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Asymptotics of Perimeter-Minimizing Partitions

Quinn Maurmann, Max Engelstein, Anthony Marcuccio, and Taryn Pritchard

Abstract. We prove that the least perimeter $P(n)$ of a partition of a smooth, compact Riemannian surface into n regions of equal area A is asymptotic to $n/2$ times the perimeter of a planar regular hexagon of area A. Along the way, we derive tighter estimates for flat tori, Klein bottles, truncated cylinders, and Möbius bands.

1 Introduction

T. Hales [3] proved that a tiling by congruent regular hexagons provides a least-perimeter way to partition the plane into unit areas. (Since the plane is infinite, this is really a statement about the optimal perimeter to area ratio inside large balls about the origin or for large, finite clusters, as stated in Theorem 2.1 below.) Theorem 2.9 shows that for any smooth, compact Riemannian surface as in Figure 1, the least perimeter $P(n)$ of a partition into n equal areas is asymptotic to the perimeter of a partition into regular planar hexagons, *i.e.*, $n/2$ times the perimeter of a planar regular hexagon of the given area. (Divide by 2 because each interface is shared by two regions.)

It is known that least-perimeter partitions exist and are given by finitely many constant-curvature curves meeting at 120 degrees at finitely many points [7, 2.3, 2.4, 3.3].

An early result, Lemma 2.4, shows that a planar disk of finite perimeter can be almost filled by a union of small, congruent regular hexagons with bounded perimeter. Corollaries 2.2, 2.3, and 2.5 find that for many flat surfaces, the estimate for least perimeter $P(n)$ for partitions holds not only asymptotically but also with bounded error.

Lemma 2.6 (refined in Lemma 2.7) provides area-preserving near-isometries between small open disks in a surface M and in the plane. The first application, Lemma 2.8, provides fairly efficient partitions of any measurable subset of M. The second application is our main asymptotic formula (Theorem 2.9) for least perimeter $P(n)$. Most of M is partitioned by Lemmas 2.4 and 2.7, and the small remaining space in between is partitioned by Lemma 2.8, yielding the requisite bound on $P(n)$ from above. The bound from below follows from Lemma 2.7 and Hales's Theorem 2.1.

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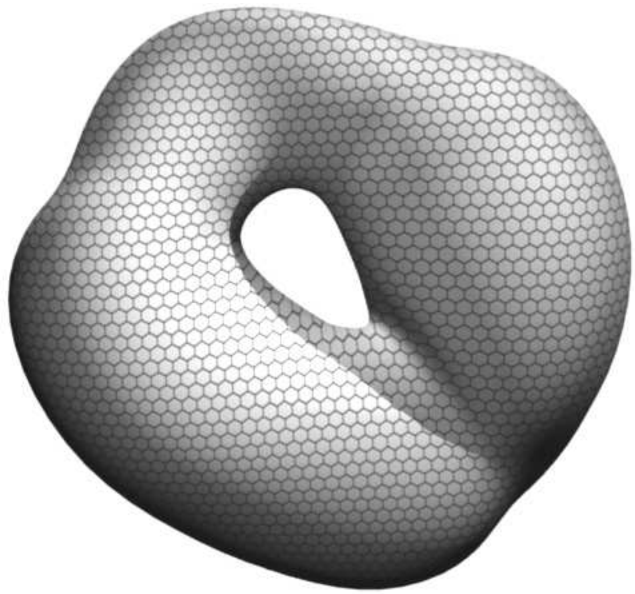

Figure 1: Theorem 2.9 states that the least perimeter $P(n)$ for partitioning a compact surface into n equal areas is asymptotic to that corresponding to planar regular hexagons.

Even for the round sphere, the least-perimeter partition into n equal areas is known only for $n = 2$ (Bernstein 1905, [1]), $n = 3$ (Masters 1996, [5]), $n = 4$ (Engelstein 2010 [2], and $n = 12$ (Hales 1999, [4]). The main difficulty is that a region could in principle have many components.

