

OCCUPATION TIMES, DRAWDOWNS, AND DRAWUPS FOR ONE-DIMENSIONAL REGULAR DIFFUSIONS

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Abstract

The drawdown process of a one-dimensional regular diffusion process X is given by X reflected at its running maximum. The drawup process is given by X reflected at its running minimum. We calculate the probability that a drawdown precedes a drawup in an exponential time-horizon. We then study the law of the occupation times of the drawdown process and the drawup process. These results are applied to address problems in risk analysis and for option pricing of the drawdown process. Finally, we present examples of Brownian motion with drift and three-dimensional Bessel processes, where we prove an identity in law.

Keywords: Drawdown; drawup; diffusion process; occupation time

2010 Mathematics Subject Classification: Primary 60J60
Secondary 60G17

1. Introduction

In a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$, with filtration $\mathbb{F} = \{\mathcal{F}_t\}$, we consider a one-dimensional regular time-homogenous diffusion X on $I := (l, \infty)$ with natural (or entrance) boundaries. Its evolution is governed by the stochastic differential equation

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t, \quad X_0 = x \in I, \quad (1)$$

with an infinitesimal generator

$$\mathcal{L}_X = \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} + \mu(x) \frac{\partial}{\partial x}.$$

Here B is a standard \mathbb{F} -Brownian motion, and $(\mu(\cdot), \sigma(\cdot))$ is a pair of real-valued continuous functions, such that $\sigma(x) > 0$ for all $x \in I$. We introduce the running maximum and minimum processes of X by

$$\overline{X}_t := \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad \underline{X}_t := \inf_{0 \leq s \leq t} X_s, \quad t \geq 0.$$

The drawdown process of X , and its dual, the drawup process, are then defined as $Y := \overline{X} - X$ and $\hat{Y} := X - \underline{X}$, respectively. The first passage times of Y and \hat{Y} above a positive threshold are respectively called the drawdown and drawup times. The occupation time of a stochastic process is the amount of time the stochastic process stays within a certain range.

Received 17 September 2013; revision received 9 February 2014.

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Laplace transforms of stopping times and occupation times for diffusion processes are well studied due to analytical tractability and their applications in risk theory, mathematical finance, and engineering. Some classical results for general diffusion processes using the Feynman–Kac representation and excursion theory can be found in [22], [23]. Recent advances on this topic include jumps in the underlying model [3], [12], [13], or incorporate memory into the model [7]. Among all the stopping times studied, one that is especially interesting is the drawdown time, which finds applications in financial risk management [4], [9], [16], [26], [33], the theory of optimal stopping [17], [21], [30] and the problem of the quickest change-point detection [11], [24], [31], [38], [40]. The Laplace transform of the drawdown time for general diffusion processes was first derived in [14]. Recently, [29] derived the probability that the minimum of the drawdown and drawup times precedes an exponential random variable for a standard Brownian motion. In [37], the authors derived the joint Laplace transform of the drawdown time and the so-called speed of market crash for a general diffusion process, using progressive enlargement of filtrations. Laplace transforms involving drawdowns for general spectrally negative Lévy processes are studied, using excursion theory in [18], which generalizes a result for diffusion-type processes in [10], [27], [35], [36].

In this paper we obtain a class of results regarding the sequential order of the drawdown and drawup times in a finite time horizon, and laws of occupation times of the drawdown process Y and the drawup process \hat{Y} , for a general one-dimensional regular diffusion X . In particular, we derive the probability that the drawdown time precedes the drawup time in an exponential time horizon. We then compute the Laplace transforms of the occupation times of X below y and below the starting point, until the first exit time and the drawdown time, respectively. Using these results, we proceed to study the law of occupation time of the drawdown process Y above y , and of the drawup process \hat{Y} below y , until the drawdown time or an exponential time independent of X .

Rather than relying on standard techniques in excursion theory, we use a perturbation approach to obtain the Laplace transform results. To the author's knowledge, the first use of this approach dates back to [14], where the conditional Laplace transform of the drawdown time is obtained by an approximation argument based on the Laplace transforms for the first hitting times. Recent applications of this approach can be seen in [12], [13] for (refracted) Lévy process and [15] for diffusion process. In this paper we subsequently reduce the laws of complicated stopping times and occupation times at hand to those of the simpler case. This reduction is made possible using an argument with a strong Markov property and progressive enlargement of filtration [28], [37].

