Canad. Math. Bull. Vol. **41** (2), 1998 pp. 240–244

ON CERTAIN K-GROUPS ASSOCIATED WITH MINIMAL FLOWS

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ABSTRACT. It is known that the Toeplitz algebra associated with any flow which is both minimal and uniquely ergodic always has a trivial K_1-group. We show in this note that if the unique ergodicity is dropped, then such K_1-group can be non-trivial. Therefore, in the general setting of minimal flows, even the K-theoretical index is not sufficient for the classification of Toeplitz operators which are invertible modulo the commutator ideal.

Let X be a compact Hausdorff space on which $\mathbf{R}$ acts continuously as a group of homeomorphisms. That is, there is a continuous map $(x, t) \longmapsto x + t$ from $X \times \mathbf{R}$ onto X such that $(x + t) + s = x + (t + s)$, $x + 0 = x$, and such that, for each t, $x \longmapsto x + t$ is a homeomorphism on X. Such an action of $\mathbf{R}$ on X is usually referred to as a flow. Given $f \in C(X)$ and $x \in X$, we denote the function $t \longmapsto f(x + t)$ on $\mathbf{R}$ by f_x. Let $A(X)$ denote the collection of $\varphi \in C(X)$ such that $\varphi_x \in H^\infty(\mathbf{R})$ for every $x \in X$.

Recall that a flow $(X, \mathbf{R})$ is said to be *minimal* if for every $x \in X$, the orbit $\{x+t : t \in \mathbf{R}\}$ is dense in X. A flow $(X, \mathbf{R})$ is said to be *uniquely ergodic* if there exists only one invariant probability measure on X.

For a given flow $(X, \mathbf{R})$, an *analytic representation* [5, 6] (also called Silov representation [7]) is a pair (π, P) where π is a unital C^*-algebra representation of $C(X)$ on some Hilbert space K and P is an orthogonal projection on K such that PK is invariant under $\pi\big(A(X)\big)$. That is, $\pi(\varphi)P = P\pi(\varphi)P$ for every $\varphi \in A(X)$. An analytic representation (π, P) is said to be *non-degenerate* if the C^*-algebra generated by P and $\pi\big(C(X)\big)$ is non-commutative. Each analytic representation (π, P) gives rise to a Toeplitz algebra $\mathcal{T}(\pi, P)$ on PK. That is, $\mathcal{T}(\pi, P)$ is the C^*-algebra generated by the abstract Toeplitz operators $\{P\pi(f)P : f \in C(X)\}$. We denote the commutator ideal of $\mathcal{T}(\pi, P)$ by $\mathcal{C}(\pi, P)$. For a minimal flow, $\mathcal{C}(\pi, P)$ coincides with the ideal in $\mathcal{T}(\pi, P)$ generated by the semi-commutators $\{P\pi(fg)P - P\pi(f)P\pi(g)P : f, g \in C(X)\}$ [6]. We next recall some familiar examples of analytic representation.

Let $K = L^2(\mathbf{R})$ and let P_{Hardy} be the orthogonal projection from $L^2(\mathbf{R})$ onto $H^2(\mathbf{R})$, the Hardy space associated with the upper half-plane. For each fixed $x \in X$, define $\pi_x(f) = M_{f_x}$, the multiplication by f_x on $L^2(\mathbf{R})$, $f \in C(X)$. Then $(\pi_x, P_{\mathrm{Hardy}})$ is an analytic representation of the flow.

Consider $K = L^2(X, dm)$, where dm is an invariant probability measure of the flow. Define $\pi_{dm}(f) = M_f$, the multiplication by f on $L^2(X, dm)$, $f \in C(X)$. For each $t \in \mathbf{R}$,

Received by the editors October 8, 1996.
Research supported in part by the National Science Foundation.
AMS subject classification: 46L80, 47B35, 47C15.

$\big(u_{dm}(t)\xi\big)(x) = \xi(x+t)$, $\xi \in L^2(X, dm)$, is a unitary operator. Let δ_{dm} denote the infinitesimal generator of the unitary group $\{u_{dm}(t) : t \in \mathbf{R}\}$. That is, $u_{dm}(t) = \exp(it\delta_{dm})$ for every $t \in \mathbf{R}$. Suppose that e_{dm} is the spectral measure for δ_{dm}. Then for each $\lambda \in \mathbf{R}$, $\Big(\pi_{dm}, e_{dm}\big((\lambda,\infty)\big)\Big)$ and $\Big(\pi_{dm}, e_{dm}\big([\lambda,\infty)\big)\Big)$ are analytic representations of $(X, \mathbf{R})$ [5, 6, 7].

In this note we address a question regarding the K-groups of $\mathcal{T}(\pi, P)$ and $\mathcal{C}(\pi, P)$ raised by the following two theorems from previous investigations.