2 Definitions

Let M be a smooth, compact Riemannian surface of area $|M|$, possibly with boundary. Let H denote half the perimeter of a planar regular hexagon of area $|M|$, namely $12^{1/4}\sqrt{|M|}$. Following Morgan [6, §15.2], define a planar *cluster* as a smooth, locally finite graph with each closed face included in a unique region (nonempty union of faces). The *external perimeter* of a cluster is the perimeter of the union of all the faces or regions. Occasionally, we will abuse terminology and refer to the boundary of this union of faces as a cluster's external perimeter.

We begin the march toward our main asymptotic Theorem 2.9 with a statement of Hales' theorem for planar clusters.

Theorem 2.1 (Hales, [3, Theorem 2], [6, Proposition 15.6]) *Any cluster of planar regions with areas $a_1, \dots, a_n$, where $0 < a_i \leq 1$ for $i = 1, \dots, n$ has perimeter greater than $12^{1/4} \sum_{i=1}^{n} a_i$. In particular, if each $a_i = 1$, then the perimeter is greater than $12^{1/4} n$.*

One immediate consequence is a constant lower bound for the least perimeter of

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partitions of several flat surfaces.

Corollary 2.2 *If M is a flat torus, Klein bottle, truncated cylinder, or Möbius band, then $P(n) > H\sqrt{n} - L$ for all n, where L is the length of the boundary of a fundamental planar parallelogram of M.*

Proof We view a flat surface M as a parallelogram in the plane with appropriate sides identified. Given a perimeter-minimizing partition of M into n equal areas, adding the boundary of the parallelogram yields a cluster of n (not necessarily connected) planar regions of area $|M|/n$ and perimeter at most $P(n) + L$. Scale the cluster by a factor of $\sqrt{n/|M|}$ to obtain a cluster of unit-area regions with perimeter at most $(P(n)+L)\sqrt{n/|M|}$, which by Theorem 2.1 must be greater than $12^{1/4}n$. In particular, $P(n) > 12^{1/4}\sqrt{|M|n} - L = H\sqrt{n} - L$. ■

Though we are content bounding $P(n) - H\sqrt{n}$ below by a constant, we do note that the lower bound may be tightened to $P(n) \geq H\sqrt{n} - L/2$ by constructing a large grid of copies of M so that only half the boundary of the fundamental parallelogram need be included to separate the copies. Only copies on the very edge of the grid will need the entire boundary, and they are a small fraction of the total number. The next corollary uses this type of technique to improve the lower bound for flat tori and Klein bottles.

Corollary 2.3 *If M is a flat torus or Klein bottle,* $\liminf_{n\to\infty}(P(n) - H\sqrt{n}) \geq 0$.

Proof For each n, let X_n be a perimeter-minimizing graph which partitions M into n regions of equal area. The main difficulty will be treating noncontractible components should they arise in such a partition. Topologically, a noncontractible component must be an annular band, a Möbius band, or the complement of a disk, possibly with other (contractible) holes in it. Actually, such holes will not arise in a perimeter-minimizing partition: holes may be slid, preserving area and perimeter, to eventually contradict the regularity results of [7].

At this point, we assume without loss of generality that M has area 1 and is generated by the rectangle $[0,1) \times [0,1)$. In the case that M is generated by a general parallelogram, our calculations will change only by constant factors. It is easy to see that for flat surfaces $P(n)$ is $O(\sqrt{n})$; for example, skip to Corollary 2.5 or simply partition M by a grid of line segments. Since the perimeter of any annular band or Möbius band in M is at least 2, we must have that the number of bands in the partition by X_n is at most $O(\sqrt{n})$. That is, for some $K > 0$ and large n, the number of bands is less than $K\sqrt{n}$. Note that if the partition by X_n contains the complement of a disk, then certainly it is the only noncontractible component. In either case, the number of noncontractible components is less than $K\sqrt{n}$.