The results obtained can be applied in risk analysis and for option pricing of the drawdown processes. In particular, we consider a time-homogenous diffusion with reduced form default model. The probability of realizing a drawdown before a drawup before the default time can be computed when the hazard rate is a constant. Using the so-called Omega model [1], [8], we can describe the hazard rate of the default in such a way that it depends on the asset process, its drawdown, or drawup processes. The probability of default before a large drawdown can then be computed. Moreover, our results can be used to price Parisian-like digital call options and α -quantile options of the drawdown process, a non-trivial extension of the option pricing problem for maximum drawdowns [4], [25], [32], [33], [39].

As examples of our general result, we present explicit equations for some of the main results in the cases of Brownian motions with drift and three-dimensional Bessel processes. Moreover, we prove through Laplace transform that, in these two models, the law of the occupation time of the drawdown process above a level is the same as that of the drawdown time of a certain threshold.

The paper is structured in the following way: standard identities regarding the first hitting time and the first exit time of a one-dimensional regular diffusion are reviewed in Section 2. In Section 3 we compute the probability that a drawdown precedes a drawup in a finite time-horizon. In Section 4 we derive the Laplace transforms of several occupation times related to the drawdown and drawup processes. In Section 5 we discuss applications in risk analysis and for option pricing of the results obtained. In Section 6 we present explicit equations for some of the main results in the cases of Brownian motion with drift and three-dimensional Bessel processes. The proofs of omitted propositions and theorems can be found in Appendix A.

2. Preliminaries

Let X be the linear diffusion process defined in (1). The first hitting time and the first passage times to a level $y \in I$ by X are, respectively, given by

$$\tau_y := \inf\{t > 0: X_t = y\}, \quad \tau_y^\pm := \inf\{t > 0: X_t \begin{matrix} \geq \\ \leq \end{matrix} y\}.$$

Throughout the paper we use $\mathbb{P}_x(\cdot)$ to denote the measure $\mathbb{P}(\cdot \mid X_0 = x)$, and \mathbb{E}_x the expectation under \mathbb{P}_x . Moreover, for $q \geq 0$, we denote by e_q an exponential random variable with parameter q , which is independent of X . (As a convention, we assume that $\mathbb{P}(e_0 = \infty) = 1$.)

It is well known that, for any $q > 0$, the *Sturm-Liouville equation* $(\mathcal{L}_X f)(x) = qf(x)$ has a positive increasing solution $\phi_q^+(\cdot)$ (decreasing solution $\phi_q^-(\cdot)$, respectively). In fact, for an arbitrary fixed $\kappa \in I$, we can choose

$$\phi_q^+(x) = \begin{cases} \mathbb{E}_x\{e^{-q\tau_\kappa}\} & \text{if } x \leq \kappa, \\ \frac{1}{\mathbb{E}_\kappa\{e^{-q\tau_x}\}} & \text{if } x > \kappa, \end{cases} \quad \phi_q^-(x) = \begin{cases} \frac{1}{\mathbb{E}_\kappa\{e^{-q\tau_x}\}} & \text{if } x \leq \kappa, \\ \mathbb{E}_x\{e^{-q\tau_\kappa}\} & \text{if } x > \kappa, \end{cases} \tag{2}$$

for all $x \in I$. A scale function of X , $s(\cdot)$, is an increasing function from I to \mathbb{R} , such that $(\mathcal{L}_X s)(x) = 0$ for all $x \in I$. In particular, we can choose $s'(x) = \exp(-\int_{\kappa'}^x (2\mu(y)/\sigma^2(y)) dy)$ for some $\kappa' \in I$. Fix a scale function $s(\cdot)$, there exists a constant $w_q > 0$ such that (see, for example, page 19 of [2])

$$w_q \cdot s'(x) = (\phi_q^+)'(x)\phi_q^-(x) - (\phi_q^-)'(x)\phi_q^+(x). \tag{3}$$

Furthermore, we define the function

$$W_q(x, y) := w_q^{-1} \det \begin{bmatrix} \phi_q^+(x) & \phi_q^+(y) \\ \phi_q^-(x) & \phi_q^-(y) \end{bmatrix} \quad \text{for all } x, y \in I,$$

with its derivatives

$$W_{q,1}(x, y) := \frac{\partial}{\partial x} W_q(x, y), \quad W_{q,2}(x, y) := \frac{\partial}{\partial y} W_q(x, y).$$

When $q = 0$, we extend the definition of W_q using

$$W_0(x, y) := s(x) - s(y). \tag{4}$$

The functions W_q and ϕ_q^- have the following properties.