THEOREM 1 [5, THEOREMS 3.1.1 AND 4.3.1]. *Suppose that $(X, \mathbf{R})$ is a minimal and uniquely ergodic flow and that (π, P) is a non-degenerate analytic representation of $(X, \mathbf{R})$.*

(i) The Toeplitz algebra $\mathcal{T}(\pi, P)$ is universal in the sense that if (π', P') is another analytic representation of the flow, then the map

$$P\pi(f)P \longmapsto P'\pi'(f)P', \quad f \in C(X),$$

extends to a C^*-algebra homomorphism from $\mathcal{T}(\pi, P)$ onto $\mathcal{T}(\pi', P')$. It is, therefore, an isomorphism if (π', P') is also non-degenerate.

(ii) In the K-theory six-term exact sequence induced by the short exact sequence

$$0 \longrightarrow \mathcal{C}(\pi, P) \longrightarrow \mathcal{T}(\pi, P) \longrightarrow C(X) \longrightarrow 0,$$

we have

$$K_0\big(\mathcal{T}(\pi, P)\big) = \mathbf{Z}[1] \quad \text{and} \quad K_1\big(\mathcal{T}(\pi, P)\big) = \{0\}.$$

Therefore $K_1\big(C(X)\big) \cong K_0\big(\mathcal{C}(\pi, P)\big)$ and $K_0\big(C(X)\big)/\mathbf{Z}[1] \cong K_1\big(\mathcal{C}(\pi, P)\big)$ via the index map and the exponential map respectively.

If the unique ergodicity is dropped, it is not known whether $\mathcal{T}(\pi, P)$ is universal in general. However we do know the following:

THEOREM 2 [7, SECTIONS 4 AND 8]. *Suppose that $(X, \mathbf{R})$ is a minimal flow and that (π, P) is a non-degenerate analytic representation of $(X, \mathbf{R})$. Then there is a C^*-algebra homomorphism from $\mathcal{T}(\pi, P)$ onto $\mathcal{T}(\pi', P')$ which extends the map $P\pi(f)P \longmapsto P'\pi'(f)P'$, $f \in C(X)$, in the following situations:*

(a) $(\pi', P') = (\pi_x, P_{\mathrm{Hardy}})$ for any $x \in X$.

(b) $(\pi', P') = \Big(\pi_{dm}, e_{dm}\big((\lambda,\infty)\big)\Big)$ for any invariant probability measure dm of the flow and for any $\lambda \in \mathbf{R}$.

(c) $(\pi', P') = \Big(\pi_{dm}, e_{dm}\big([\lambda,\infty)\big)\Big)$ for any invariant probability measure dm of the flow and for any $\lambda \in \mathbf{R}$.

In particular, $\mathcal{T}(\pi_x, P_{\mathrm{Hardy}})$, $\mathcal{T}\Big(\pi_{dm}, e_{dm}\big((\lambda,\infty)\big)\Big)$ and $\mathcal{T}\Big(\pi_{dm}, e_{dm}\big([\lambda,\infty)\big)\Big)$ are naturally isomorphic.

When $(X, \mathbf{R})$ is only assumed to be minimal, K-theory results are conspicuously missing despite of efforts to extend Theorem 1 to this case. A close examination of [5] tells

https://doi.org/10.4153/CMB-1998-034-2 Published online by Cambridge University Press

us that the techniques employed there seem to break down drastically when the unique ergodicity is dropped. The purpose of this note is to show that there is a good reason for this break-down. In fact results stated in Theorem 1(ii) in general do not hold for minimal flows which are not uniquely ergodic. In other words, the unique ergodicity assumption in [5] is not for technical convenience; it is demanded by the results.

More specifically, we will show that $K_1\big(\mathcal{T}(\pi, P)\big) = \{0\}$ is not true in general for minimal flows. Given a pair (π, P), the map

$$\iota_{\mathrm{op}}\colon \varphi \longmapsto \pi(\varphi)P = P\pi(\varphi)P, \quad \varphi \in A(X),$$

is a Banach algebra homomorphism from $A(X)$ into $\mathcal{T}(\pi, P)$. We will present minimal flows for which the map

$$\iota_{\mathrm{op}\,*}\colon K_1\big(A(X)\big) \longrightarrow K_1\big(\mathcal{T}(\pi, P)\big)$$

has a non-trivial image. It follows from the results cited in Theorem 2 and [2, Lemma 24.2] that, if the flow is minimal, then the symbol map $s(P\pi(f)P + B) = f$, $f \in C(X)$, $B \in \mathcal{C}(\pi, P)$, extends to a C^*-algebra homomorphism from $\mathcal{T}(\pi, P)$ onto $C(X)$. If we compose $\iota_{\mathrm{op}\,*}$ with $s_*\colon K_1\big(\mathcal{T}(\pi, P)\big) \longrightarrow K_1\big(C(X)\big)$, then

$$s_* \circ \iota_{\mathrm{op}\,*} = \iota_*,$$

where $\iota\colon A(X) \longrightarrow C(X)$ is the natural inclusion. Thus to produce a minimal flow for which $\iota_{\mathrm{op}\,*}$ has a non-trivial range, we will simply present one for which the natural inclusion map

$$\iota_*\colon K_1\big(A(X)\big) \longrightarrow K_1\big(C(X)\big)$$

has a non-trivial image.