With a horizontal line segment and a vertical line segment each of length 1, we could cut every noncontractible component into contractible pieces; by placing the segments more carefully and by intersecting them with a given noncontractible component, we can do better. For any noncontractible component A and for each $x \in [0,1)$, let S_x be the intersection of the vertical line at x with A. When $|S_x|$ denotes the length of S_x and $|A|$ denotes the area of A, we have

$$1/n \geq |A| = \int_0^1 |S_x| dx$$

so that we may choose some $\bar{x}$ with $|S_{\bar{x}}| \leq 1/n$. Similarly, we may let T_y be the intersection of the horizontal line at y with A, and choose $\bar{y}$ so that $|T_{\bar{y}}| \leq 1/n$. For each noncontractible component, take the union of the corresponding $S_{\bar{x}}$ and $T_{\bar{y}}$ with X_n to get a graph X_n^*. Since the number of noncontractible components is less than $K\sqrt{n}$, the perimeter of X_n^* is less than $P(n) + 2K/\sqrt{n}$ and can be made as close as desired to $P(n)$ for large n.

Now suppose $\liminf_{n\to\infty}(P(n) - H\sqrt{n}) < 0$; that is, for some $\varepsilon > 0$ and some increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$, we have $P(n_k) - H\sqrt{n_k} < -\varepsilon$. Take k sufficiently large so that $2K/\sqrt{n_k} < \varepsilon/2$, and hence the perimeter of $X_{n_k}^*$ is less than $H\sqrt{n_k} - \varepsilon/2$. Copies of M (partitioned by $X_{n_k}^*$) tile the plane, and components along the boundary of the fundamental rectangle can be identified. Since each component is contractible, the identification does not produce any regions with area greater than $1/n_k$. For any integer m, we may construct an $m \times m$ grid in the plane of copies of M. Including only the outermost boundaries of rectangles in the grid (with total length m), we get a cluster of $m^2 n_k$ regions in the plane of area $1/n_k$ and perimeter less than $Hm^2\sqrt{n_k} - \varepsilon m^2/2 + m$. Scaling up by a factor of $\sqrt{n_k}$, we obtain a cluster of $m^2 n_k$ unit-area regions with perimeter less than

$$12^{1/4} m^2 n_k - \frac{\varepsilon m^2}{2}\sqrt{n_k} + m\sqrt{n_k}.$$

For large m, this perimeter can be made less than $12^{1/4} m^2 n_k$, contradicting Theorem 2.1. ■

Lemma 2.4 shows that a planar disk of finite perimeter can be almost filled by a union of small, congruent, regular hexagons with bounded perimeter.

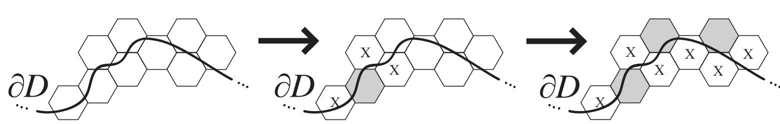

Figure 2: To limit the number of hexagons intersected by the boundary curve ∂D, we see that at least 1/7 of them (shaded in figure) are nonadjacent, because any hexagon is adjacent to just six others. Illustrated is the process of following ∂D, repeatedly choosing the first available hexagon and excluding its neighbors.

Lemma 2.4 *Let D be a disk in $\mathbf{R}^2$ with rectifiable boundary ∂D, and fix $A > 0$. For each n, let T_n denote a tiling of the plane by regular hexagons of area A/n. Then there exists a constant B such that for large n, the number of hexagons of T_n intersected by ∂D is less than $B\sqrt{n}$. Furthermore, letting C_n denote the cluster of hexagons contained in D, the total area covered by the hexagons in C_n approaches $|D|$ as $n \to \infty$, and the external perimeters of the C_n are bounded.*

Proof Let k_n be the number of hexagons of T_n intersected by ∂D. We will see that ∂D must intersect at least $k_n/7$ nonadjacent hexagons as illustrated in Figure 2. Pick any of these hexagons and include it in our collection of nonadjacent hexagons, excluding its neighbors from the process. Follow ∂D until it intersects another hexagon that has neither been marked for inclusion nor exclusion in our collection. Mark it for inclusion, its neighbors for exclusion, and repeat until the entire length of ∂D has been traversed.

The minimum distance between two nonadjacent hexagons is the length of one side of one hexagon, $s/\sqrt{n}$, where $s = 3^{-3/4}\sqrt{2A}$. Let L denote the length of ∂D, and so long as $k_n/7 \geq 2$, we have $L \geq k_n s/7\sqrt{n}$, so that $k_n \leq 7L\sqrt{n}/s$; take $B = 7L/s$.

