorization

$a = p^n u$, where u is a unit, if and only if $\bigcap_{m=1}^{\infty} p^m R = 0$.

Proof. Suppose $\bigcap_{m=1}^{\infty} p^m R = 0$. If $0 \neq a \in M$, then

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$a = pd$. If d is not a unit, then $d = pe$ and $a = p^2 e$. This process must end or a would be in $\bigcap_{m=1}^{\infty} p^m R = 0$. The converse follows from 3.4.

3.6 COROLLARY. If H is a proper right ideal of (R, M) , then $H = p^n R$ for some integer n and thus R is right noetherian. Thus if M is nil, it is nilpotent.

Proof. Let n be the least positive integer such that $p^n u \in H$. Then $H \subseteq p^n R$. Since $p^n u \in H$, then $p^n \in H$ and $H = p^n R$.

3.7 PROPOSITION. Let (R, M) be a local ring and $M = pR$. Then $\bigcap_{n=1}^{\infty} p^n R = 0$ for each of the following conditions.

(a) M is nil.

(b) R is left noetherian.

Proof. The first condition is obvious. Suppose R is left noetherian. If $a = pb$ and if b is not a unit $Ra \subset Rb$ and $b = pd$. If d is not a unit then $Rb \subset Rd$. This process must end by the left noetherian condition. Thus $a = p^n u$, and our result follows from 3.4.

3.8 PROPOSITION. Let (R, M) be a local ring, where M is a nil ideal and $M = pR$. Then for every right ideal $p^k R$, we have $(p^k R)^{lr} = p^k R$.

Proof. Let $p^n = 0$, $p^{n-1} \neq 0$. Let $H = p^k R$. Then $(p^k R)^l \ni p^{n-k}$. Suppose $d > 1$, then $p^{k-d} p^{n-k} = 0$ implies $p^{n-d} = 0$, a contradiction.

3.9 THEOREM. Let (R, M) be a local ring and $M = pR$ be nil. Then R is quasi-Frobenius if and only if $pR = Rp$. In addition, R is completely indecomposable.

Proof. The fact that R is quasi-Frobenius follows from the preceding discussion and the definition of a quasi-Frobenius ring as given in [1]. If now R is quasi-Frobenius, then $(Rp)^{rl} = Rp$. Now $(Rp)^r = p^{n-1}R$, which is a two-sided ideal.[1] Hence $(p^{n-1}R)^l = pR$ and $Rp = pR$. The fact that R is completely indecomposable follows since the ideals are linearly ordered by inclusion. This type of ring is referred to as a valuation ring in the literature.

3.10 THEOREM. Let (R, M) be a local ring and $M = pR$ be nil. If R is right noetherian and right injective, then it is completely indecomposable.

Proof. Since R is right injective, then $(Rp)^{rl} = Rp$. From the proof of 3.9 we can conclude that $pR = Rp$. Now apply theorem 3.9.

Not in all local right noetherian rings does $M = pR$ imply $M = Rp$. Following Goldie [4], consider (R, M) as the set of all elements of the form $a + xb$, with $x^2 = 0$, $ax = x\bar{a}$, where $a \rightarrow \bar{a}$ is an isomorphism of the coefficient field F, which is not an automorphism. Then $xR = M$, but $Rx \neq xR$. Note that if the mapping is an automorphism, then R is completely indecomposable.

Not every completely indecomposable ring (R, M) has the property that $M = pR = Rp$. See for example 3'.2 of [2, p.357].

4. Problems. We conclude this paper with a few problems.

4.1 Are the results in this paper and [3] true if one uses only a restricted chain condition as given in [4]?

4.2 For what class of finite groups are the group rings quasi-Frobenius X rings?
This is obviously true for a finite commutative group. For if R is commutative in 2.7, then R is a direct sum of completely indecomposable rings which are two-sided ideals.

4.3 Determine the properties of completely indecomposable rings and matrix rings over completely indecomposable rings. See [2] and [8] for some properties.

[1] Here $p^n = 0$, $p^{n-1} \neq 0$.

4.4 Determine the properties of modules over quasi-Frobenius X rings.

4.5 Theorem 2.7 shows that the ring of endomorphisms of the uniform components is completely indecomposable. Is this true for arbitrary uniform and injective modules with chain conditions?

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