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ON EXTENDING PROJECTIVES OF FINITE GROUP-GRADED ALGEBRAS

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ABSTRACT. Let G be a finite group, let k be a field and let R be a finite dimensional fully G-graded k-algebra. Also let L be a completely reducible R-module and let P be a projective cover of R. We give necessary and sufficient conditions for $P|_{R_1}$ to be a projective cover of $L|_{R_1}$ in Mod (R_1). In particular, this happens if and only if L is R_1-projective. Some consequences in finite group representation theory are deduced.

1. **Introduction and Statements.** Our notation and terminology are standard and tend to follow the conventions of [5]. In particular, in this article, all rings have identities, all modules over a ring are right. unital and finitely generated and all algebras over a commutative ring are finitely generated as modules over the commutative ring. Also if n is a positive integer and V is a module over a ring A, then nV denotes the A-module direct sum of n copies of V and the head of V is $\mathcal{H}(V) = V/\operatorname{Rad}(V)$. Moreover if B is a subring of A, then, by assumption, the identity of A lies in B and $V|_B$ denotes the restriction of V to B.

Throughout this article, k denotes a field, p denotes a prime integer, G and H are finite groups and N is a normal subgroup of H. Also O is a commutative ring and R denotes a fully G-graded O-algebra (i.e., R is an O-algebra and $R = \oplus_{g\in G} R_g$ in Mod (O) where R_g is a (finitely generated) O-submodule of R for each $g \in G$ and such that $R_gR_h = R_{gh}$ for all $g, h \in G$). Thus R_1 is an O-subalgebra of R by [5, Proposition 1.4]. Here $J(R_1)R = \oplus_{g\in G} (J(R_1)R_g) = \oplus_{g\in G} (R_gJ(R_1)) = RJ(R_1)$ and $J(R_1)R$ is a G-graded 2-sided ideal of R contained in $J(R)$ by [4, Proposition 1.11] and [2, Corollary 4.2 and Theorem 4.4(1)]. Thus if V is an R-module, then $VJ(R_1)R = VJ(R_1)$ is an R-submodule of V contained in $VJ(R)$. Also if K is a subgroup of G, then $R_K = \oplus_{g\in K} R_g$ is a fully K-graded O-subalgebra of R. As usual, $O[H]$ is the group algebra of H over O and if $G = H/N$, then $O[H]$ is a fully G-graded O-algebra with $O[H]_{gN} = \oplus_{x\in gN} Ox$ for all $gN \in G = H/N$. Note here that $O[H]_N = \mathbf{O}[N]$.

Suppose that P is a projective cover of a completely reducible R-module L, so that $P|_{R_1}$ is a projective R_1-module (cf. Lemma 2.3). It is natural to ask: when is $P|_{R_1}$ a projective cover of $L|_{R_1}$?

It is well-known that, in general, there may not exist any R-module X such that $X|_{R_1}$ is a projective cover of $L|_{R_1}$.

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EXAMPLE 1. Assume that char $(k) = p$, let H be a cyclic group of order p^2, let N denote the unique subgroup of H of order p and let L denote a trivial $k[H]$-module. Thus $L|_{k[N]}$ is a trivial $k[N]$-module and $P = k[N]_{k[N]}$ (the regular $k[N]$-module) is a projective cover of $L|_{k[N]}$ by [11, VII, Theorem 5.2]. We claim that there is no $k[H]$-module Q such that $Q|_{k[N]} \cong P$ in Mod $(k[N])$. To see this suppose that X is such a $k[N]$-module. Then X is indecomposable as $X|_{k[N]} \cong P$ is indecomposable in Mod $(k[N])$. Since dim $(X/k) =$ dim $(P/k) = p$. [11, VII, Theorem 5.3] implies that every element of N acts trivially on X. As $X|_{k[N]} \cong k[N]_{k[N]}$, we have a contradiction and our claim is demonstrated.

Note, in this example, that the regular $k[H]$-module $M = k[H]_{k[H]}$ is a projective cover of L, $M|_{k[N]} \cong pP$ in Mod $(k[N])$ and hence $MJ(k[N]) \neq MJ(k[H])$ and that P has precisely p descending Loewy factors all of which are isomorphic to $L|_{k[N]}$ in Mod $(k[N])$ in accordance with [9].