Let c_n be the number of hexagons in C_n so that the area covered by C_n is Ac_n/n. To see that this area converges to $|D|$, we have

$$|D| - \frac{Ac_n}{n} \leq \frac{Ak_n}{n} \leq \frac{AB}{\sqrt{n}},$$

which approaches 0 as $n \to \infty$. And certainly the external perimeter of the cluster is less than $6k_n s/\sqrt{n} \leq 42L$, which is independent of n as desired. ∎

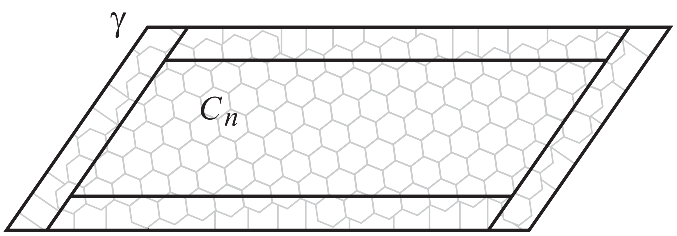

Figure 3: To partition a flat surface with perimeter $H\sqrt{n}$ plus a constant, start with the cluster C_n of hexagons inside the fundamental parallelogram γ. By including four long line segments parallel to the sides of γ, it is easy to partition the $O(\sqrt{n})$ remaining regions with perimeter at most $O(1/\sqrt{n})$ each.

Corollary 2.5 *If M is a flat torus, Klein bottle, truncated cylinder, or Möbius band, then $P(n)$ is no greater than $H\sqrt{n}$ plus a constant.*

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Proof Let γ be the boundary of a parallelogram that generates M, and let L be its length. For large n, lay down the tiling T_n of regular hexagons of area $|M|/n$, so that, by Lemma 2.4, the number of hexagons intersected by γ is less than $B\sqrt{n}$ for some constant B. Let $d_n = 2^{3/2}3^{-3/4}\sqrt{|M|/n}$, the diameter of a regular hexagon of area $|M|/n$. When C_n denotes the cluster of hexagons inside the parallelogram, it is easy to see that at each point on the external perimeter of C_n, there is some side of γ which is not further than d_n from that point, otherwise another hexagon from T_n would have fit inside the parallelogram.

Each C_n has total perimeter no greater than $H\sqrt{n} + K$ for some constant K, and will be the starting point for a partition of M into n regions of equal area. Fewer than $B\sqrt{n}$ regions remain to be separated in the partition. As in Figure 3, we may include four line segments parallel to each side and at orthogonal distance d_n with total length less than L. Then each remaining region can be separated by a segment with length at most d_n (without trouble at the parallelogram's corners). Including γ in the partition to separate regions on the boundary, we have a partition of M into n regions of area $|M|/n$ with total perimeter less than

$$H\sqrt{n} + K + 2L + d_nB\sqrt{n},$$

where $d_nB\sqrt{n} = 2^{3/2}3^{-3/4}B\sqrt{|M|}$ and in particular is constant. ∎

Lemmas 2.6 and 2.7 partition M into disks with area-preserving near-isometries with planar regions.

Lemma 2.6 *Let M be a smooth Riemannian surface, possibly with boundary, and let $\varepsilon > 0$. Then every interior point p of M has an open disk neighborhood V_p with smooth boundary ∂V_p and an area-preserving diffeomorphism Φ_p mapping V_p to an open disk in $\mathbf{R}^2$ while distorting length by no more than a factor of $1 + \varepsilon$. Similarly, each point in ∂M has an open disk neighborhood with piecewise smooth boundary and an area-preserving diffeomorphism to an open disk in the upper half-plane $\mathbf{R}^2_+ = \{(x, y) \in \mathbf{R}^2 : y \geq 0\}$ which distorts length by no more than $1 + \varepsilon$.*

Proof As in Figure 4, we define Φ_p for interior points p by mapping some geodesic through p, parameterized by arc length, into the x-axis. Similarly, for $p \in \partial M$, map ∂M into the x-axis at unit speed. In either case, map orthogonal geodesics into vertical lines, parameterized at a speed to preserve area. The stretch factor is continuous, so we may restrict Φ_p to a small open disk V_p on which the stretch is between $1 + \varepsilon$ and $\frac{1}{1+\varepsilon}$. ∎