Lemma 1. For any $x, y, z \in I$, and $q > 0$

$$W_q(x, y) = -W_q(y, x), \quad \frac{\partial}{\partial x} \frac{W_q(x, y)}{W_q(x, z)} = \frac{s'(x)}{W_q^2(x, z)} W_q(y, z),$$

$$\lim_{x \downarrow l} \phi_q^-(x) = \infty, \quad \lim_{q \downarrow 0} W_q(x, y) = W_0(x, y), \quad \lim_{q \downarrow 0} W_{q,1}(x, y) = s'(x).$$

Proof. The proof can be found in Appendix A.

We recall the following result regarding the first exit of X from page 603 of [14].

Lemma 2. Suppose that $x, y, z \in I$ and $(x - y)(z - x) > 0$, for $q \geq 0$, we have

$$\mathbb{E}_x \{e^{-q\tau_y}; \tau_y < \tau_z\} = \mathbb{P}_x(\tau_y < \tau_z \wedge e_q) = \frac{W_q(x, z)}{W_q(y, z)}.$$

3. Drawdowns and drawups

In this section we study the law of drawdowns and drawups of X , where X is the linear diffusion process defined in (1). To this end, we introduce the running maximum and minimum processes of X by

$$\bar{X}_t := \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad \underline{X}_t := \inf_{0 \leq s \leq t} X_s, \quad t \geq 0.$$

The drawdown and drawup processes of X are defined respectively as

$$Y_t := \bar{X}_t - X_t \quad \text{and} \quad \hat{Y}_t := X_t - \underline{X}_t, \quad t \geq 0,$$

and the drawdown of a units and the drawup of b units are the following first passage times:

$$\sigma_a := \inf\{t \geq 0, : Y_t \geq a\}, \quad \hat{\sigma}_b := \inf\{t \geq 0: \hat{Y}_t \geq b\}, \quad \text{for all } a, b > 0.$$

In the remainder of this section we will derive the probability that σ_a precedes $\hat{\sigma}_b$ in a finite time interval $[0, t]$ for any $t > 0$. This is a nontrivial extension of the infinite time-horizon result in [27], which enables us to study the drawdowns and drawups for diffusions with killing, as well as limiting conditional distributions of σ_a as the thresholds a, b tend to infinity in a proper manner. The task is accomplished by computing the Laplace transform of the function $p_{x;a,b}(t) := \mathbb{P}_x(\sigma_a < \hat{\sigma}_b \wedge t)$. In particular, for any x, a such that $x - a \in I$, we will calculate

$$\mathbb{P}_x(\sigma_a < \hat{\sigma}_b \wedge e_q) = q \int_0^\infty e^{-qt} \mathbb{P}_x(\sigma_a < \hat{\sigma}_b \wedge t) dt.$$

3.1. The case of $a \geq b > 0$

Theorem 1. On the event $\{\sigma_a < \hat{\sigma}_b \wedge e_q\}$, we have $\sigma_b < \hat{\sigma}_b$ and $X_{\sigma_b} \in (x - b, x)$. Moreover, for $u \in (x - b, x)$ we have,

$$\begin{aligned} &\mathbb{P}_x(\sigma_a < \hat{\sigma}_b \wedge e_q, X_{\sigma_a} \in b - a + du) \\ &= \frac{s'(u + b)W_q(x, u)}{W_q^2(u + b, u)} \exp\left(\int_{u+b-a}^u \frac{W_{q,1}(v, v + b)}{W_q(v, v + b)} dv\right) du. \end{aligned} \tag{5}$$

On the event $\{\hat{\sigma}_a < \sigma_b \wedge e_q\}$, we have $\hat{\sigma}_b < \sigma_b$ and $X_{\hat{\sigma}_b} \in (x, x+b)$. Moreover, for $u \in (x, x+b)$ we have,

$$\begin{aligned} & \mathbb{P}_x(\hat{\sigma}_a < \sigma_b \wedge e_q, X_{\hat{\sigma}_a} \in a - b + du) \\ &= \frac{s'(u - b)W_q(u, x)}{W_q^2(u, u - b)} \exp\left(-\int_u^{u+a-b} \frac{W_{q,1}(v, v - b)}{W_q(v, v - b)} dv\right) du. \end{aligned} \tag{6}$$