This also has implications at the level of Toeplitz algebras themselves. The failure of Theorem 1(ii) for minimal flows in general alerts us to the possibility that Theorem 1(i) may also fail without unique ergodicity. That is, for a given minimal flow, there might be pairs (π, P) and (π', P') such that neither of $\mathcal{T}(\pi, P)$ and $\mathcal{T}(\pi', P')$ is a homomorphic image of the other. The non-triviality of $K_1\big(\mathcal{T}(\pi, P)\big)$ means that not only is numerical index [3] insufficient for classifying Toeplitz systems which are invertible modulo $\mathcal{C}(\pi, P) \otimes M_n$, even the K-theoretical index map $K_1\big(C(X)\big) \longrightarrow K_0\big(\mathcal{C}(\pi, P)\big)$ will not do the job.

After the above background information, we are ready to present the results of the paper. Denote the Poisson kernel for $\mathbf{R}$ by $P_z(t) = \operatorname{Im} z/\pi|t - z|^2$. For any $f \in C(X)$ and $\xi \in L^1(\mathbf{R})$, denote

$$(f * \xi)(x) = \int_{\mathbf{R}} f(x + t)\xi(t)\, dt,$$

which is a continuous function on X. Recall that if φ, $\psi \in A(X)$, then $(\varphi\psi) * P_z = (\varphi * P_z)(\psi * P_z)$ if $\operatorname{Im} z > 0$.

https://doi.org/10.4153/CMB-1998-034-2 Published online by Cambridge University Press

THEOREM 3. *Suppose that* $(X, \mathbf{R})$ *is a flow such that* $A(X)$ *is a logmodular algebra but not a Dirichlet algebra in* $C(X)$*. That is,* $\{\log|u| : u \in A(X),\ u \text{ is invertible in } A(X)\}$ *is dense in the collection of real-valued functions in* $C(X)$ *and* $A(X)+\overline{A(X)}$ *is not dense in* $C(X)$*. Then there is a* $w \in A(X)$ *with* $1/w \in A(X)$ *such that its* K_1*-class* $[w]$ *in* $K_1\big(C(X)\big)$ *is not trivial.*

PROOF. Since $A(X)+\overline{A(X)}$ is invariant under the multiplication by i, not every real-valued function in $C(X)$ can be approximated by functions in $A(X)+\overline{A(X)}$. Thus there is a $w \in A(X)$ with $1/w \in A(X)$ such that

$$\inf\{\|\log|w| - (\eta+\bar{\zeta})\|_\infty : \eta, \zeta \in A(X)\} = \delta > 0. \tag{1}$$

We will show that $[w] \neq [1]$ in $K_1\big(C(X)\big)$.

Let us first show that $w \notin \exp\big(C(X)\big)$. Since $\lim_{r\downarrow 0}\|w * P_{ir} - w\|_\infty = 0$ and $|w|$ is bounded away from 0, we have $\lim_{r\downarrow 0}\|\log|w * P_{ir}| - \log|w|\|_\infty = 0$. Now if it were true that $w \in \exp\big(C(X)\big)$, then there would be an $r_0 > 0$ such that

$$w * P_{ir_0} = \exp(h) \tag{2}$$

for some $h \in C(X)$ and

$$\|\log|w| - \log|w * P_{ir_0}|\|_\infty \leq \delta/2. \tag{3}$$

For $f \in C(X)$, denote $f_a(x) = f(x+a)$ for $a \in \mathbf{R}$. Then

$$\exp(h_\epsilon - h) - 1 = \big((w * P_{ir_0})_\epsilon - w * P_{ir_0}\big)e^{-h} = w * (P_{\epsilon+ir_0} - P_{ir_0})e^{-h}.$$

That is, for $\epsilon \neq 0$,

$$\frac{1}{\epsilon}(h_\epsilon - h)\Big[1 + \frac{1}{2!}(h_\epsilon - h) + \cdots\Big] = w * \Big[\frac{1}{\epsilon}(P_{\epsilon+ir_0} - P_{ir_0})\Big]\Big/ w * P_{ir_0}.$$

Let $g_0 = dP_{\epsilon+ir_0}/d\epsilon|_{\epsilon=0}$, which obviously belongs to $L^1(\mathbf{R})$. It follows from the above and a routine limit argument that

$$Dh = \frac{w * g_0}{w * P_{ir_0}},$$

where $(Dh)(x) = dh(x+\epsilon)/d\epsilon|_{\epsilon=0}$. Since w is invertible in $A(X)$, $1/(w * P_{ir_0}) = (1/w) * P_{ir_0} \in A(X)$. Hence $Dh \in A(X)$.