Lemma 2.7 *Let M be a smooth, compact, Riemannian surface, possibly with boundary, and let $\varepsilon > 0$. Then for some integer n there exists a partition of M into disjoint regions E_i for $i = 1, \ldots, n$ with piecewise smooth boundaries such that on each E_i there exists an area-preserving diffeomorphism Φ_i mapping E_i to a region in $\mathbf{R}^2$ while distorting length by no more than a factor of $1 + \varepsilon$. Moreover, the E_i may be taken so that each consists of finitely many disks.*

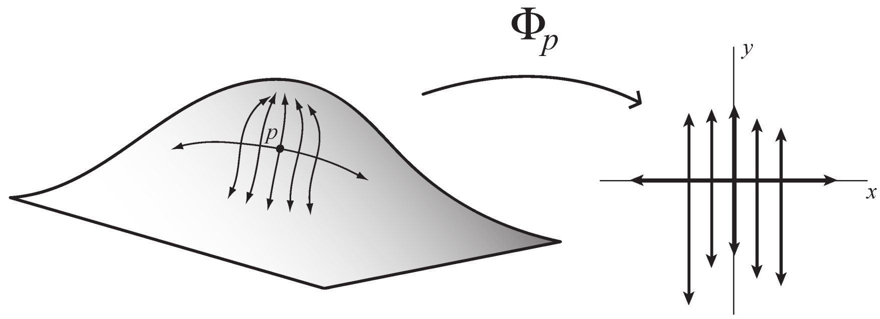

Figure 4: The area-preserving map Φ_p takes some geodesic through p to the x-axis at unit speed and takes orthogonal geodesics to vertical intervals at a speed chosen to preserve area.

Proof At each point $p \in M$, let V_p be an open disk neighborhood of p in M with piecewise smooth boundary ∂V_p such that some area-preserving diffeomorphism Φ_p maps V_p to an open disk in $\mathbf{R}^2$ $[\mathbf{R}^2_+]$ and distorts length by no more than a factor of $1 + \varepsilon$. These disks may be taken to be sufficiently small and round so that the boundary of one intersects the boundary of any other in at most two points. Since M is compact and $\{V_p : p \in M\}$ is an open cover of M, there exists some finite set of points $\{p_1, \dots, p_n\}$ in M with $\bigcup_i V_{p_i} = M$. We may assume that none of the V_{p_i} are completely contained in any other; otherwise, such a disk need not be included in the finite subcover. Now for each $i = 1, \dots, n$, set $E_i = V_{p_i} \setminus \bigcup_{j=1}^{i-1} V_{p_j}$, so that the E_i are pairwise disjoint with $\bigcup_i E_i = M$ and $\bigcup_i \partial E_i \subset \bigcup_i \partial V_{p_i}$, and each E_i is a finite union of disks. Take each Φ_i as the restriction of Φ_{p_i} to E_i. ■

The following lemma provides fairly efficient partitions of any measurable subset of M.

Lemma 2.8 *Let M be a smooth, compact Riemannian surface. Then there exist constants c_1 and c_2 such that for any integer n and any measurable subset $A \subset M$, there exists a partition of M into n regions $R_1, \dots, R_n$ with total perimeter less than $c_1\sqrt{n} + c_2$ such that the measure of each $A \cap R_i$ equals $|A|/n$.*

Proof By Lemma 2.7, there is a partition of M into m regions E_i with area-preserving diffeomorphisms Φ_i into planar regions which each distort length by no more than a factor of 2. The partition has finite perimeter Q. Set $d = \max \operatorname{diam} E_i$, so that each $\Phi_i(E_i)$ has diameter no greater than $2d$. We partition each planar region $\Phi_i(E_i)$ into n regions containing equal portions of $\Phi_i(A)$ by horizontal and vertical line segments, each of length no greater than $2d$. First, take k as the unique positive integer such that $k^2 \leq n \leq k^2 + 2k$, and set $A_i = \Phi_i(A \cap E_i)$.