Our main result is:

PROPOSITION 2. *Suppose that $O = k$ is a field. Let L be a completely reducible R-module and let P be a projective cover of L in* Mod (R). *Then the following three conditions are equivalent:*

(a) $P|_{R_1}$ is a projective cover of the (completely reducible) R_1-module $L|_{R_1}$:

(b) L is R_1-projective: and

(c) $PJ(R) = PJ(R_1)$.

Clearly [6, Proposition 3.3] yields:

COROLLARY 3. *As in Proposition 2. assume that $O = k$ is a field and also that $|G|\, 1_k$ is a unit in k. Let L be a completely reducible R-module and let P be a projective cover of L in* Mod (R). *Then $P|_{R_1}$ is a projective cover of $L|_{R_1}$ in* Mod (R_1).

REMARK 4. As in Proposition 2, let $O = k$ be a field. Let W be an irreducible R_1-module and let P be a projective cover of W in Mod (R_1), so that $P/(PJ(R_1)) \cong W$ in Mod (R_1). Assume that Q is an R-module such that $Q|_{R_1} \cong P$ in Mod (R_1). Then Q is indecomposable and $Q/(QJ(R_1)) = V$ is an R-module such that $V|_{R_1} \cong W$. Thus V is an irreducible R-module and $QJ(R_1) = QJ(R)$. If also $|G|\, 1_k$ is a unit in k, then Q is a projective cover of V by [6, Proposition 3.3].

Next we present an application of our results to the classical case in Stable Clifford Theory in finite group representation theory (cf. [10, V, Satz 17.5]).

Let M be an irreducible $k[N]$-module. Assume that $c\colon H \times H \longrightarrow k^{\times}$ is a 2-cocycle with H acting trivially on $k^{\times}$ such that $c\colon N \times N \longrightarrow \{1\}$ and c is constant on (gN, hN) for all $g, h \in H$. Let $k[H](c)$ denote the corresponding twisted group algebra, so that $k[N]$ is a subalgebra of $k[H](c)$. Since $k[H](c)$ can be viewed as a fully $G = H/N$-graded k-algebra with $(k[H](c))_{gH} = \oplus_{x \in gN}\, kx$ for all $g \in H$. Proposition 2 yields:

COROLLARY 5. *Suppose that there is a $k[H](c)$-module L such that $L|_{k[N]} \cong M$ in* Mod $(k[N])$. *Let P be a projective cover of L in* Mod $(k[H](c))$. *Then the following three conditions are equivalent:*

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(a) $P|_{k[N]}$ is a projective cover of M;
(b) L is $k[N]$-projective; and
(c) $PJ(k[H](c)) = PJ(k[N])$.

As another application, we observe that [12, Proposition 2.8] is a special case of our results. For, [12, Proposition 2.8(a)] is a special case of Corollary 5 and [6, Proposition 3.3], and [12, Proposition 2.8(b)] is a special case of Remark 4, Corollary 5 and [6, Proposition 3.3].

REMARK 6. Suppose that O is a complete discrete valuation ring and let $k = O/J(O)$. Then

$$J(O)R = RJ(O) = \oplus_{g\in G}(J(O)R_g) = \oplus_{g\in G}(R_gJ(O))$$

is a G-graded 2-sided ideal of R contained in $J(R)$ (cf. [7, I, Lemma 8.15]) and $\bar{R} = R/(RJ(O))$ is a fully G-graded finite dimensional k-algebra with $(\bar{R})_g = (R_g + RJ(O))/(RJ(O))$ for all $g \in G$. Let L be a finitely generated completely reducible R-module and let $f : P \longrightarrow L$ be a projective cover of L in Mod (R) where P is a projective R-module and $f \in \mathrm{Hom}_R(P, L)$ is essential (cf. [3, Section 6C]). Since $LJ(O) \subseteq LJ(R) = (0)$, L may be viewed as a completely reducible $\bar{R}$-module and $PJ(O) \subseteq \mathrm{Ker}(f)$. Also $\bar{P} = P/(PJ(O))$ is a projective $\bar{R}$-module and f induces the projective cover $\bar{f} : \bar{P} \longrightarrow L$ in Mod $(\bar{R})$. Here $(\bar{R})_1 = (R_1 + RJ(O))/(RJ(O)) \cong R_1/(R_1J(O))$ as rings and, using [3, Section 6C], it is easy to see that $f: P \longrightarrow L$ is a projective cover of L in Mod (R_1) if and only if $\bar{f}: \bar{P} \longrightarrow L$ is a projective cover of L in Mod $((\bar{R})_1)$.