Proof. First, it is easily seen that for $t > 0$, and $u \in (x - b, x)$,

$$\{\sigma_b \in dt, X_t \in du, \hat{\sigma}_b > t\} = \{\tau_u \in dt, \bar{X}_t \in b + du\}, \quad \mathbb{P}_x\text{-almost surely } (\mathbb{P}_x\text{-a.s.}).$$

It follows from integration by parts that,

$$\begin{aligned} \mathbb{P}_x(\sigma_b < \hat{\sigma}_b \wedge e_q; X_{\sigma_b} \in du) &= \int_0^\infty qe^{-qt} \mathbb{P}_x(\sigma_b < \hat{\sigma}_b \wedge t, X_{\sigma_b} \in du) dt \\ &= \int_0^\infty e^{-qt} \mathbb{P}_x(\sigma_b \in dt, X_t \in du, \hat{\sigma}_b > t). \end{aligned}$$

On the other hand, we observe the fact that $\sigma_b \wedge \hat{\sigma}_b = \inf\{t \geq 0: \bar{X}_t - \underline{X}_t \geq b\}$, \mathbb{P}_x -a.s., is the first range time. From page 117 of [2], we have,

$$\begin{aligned} \mathbb{P}_x(\sigma_b \in dt, X_t \in du, \hat{\sigma}_b > t) &= \frac{\partial}{\partial u} \mathbb{P}_x\{\sigma_b \wedge \hat{\sigma}_b \in dt, X_t \leq u\} du \\ &= \frac{\partial}{\partial b} \mathbb{P}_x(\tau_u \in dt, \tau_{u+b} > t) du. \end{aligned}$$

From Lemmas 1 and 2 we obtain (5) for the case $a = b$.

If $a > b$, then any path in the event $\{\sigma_a < \hat{\sigma}_b \wedge e_q\}$ can be decomposed into two fragments: $\{X_t\}_{0 \leq t \leq \sigma_b}$ and $\{X_t\}_{\sigma_b \leq t \leq \sigma_a}$. Conditioning on $\{X_{\sigma_b} = u\}$, the second fragment is a process starting at u , and decreasing to $u + b - a$ before the drawup of b units. Formally, using the Markov shifting operator $(X_t \circ \theta(s) = X_{t+s})$, we have,

$$\sigma_a = \sigma_b + \tau_{u-a+b}^- \circ \theta(\sigma_b) \quad \text{for all } u \in (x - b, x).$$

Using the strong Markov property and the memoryless property of e_q , we have,

$$\begin{aligned} & \mathbb{P}_x(\sigma_a < \hat{\sigma}_b \wedge e_q, X_{\sigma_a} \in b - a + du) \\ &= \mathbb{P}_x\{\sigma_b < \hat{\sigma}_b \wedge e_q, X_{\sigma_b} \in du\} \cdot \mathbb{P}_u(\tau_{u-a+b}^- < \hat{\sigma}_b \wedge e_q). \end{aligned}$$

Now (5) follows from Proposition 1 below. Equation (6) can be proved using a similar argument.

Proposition 1. For $x, y \in I$, we define $m = \max(x, y)$ and $n = \min(x, y)$. Then we have

$$\mathbb{P}_m(\tau_n^- < \hat{\sigma}_b \wedge e_q) = \exp\left(\int_n^m \frac{W_{q,1}(v, v + b)}{W_q(v, v + b)} dv\right), \tag{7}$$

$$\mathbb{P}_n(\tau_m^+ < \sigma_a \wedge e_q) = \exp\left(-\int_n^m \frac{W_{q,1}(v, v - a)}{W_q(v, v - a)} dv\right). \tag{8}$$

3.2. The case of $b > a > 0$

To obtain the result for general $b > a$, we recall the following result for the limiting case $b = \infty$ from page 602 of [14].

Proposition 2. *It holds that*

$$\mathbb{P}_x(\sigma_a < e_q) = \int_x^\infty \frac{s'(u)}{W_q(u, u - a)} \exp\left(-\int_x^u \frac{W_{q,1}(v, v - a)}{W_q(v, v - a)} dv\right) du.$$

A useful observation is that

$$\mathbb{P}_x(\sigma_a < \hat{\sigma}_b \wedge e_q) = \mathbb{P}_x(\sigma_a < e_q) - \mathbb{P}_x(\hat{\sigma}_b < \sigma_a < e_q). \tag{9}$$

The second term on the right-hand side of the above equation is computed in the following result.