As in Figure 5, we partition each $\Phi_i(E_i)$ into $k+1$ regions $B_i^0, \dots, B_i^k$ by placing k horizontal lines at appropriate heights so that the measures satisfy

$$|A_i \cap B_i^0| = \frac{n-k^2}{n}|A_i| \quad \text{and} \quad |A_i \cap B_i^j| = \frac{k}{n}|A_i| \quad \text{for} \quad j = 1, \dots, k.$$

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The regions may be situated so that B_i^α lies below B_i^β whenever $\alpha < \beta$. The k horizontal lines can be intersected with $\Phi_i(E_i)$ to obtain segments with total length no more than $2dk$. The regions $B_i^1, \dots, B_i^{k-1}$ can each be partitioned into k regions by taking $k - 1$ vertical segments between the highest and lowest parallel lines used to define the B_i^j. Each of these segments can be broken wherever it intersects any of these parallel horizontal lines, and the resulting segments can be translated left or right so that each region contains the same portion of A_i. These segments again have total length no more than $2dk$. The regions B_i^k and B_i^0 can also be partitioned by $k - 1$ and by $n - k^2 - 1 < 2k$ vertical lines, respectively. These segments have total length less than $6dk$, so that all the segments placed in $\Phi_i(E_i)$ have total length less than $10dk \le 10d\sqrt{n}$. For each $i = 1, \dots, m$, we map these segments by the appropriate Φ_i^{-1} into M, stretching edges by no more than a factor of 2. These curves together with the ∂E_i partition M into n (disconnected) regions with perimeter less than $20md\sqrt{n} + Q$, and each region contains an equal portion of A. ■

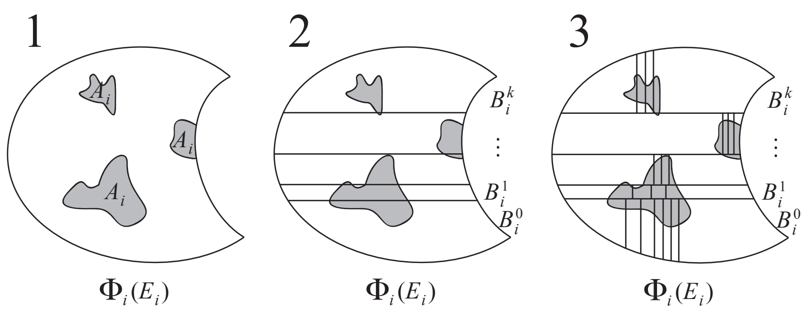

Figure 5: To bound the cost of partitioning an arbitrary subset A of the surface M into n equal areas by a multiple of $\sqrt{n}$, map portions of the surface ($E_i : i = 1, \dots, m$) into the plane and partition the part of A in E_i by horizontal and vertical line segments.

The following theorem provides our main asymptotic formula for least perimeter.

Theorem 2.9 *For any smooth, compact, Riemannian surface M with area $|M|$, and possibly with boundary, the least perimeter $P(n)$ to partition M into n regions of equal area is asymptotic to $n/2$ times the perimeter of a planar regular hexagon of area $|M|/n$:*

$$\lim_{n\to\infty} \frac{P(n)}{\sqrt{n}} = 12^{1/4}\sqrt{|M|}.$$

Proof With $H = 12^{1/4}\sqrt{|M|}$, we first prove $\limsup_{n\to\infty} P(n)/\sqrt{n} \le H$. Let $\varepsilon > 0$. As in Lemma 2.7, take regions $E_1, \dots, E_m$ and area-preserving diffeomorphisms Φ_i mapping E_i into planar regions while distorting length by no more than a factor of $1 + \varepsilon$. For each i, $\partial\Phi_i(E_i)$ is piecewise smooth, and each $\Phi(E_i)$ consists of finitely many disks, so we may apply Lemma 2.4 to find a sequence of clusters $\{C_n^i\}$ of regular

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hexagons in $\Phi_i(E_i)$, each of area $|M|/n$ whose total area approaches the area of E_i as $n \to \infty$ and whose external perimeter is bounded by some constant K_i. Since $H\sqrt{n}$ is half the total perimeter of n separated regular hexagons of area $|M|/n$, we have that the total perimeter of all these clusters is no greater than $H\sqrt{n} + \frac{1}{2}\sum_{i=1}^{m} K_i$.