Section 2 presents some basic results that are required in our proof of Proposition 2 that is given in section 3.

2. Preliminary Results.

For the convenience of the reader we present the following two well-known results (cf. [9, Lemma 2.6] and [1, II, Proposition 6.1]:

LEMMA 2.1. *(a) for each $g \in G$, R_g is a finitely generated projective R_1-module and a finitely generated projective left R_1-module; and (b) R is a finitely generated projective R_1-module and a finitely generated projective left R_1-module.*

LEMMA 2.2. *Let K be a subgroup of G and let P be a finitely generated projective $R_K = \oplus_{g\in K} R_g$-module. Then $P \otimes_{R_K} R$ is a finitely generated projective R-module.*

LEMMA 2.3. *Let K be a subgroup of G and let Q be a finitely generated projective R-module. Then $Q|_{R_K}$ is a finitely generated projective R_K-module.*

PROOF. Let T be a transversal for the left cosets of K in G. Then $R = \oplus_{x\in T} R_{xK}$ in Mod (R_K). Clearly R_{gK} is a finitely generated projective R_1-module for each $g \in G$ by Lemma 2.1. (a). Fix $g \in G$. It suffices to prove that R_{gK} is a projective R_K-module. Note that $R_g \otimes_{R_1} R_K = \oplus_{k\in K}(R_g \otimes_{R_1} R_k)$ in Mod (R_1) and that $\alpha: R_g \otimes_{R_1} R_K \longrightarrow R_{gK}$ defined by: $\alpha(r \otimes_{R_1} s) = rs$ for all $r \in R_g$ and all $s \in R_K$ is well-defined R_K-epimorphism. Since the restriction of α to $R_g \otimes_{R_1} R_K$ is one-to-one by [6, (1.4)] for all $k \in K$, α is an isomorphism. Since $R_g \otimes_{R_1} R_K$ is a projective R_K-module by Lemma 2.1(a) and [1, II, Proposition 6.1], we are done. ■

For the remainder of this section, we assume that $O = k$ is a field.

LEMMA 2.4. *Let N be an R_1-module. Then:*

(a) $\mathcal{H}(N) \otimes_{R_1} R \cong (N \otimes_{R_1} R)/((N \otimes_{R_1} R)J(R_1))$ *in* Mod (R); *and*
(b) $\mathcal{H}(\mathcal{H}(N) \otimes_{R_1} R) \cong \mathcal{H}(N \otimes_{R_1} R)$ *in* Mod (R).

PROOF. We have an exact sequence

$$(0) \longrightarrow NJ(R_1) \xrightarrow{i} N \xrightarrow{\pi} \mathcal{H}(N) = N/NJ(R_1) \longrightarrow (0)$$

in Mod (R_1) where i denotes the inclusion map and π is the canonic epimorphism. Since R is a projective and hence flat left R_1-module,

$$(0) \longrightarrow NJ(R_1) \otimes_{R_1} R \xrightarrow{i \otimes I_R} N \otimes_{R_1} R \xrightarrow{\pi \otimes I_R} \mathcal{H}(N) \otimes_{R_1} R \longrightarrow (0)$$

is exact in Mod (R). Then [8, Lemma 2.4(d)] yields (a) and (b) follows from (a) and the fact that $RJ(R_1) = J(R_1)R \subseteq J(R)$. ■

LEMMA 2.5. *Let N be an R_1-module and let Q be a projective cover of N in* Mod (R_1). *Let* Irr (R) *be a set of representatives for the types of irreducible R-modules and for each $X \in$* Irr (R)*, let $P(X)$ denote a projective cover of X in* Mod (R). *Then:*

(a) $Q \otimes_{R_1} R \cong \oplus_{X \in \mathrm{Irr}(R)} (\mathrm{mult}(X \text{ in } \mathcal{H}(N \otimes_{R_1} R)))\, P(X)$ *in* Mod (R);
(b) $(Q \otimes_{R_1} R)/((Q \otimes_{R_1} R)J(R_1)) \cong \oplus_{X \in \mathrm{Irr}(R)} (\mathrm{mult}(X \text{ in } \mathcal{H}(N \otimes_{R_1} R))(P(X)/P(X)J(R_1))$ *in* Mod (R); *and*
(c) $\mathcal{H}(N) \otimes_{R_1} R \cong \oplus_{X \in \mathrm{Irr}(R)} (\mathrm{mult}(X \text{ in } \mathcal{H}(N \otimes_{R_1} R)(P(X)/P(X)J(R_1))$ *in* Mod (R).