Theorem 2. *On the event $\{\hat{\sigma}_b < \sigma_a < e_q\}$, $X_{\hat{\sigma}_b} \in (x + b - a, x + b)$. Moreover, for $u \in (x, x + a)$ we have*

$$\begin{aligned} & \frac{\mathbb{P}_x(\hat{\sigma}_b < \sigma_a < e_q, X_{\hat{\sigma}_b} \in b - a + du)}{du} \\ &= \frac{s'(u - a)W_q(u, x)}{W_q^2(u, u - a)} \exp\left(-\int_u^{u+b-a} \frac{W_{q,1}(v, v - a)}{W_q(v, v - a)} dv\right) \\ & \times \int_{u+b-a}^\infty \frac{s'(v)}{W_q(v, v - a)} \exp\left(-\int_{u+b-a}^v \frac{W_{q,1}(w, w - a)}{W_q(w, w - a)} dw\right) dv. \end{aligned} \tag{10}$$

Proof. Note that the event $\{\hat{\sigma}_b < \sigma_a < e_q\} = \{\hat{\sigma}_b < \sigma_a \wedge e_q\} \cap \{\sigma_a < e_q\}$, \mathbb{P}_x -a.s. Using the strong Markov property and the memoryless property of e_q , we have

$$\begin{aligned} & \mathbb{P}_x(\hat{\sigma}_b < \sigma_a < e_q, X_{\hat{\sigma}_b} \in b - a + du) \\ &= \mathbb{P}_x(\hat{\sigma}_b < \sigma_a \wedge e_q, X_{\hat{\sigma}_b} \in b - a + du) \cdot \mathbb{P}_{u+b-a}(\sigma_a < e_q). \end{aligned}$$

The result follows from Theorem 1 and Proposition 2.

Remark 1. By letting $q \downarrow 0$ in (5), (6), (9), and (10), and using (4), we obtain Theorems 4.1 and 4.2 in [27]. Second, by letting $b \rightarrow \infty$ in (9) and (10), we obtain the Laplace transform of σ_a . The results in Theorems 1 and 2 also enable us to compute $\mathbb{E}_x\{\sigma_a \mid \sigma_a < \hat{\sigma}_b\}$ and the limit law of σ_a , given $\sigma_a < \hat{\sigma}_b$, as $a, b \rightarrow \infty$ in a certain way. These distributional results can be applied to the problem of sequential detections and identification, which we leave as future work.

4. Occupation time equations

In this section we begin by computing the Laplace transforms of the occupation time below a level until the first exit time for a linear diffusion process X defined in (1). Although relevant equations exist for special diffusions, the results for general linear diffusions that we provide here are new. Using this result, we then proceed to study occupation times of X , its drawdown Y , and its drawup \hat{Y} until the drawdown time σ_a or until an exponential time which is independent of X . These new results give several interesting identities and provide means of measuring financial risk and pricing options as we shall in the next section.

4.1. Occupation time below a level until the first exit time

In this subsection we study the law of occupation time until the first exit time. In particular, for $y \in (a, b) \subsetneq I$, the occupation time below y before exiting (a, b) is denoted by

$$A_y^{a,b} := \int_0^{\tau_a^- \wedge \tau_b^+} \mathbf{1}_{\{X_t < y\}} dt.$$

The law of $A_y^{a,b}$ is summarized in the following result, the proof of which can be found in Appendix A.

Proposition 3. For $x, y \in (a, b) \subsetneq I$, $q > 0$, and $p \geq 0$, we have

$$\begin{aligned} & \mathbb{E}_x \{ \exp(-qA_y^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^- \} \\ &= \frac{W_{q+p}(x, a)}{W_{q+p}(y, a)} \\ & \quad \times \frac{s'(y)}{W_{p,1}(y, b) + W_p(b, y)(W_{q+p,1}(y, a)/W_{q+p}(y, a))} \quad \text{if } x \in (a, y], \\ &= \frac{W_p(x, y)}{W_p(b, y)} + \left[\frac{W_p(b, x)}{W_p(b, y)} \right. \\ & \quad \left. \times \frac{s'(y)}{W_{p,1}(y, b) + W_p(b, y)(W_{q+p,1}(y, a)/W_{q+p}(y, a))} \right] \quad \text{if } x \in (y, b). \end{aligned} \tag{11}$$

and

$$\begin{aligned} & \mathbb{E}_x \{ \exp(-qA_y^{a,b}); \tau_b^+ > \tau_a^- \} \\ &= \frac{W_q(y, x)}{W_q(y, a)} \\ & \quad + \left[\frac{W_q(x, a)}{W_q(y, a)} \right. \\ & \quad \left. \times \frac{s'(y)}{W_{q,1}(y, a) + (s'(y))/(s(b) - s(y))W_q(y, a)} \right] \quad \text{if } x \in (a, y], \\ &= \frac{(s(b) - s(x))s'(y)}{(s(b) - s(y))W_{q,1}(y, a) + s'(y)W_q(y, a)} \quad \text{if } x \in (y, b). \end{aligned} \tag{12}$$