Let η be a C^∞-function whose support is a compact subset of $(0,\infty)$ and let $\xi(t) = -i\int_{\mathbf{R}} e^{it\lambda}\lambda^{-1}\eta(\lambda)\,d\lambda$. Then $\xi \in H^1(\mathbf{R})$ and, therefore, $\int_{\mathbf{R}}\xi(t)(Dh)(x+t)\,dt = 0$. Integrating by parts, we have

$$\int_{\mathbf{R}}\xi'(t)h(x+t)\,dt = -\int_{\mathbf{R}}\xi(t)\frac{d}{dt}h(x+t)\,dt = -\int_{\mathbf{R}}\xi(t)(Dh)(x+t)\,dt = 0.$$

Note that functions of the form $\int_{\mathbf{R}} e^{it\lambda}\eta(\lambda)\,d\lambda = \xi'(t)$ are dense in $H^1(\mathbf{R})$. This implies that $h \in A(X)$. It follows from (2) and (3) that $\|\log|w| - (h+\bar{h})/2\|_\infty \leq \delta/2$. But this contradicts (1). Hence $w \notin \exp\big(C(X)\big)$.

To complete the proof that $[w] \neq [1]$ in $K_1\big(C(X)\big)$, it now suffices to recall the well-known fact that $\mathrm{GL}_1\big(C(X)\big)/\exp\big(C(X)\big)$ is injectively embedded in $K_1\big(C(X)\big)$. ∎

Now consider $C_{\mathrm{bu}}(\mathbf{R})$, the collection of bounded, uniformly continuous functions on $\mathbf{R}$. Let $\mathbf{X}$ denote the maximal ideal space of $C_{\mathrm{bu}}(\mathbf{R})$. The translation $f(\cdot) \longmapsto f(\cdot + t)$ of functions in $C_{\mathrm{bu}}(\mathbf{R})$ induces a natural flow $(\mathbf{X}, \mathbf{R})$. According to [9, Proposition 2.3], $A(\mathbf{X})$ is a logmodular algebra. Every closed $\mathbf{R}$-invariant subset $X_0 \subset \mathbf{X}$ gives rise to a subflow $(X_0, \mathbf{R})$. Using Tietze's extension theorem, it is easy to see that $A(X_0)$ is also a logmodular algebra for any such X_0.

It is a highly non-trivial result that if $\mathbf{M}$ is a minimal set in $\mathbf{X}$, then $A(\mathbf{M})$ is not a Dirichlet algebra. See [9, Theorem 4.1] and the remarks following it. Thus Theorem 3 tells us that the natural inclusion map

$$\iota_*: K_1\big(A(\mathbf{M})\big) \longrightarrow K_1\big(C(\mathbf{M})\big)$$

has a non-trivial image.

The space $\mathbf{M}$ need not be metrizable. But from the above we can easily produce a minimal flow $(X, \mathbf{R})$ with a metrizable X such that the image of $\iota_*: K_1\big(A(X)\big) \longrightarrow K_1\big(C(X)\big)$ is non-trivial. Indeed take any invertible $w \in A(\mathbf{M})$ such that $[w] \neq [1]$ in $K_1\big(C(\mathbf{M})\big)$ and let C_w be the C^*-subalgebra of $C(\mathbf{M})$ generated by 1 and the translations $\{w_t : t \in \mathbf{R}\}$ of w. Being a C^*-subalgebra of $C(\mathbf{M})$, C_w necessarily contains the inverse of w. Let X be the maximal ideal space of C_w. Then X is metrizable. As usual, the translation on C_w induces a flow $(X, \mathbf{R})$. We can regard X as the quotient space $\mathbf{M}/\sim$, where $m \sim m'$ if and only if $f(m) = f(m')$ for every $f \in C_w$. Also it is obvious that $[m] + t = [m + t]$ for every $m \in \mathbf{M}$, where $[n]$ denotes the equivalence class of n. Hence every orbit is dense in X. That is, $(X, \mathbf{R})$ is a minimal flow. If $\rho: C_w \longrightarrow C(X)$ is the Gelfand transform, then $\rho(w)$ is an invertible element in $A(X)$. Since C_w is a subalgebra of $C(\mathbf{M})$, $[w] \neq [1]$ in $K_1(C_w)$. This implies $[\rho(w)] \neq [1]$ in $K_1\big(C(X)\big)$.

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https://doi.org/10.4153/CMB-1998-034-2 Published online by Cambridge University Press