REMARK 2.6. Let M be an irreducible R_1-module and let L be an irreducible R-module. Then $L|_{R_1}$ is a completely reducible R_1-module since $LJ(R_1) \leqq LJ(R) = (0)$ and

$$\mathcal{H}om_R(\mathcal{H}(M \otimes_{R_1} R), L) \cong \mathcal{H}om_R(M \otimes_{R_1} R, L) \cong \mathcal{H}om_{R_1}(M, L)$$

as k-spaces by [1, II, Section 6, (3′)]. Thus

$$\dim(\mathrm{End}_R(L)/k)(\mathrm{mult}(L \text{ in } \mathcal{H}(M \otimes_{R_1} R))) =$$
$$\dim(\mathrm{End}_{R_1}(M)/k)(\mathrm{mult}(M \text{ in } L|_{R_1})).$$

PROOF. Here $\mathcal{H}(N) \cong \mathcal{H}(Q)$ in Mod (R_1) (cf. [11, VII, Section 10]) and hence

$$\mathcal{H}(Q \otimes_{R_1} R) \cong \mathcal{H}(\mathcal{H}(Q) \otimes_{R_1} R) \cong \mathcal{H}(\mathcal{H}(N) \otimes_{R_1} R)$$
$$\cong \mathcal{H}(N \otimes_{R_1} R) \text{ in Mod } (R)$$

by Lemma 2.4(b). Now [11, VII, Section 10] implies (a) and (b) is immediate. Also (b), Lemma 2.4(a) and the fact that $\mathcal{H}(N) \cong (Q)$ in Mod (R_1) yield (c) and we are done. ■

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3. **A Proof of Proposition 2..** In this section, we present a proof of Proposition 2 and consequently we assume its hypotheses and we set $M = L|_{R_1}$, so that $\mathcal{H}(M) = M$ in Mod (R_1).

Suppose that (c) holds. Then $P(L)/(P(L)J(R_1)) = P(L)/P(L)J(R) \cong L$ in Mod (R) and $P(L)|_{R_1}$ is a projective R_1-module by Lemma 2.3. Since

$$\mathcal{H}(P(L)|_{R_1}) \cong (P(L)/P(L)J(R_1))|_{R_1} \cong L|_{R_1} \text{ in Mod } (R_1),$$

(a) follows. Also

$$(0) \neq \mathrm{Hom}_{R_1}(M, L|_{R_1}) \cong \mathrm{Hom}_R(M \otimes_{R_1} R, L) \cong \mathrm{Hom}_R(\mathcal{H}(M \otimes_{R_1} R), L)$$

over k by [1, II, Section 6, (3′)]. Hence

$$L \cong P(L)/(P(L)J(R)) = P(L)/P(L)J(R_1))|M \otimes_{R_1} R$$

by Lemma 2.5(c). Thus (b) also holds. Assume (a) and observe that $(P(L)/(P(L)J(R_1)))|_{R_1} \cong L|_{R_1}$ in Mod (R_1). Since $P(L)/(P(L)J(R)) \cong L$ in Mod (R) and $P(L)J(R_1) \subseteq P(L)J(R)$, a dimension argument forces (c). Assume (b). Thus

$$L|(M \otimes_{R_1} R) \cong \oplus_{X \in \mathrm{Irr}(R)}(\mathrm{mult}(X \text{ in } \mathcal{H}(M \otimes_{R_1} R)))(P(X)/(P(X)J(R_1)))$$

by Lemma 2.5(c). The Krull-Schmidt Theorem implies that $L \cong P(L)/P(L)J(R_1))$ in Mod (R). Thus (c) follows and our proof is complete. ∎

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