Proof. The proof can be found in Appendix A.

Remark 2. Lemma 2 can be recovered from Proposition 3 by the limit $y \uparrow b$ in (11) and (12). Moreover, Corollary 3.1 in [15] can be easily obtained by letting $x = y$, $b \uparrow \infty$, and $a \downarrow l$.

Letting $a \downarrow l$ in (11) and using Lemma 1, we obtain the following result.

Corollary 1. For $x \in (y, b) \subsetneq I$, $p > 0$, and $q \geq 0$, we have

$$\begin{aligned} & \mathbb{E}_x \left\{ \exp \left(-q \int_0^{\tau_b^+} \mathbf{1}_{\{X_t < y\}} dt - p\tau_b^+ \right) \right\} \\ &= \frac{W_p(x, y)}{W_p(b, y)} + \frac{(W_p(b, x)/W_p(b, y))s'(y)}{W_{p,1}(y, b) + W_p(b, y)(\phi_{q+p}^+(y)/\phi_{q+p}^+(y))}. \end{aligned}$$

4.2. Occupation time below a level until the drawdown time

For $x - a, y \in I$, the occupation time below y before the drawdown process Y hits a is denoted by

$$B_y^a := \int_0^{\sigma_a} \mathbf{1}_{\{X_t < y\}} dt.$$

While the occupation time of X below y until an exponential random variable independent of X is well-studied in the literature [15], the new quantity B_y^a defined as above relates the occupation time of X to its drawdown process Y , which can be used to characterize the drawdown risk of X . The law of B_y^a is summarized in the following results, the proofs of which can be found in Appendix A.

Proposition 4. For $q \geq 0$ and $x \in I$,

$$\begin{aligned} & \mathbb{E}_x \{ \exp(-q B_x^a) \} \\ &= \exp \left(- \int_x^{x+a} \frac{(s'(u)/s'(x))(W_{q,1}(x, u - a)/W_q(x, u - a)) du}{1 + ((s(u) - s(x))/s'(x))(W_{q,1}(x, u - a)/W_q(x, u - a))} \right) \\ &+ \int_x^{x+a} \frac{1}{(1 + ((s(u) - s(x))/s'(x))(W_{q,1}(x, u - a)/W_q(x, u - a)))^2} \frac{s'(u) du}{W_q(x, u - a)}. \end{aligned}$$

Proof. The proof can be found in Appendix A.

4.3. Occupation time of the drawdown process until the drawdown time

For any $y \in (0, a)$ and $x - a \in I$, the occupation time of the drawdown process Y above y before Y hits a is denoted by

$$C_y^a := \int_0^{\sigma_a} \mathbf{1}_{\{Y_t > y\}} dt. \tag{13}$$

The occupation time C_y^a measures the amount of time for the drawdown process Y to finish the ‘last trip’ from y to a . It can be used as a measurement of performance for the CUSUM-type (Cumulative sum control chart) stopping rule in change-point detection problems [24]. Because of its obvious financial interpretation, C_y^a can also be used as a measure for drawdown risks.

To obtain the law of C_y^a , we first condition on \bar{X}_{σ_a} , and then count separately the occupation time before and after the moment when the peak \bar{X}_{σ_a} is realized. To this end, we recall the following result from page 602 of [14].