Each Φ_i^{-1} maps the appropriate cluster of planar hexagons into E_i, increasing perimeter by at most a factor of $1 + \varepsilon$ while preserving area. After the mapping, the total length of these arcs is less than $(1+\varepsilon)\left(H\sqrt{n} + \frac{1}{2}\sum_{i=1}^{m} K_i\right)$. To finish the partition, we must separate the remaining area into some number r_n of regions of area $|M|/n$, which can be partitioned as in Lemma 2.8 with additional perimeter less than $c_1\sqrt{r_n} + c_2$ for some constants c_1 and c_2. The area covered by the hexagons in $\bigcup_i \Phi_i^{-1}(C_n^i)$ is $|M| - r_n|M|/n$, and must converge to $|M|$, hence $r_n/n \to 0$ as $n \to \infty$. Thus we have an upper bound on the perimeter of this partition. In particular, we may bound the minimal perimeter of an equal-area partition:

$$\frac{P(n)}{\sqrt{n}} < \frac{(1+\varepsilon)\left(H\sqrt{n} + \frac{1}{2}\sum_{i=1}^{m} K_i\right) + c_1\sqrt{r_n} + c_2}{\sqrt{n}}.$$

As $n \to \infty$, all terms but the first approach 0, hence $\limsup_{n\to\infty} P(n)/\sqrt{n} \le (1+\varepsilon)H$; since ε was arbitrary, we have $\limsup_{n\to\infty} P(n)/\sqrt{n} \le H$.

It now suffices to prove $\liminf_{n\to\infty} P(n)/\sqrt{n} \ge H$. Suppose on the contrary that $\liminf_{n\to\infty} P(n)/\sqrt{n} = G < H$, so for some increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$, we have $P(n_k)/\sqrt{n_k} \to G$ as $k \to \infty$. Fix $\varepsilon > 0$, sufficiently small so that $(1+\varepsilon)^2 G < H$. Take N so that $k > N$ gives $P(n_k)/\sqrt{n_k} < (1+\varepsilon)G$. For each $k > N$, let X_k be a perimeter-minimizing graph which partitions M into n_k regions of equal area, so that the perimeter of X_k is less than $(1+\varepsilon)G\sqrt{n_k}$.

By Lemma 2.7, we take a partition of M into some finite collection of regions $E_1, \dots, E_m$ with finite total perimeter on which area-preserving diffeomorphisms Φ_i into planar regions distort length by no more than a factor of $1+\varepsilon$. Let Q be the sum of lengths of the $\partial\Phi_i(E_i)$, and Q is also finite. We may assume each Φ_i maps each E_i into a different region of the plane. Then $\bigcup_i(\partial\Phi_i(E_i) \cup \Phi_i(X_k \cap E_i))$ is a cluster of n_k (not necessarily connected) regions of area $|M|/n_k$, and this cluster has perimeter less than $(1+\varepsilon)^2 G\sqrt{n_k} + Q$. The regions can be scaled to have area 1, giving a cluster of n_k unit regions in the plane with total perimeter less than

$$\left((1+\varepsilon)^2 G\sqrt{n_k} + Q\right)\sqrt{\frac{n_k}{|M|}} = (1+\varepsilon)^2 \frac{G}{H} 12^{1/4} n_k + Q\sqrt{\frac{n_k}{|M|}}.$$

Such a construction must be valid for every $k > N$, and since $(1+\varepsilon)^2 G/H < 1$, the above can be made less than $12^{1/4} n_k$ for large k, contradicting Theorem 2.1. The contradiction ensures $\liminf_{n\to\infty} P(n)/\sqrt{n} \ge H$, proving our result. ■

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Department of Mathematics, Yale University, New Haven, CT, U.S.A.
e-mail: max.engelstein@yale.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA, U.S.A.
e-mail: (Anthony Marcuccio) 08anm@williams.edu
(Taryn Pritchard) 08tbp@williams.edu

Department of Mathematics, UCLA, Los Angeles, CA, U.S.A.
e-mail: qmaurmann@math.ucla.edu