Lemma 3. For $m > x$,

$$\mathbb{P}_x(\bar{X}_{\sigma_a} \geq m) = \exp \left(- \int_x^m \frac{s'(v)}{s(v) - s(v - a)} dv \right).$$

Theorem 3. For $q \geq 0$, and $0 < y < a$,

$$\begin{aligned} & \mathbb{E}_x \{ \exp(-q C_y^a); \sigma_a < \infty \} \\ &= \int_x^\infty \frac{(s'(m)/W_q(m - y, m - a))}{1 + ((s(m) - s(m - y))/s'(m - y))(W_{q,1}(m - y, m - a)/W_q(m - y, m - a))} \\ &\times \exp \left(- \int_x^m \frac{(s'(u)/s'(u - y))(W_{q,1}(u - y, u - a)/W_q(u - y, u - a)) du}{1 + ((s(u) - s(u - y))/s'(u - y))(W_{q,1}(u - y, u - a)/W_q(u - y, u - a))} \right) dm. \end{aligned}$$

Proof. Following the conditioning argument in [37], we consider the last passage time $g_a := \sup\{t \leq \sigma_a : X_t = \bar{X}_t\}$. Given X_{g_a} , the path fragments $\{X_t\}_{t \in [0, g_a]}$ and $\{X_t\}_{t \in [g_a, \sigma_a]}$ are two independent conditional processes. Moreover, by Proposition 1 of [37], the optional projection of nonincreasing process $\mathbf{1}_{\{g_a > t\}} : \chi_t := \mathbb{P}_x(g_a > t \mid \mathcal{F}_t)$ is a supermartingale, with a Doob–Meyer decomposition $\chi_t = M_t - L_t$, where

$$M_t = 1 + \int_0^{t \wedge \sigma_a} \frac{s'(X_s)\sigma(X_s) dB_s}{s(\bar{X}_s) - s(\bar{X}_s - a)}, \quad L_t = \int_0^{t \wedge \sigma_a} \frac{s'(\bar{X}_s) d\bar{X}_s}{s(\bar{X}_s) - s(\bar{X}_s - a)}.$$

Now introduce a nonnegative bounded optional process

$$\Gamma_t = \exp\left(-q \int_0^t \mathbf{1}_{\{Y_s > y\}} ds\right) \mathbf{1}_{\{t < \sigma_a < \infty\}}, \quad t \geq 0.$$

Using the same argument as in the proof of Theorem 15 on page 380 of [28], we have that, for any positive test function $f(\cdot)$ on $[0, \infty)$,

$$\mathbb{E}_x\{f(X_{g_a})\Gamma_{g_a}\} = \mathbb{E}_x\left\{\int_0^\infty f(X_t)\Gamma_t dL_t\right\} = \mathbb{E}_x\left\{\int_0^\infty \frac{f(X_t)\Gamma_t s'(\bar{X}_t) d\bar{X}_t}{s(\bar{X}_t) - s(\bar{X}_t - a)}\right\}.$$

By using a change of variable, $m = \bar{X}_t$ in the above equation, and also the fact that $X_t = \bar{X}_t$ on the support of measure $d\bar{X}_t$, we have that,

$$\mathbb{E}_x\{f(X_{g_a})\Gamma_{g_a}\} = \int_x^\infty f(m)\mathbb{E}_x\left\{\exp\left(-q \int_0^{\tau_m^+} \mathbf{1}_{\{Y_t > y\}} dt\right) \mathbf{1}_{\{\tau_m^+ < \sigma_a\}}\right\} \frac{s'(m) dm}{s(m) - s(m - a)}. \tag{14}$$

On the other hand, from Lemma 3 we have that, for all $u > x$,

$$\mathbb{P}_x(\tau_m^+ < \sigma_a) = \exp\left(-\int_x^m \frac{s'(v)}{s(v) - s(v - a)} dv\right), \tag{15}$$

$$\mathbb{P}_x(X_{g_a} \in dm) = \frac{s'(m)}{s(m) - s(m - a)} \exp\left(-\int_x^m \frac{s'(v)}{s(v) - s(v - a)} dv\right). \tag{16}$$

From (14) and (15) we have,

$$\begin{aligned} &\mathbb{E}_x\{f(X_{g_a})\Gamma_{g_a}\} \\ &= \int_x^\infty f(m)\mathbb{E}_x\left\{\exp\left(-q \int_0^{\tau_m^+} \mathbf{1}_{\{Y_t > y\}} dt\right) \middle| \tau_m^+ < \sigma_a\right\} \\ &\quad \times \frac{s'(m)}{s(m) - s(m - a)} \exp\left(-\int_x^m \frac{s'(v)}{s(v) - s(v - a)} dv\right) dm \\ &= \int_x^\infty f(m)\mathbb{E}_x\left\{\exp\left(-q \int_0^{\tau_m^+} \mathbf{1}_{\{Y_t > y\}} dt\right) \middle| \tau_m^+ < \sigma_a\right\} \mathbb{P}_x(X_{g_a} \in dm), \end{aligned} \tag{17}$$

where the last line follows from (16). It follows from (17) that,

$$\mathbb{E}_x\left\{\exp\left(-q \int_0^{\tau_m^+} \mathbf{1}_{\{Y_t > y\}} dt\right) \middle| \tau_m^+ < \sigma_a\right\} = \mathbb{E}_x\left\{\exp\left(-q \int_0^{g_a} \mathbf{1}_{\{Y_t > y\}} dt\right) \middle| X_{g_a} = m\right\}.$$

To obtain the conditional expectation on the left-hand side for any $m > x$, we let $\varepsilon = (m - x)/N$ for a large integer $N > 0$. Using the Lebesgue dominated convergence theorem, continuity and the strong Markov property of X , we have that

$$\begin{aligned} & \mathbb{E}_x \left\{ \exp \left(-q \int_0^{\tau_m} \mathbf{1}_{\{Y_t > y\}} dt \right) \middle| \sigma_a > \tau_m^+ \right\} \\ &= \lim_{N \rightarrow \infty} \prod_{i=0}^{N-1} \mathbb{E}_{x+i\varepsilon} \{ \exp(-q A_{x+i\varepsilon-y}^{x+i\varepsilon-a, x+(i+1)\varepsilon}) \mid \tau_{x+(i+1)\varepsilon}^+ < \tau_{x+i\varepsilon-a}^- \} \\ &= \exp \left(\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} [\mathbb{E}_{x+i\varepsilon} \{ \exp(-q A_{x+i\varepsilon-y}^{x+i\varepsilon-a, x+(i+1)\varepsilon}) \mid \tau_{x+(i+1)\varepsilon}^+ < \tau_{x+i\varepsilon-a}^- \} - 1] \right) \\ &= \exp \left(\int_x^m \left[\frac{s'(u)}{s(u) - s(u-a)} \right. \right. \\ &\quad \left. \left. - \frac{(s'(u)/s'(u-y))(W_{q,1}(u-y, u-a)/W_q(u-y, u-a))}{1 + ((s(u) - s(u-y))/s'(u-y))(W_{q,1}(u-y, u-a)/W_q(u-y, u-a))} \right] du \right), \end{aligned}$$

where we used Proposition 3 in the last equality. Similarly, for the occupation time after the random time g_a , we have that

$$\begin{aligned} & \mathbb{E}_x \left\{ \exp \left(-q \int_{g_a}^{\sigma_a} \mathbf{1}_{\{Y_t > y\}} dt \right) \middle| X_{g_a} = m \right\} \\ &= \mathbb{E}_m \left\{ \exp \left(-q \int_0^{\tau_{m-a}^-} \mathbf{1}_{\{X_t < m-y\}} dt \right) \middle| \tau_{m-a}^- < \tau_m^+ \right\} \\ &= \lim_{\varepsilon' \downarrow 0} \frac{\mathbb{E}_m \{ \exp(-q A_{m-y}^{m-a, m+\varepsilon'}) ; \tau_{m-a}^- < \tau_{m+\varepsilon'}^+ \}}{\mathbb{P}_m(\tau_{m-a}^- < \tau_{m+\varepsilon'}^+)} \\ &= \frac{((s(m) - s(m-a))/W_q(m-y, m-a))}{1 + ((s(m) - s(m-y))/s'(m-y))(W_{q,1}(m-y, m-a)/W_q(m-y, m-a))}. \end{aligned}$$

The result now follows from integration using the density in (16).

4.4. Occupation time of the drawup process until the drawdown time

For any $y \in [a, \infty)$ and $x - a \in I$, the occupation time of the drawup process \hat{Y} below y before the drawdown process Y hits a is denoted by

$$D_y^a := \int_0^{\sigma_a} \mathbf{1}_{\{\hat{Y}_t < y\}} dt.$$

The occupation time D_y^a can be considered as a counterpart of C_y^a defined in (13). The law of D_y^a is summarized in the following result.

Theorem 4. For all $q > 0$ and $y \geq a$,

$$\begin{aligned} & \mathbb{E}_x\{\exp(-qD_y^a); \sigma_a < \infty\} \\ &= \mathbb{P}_x(\sigma_a < \hat{\sigma}_y \wedge \mathbf{e}_q) \\ &+ \int_x^{x+a} \frac{s'(u-a)W_q(u,x)}{W_q^2(u,u-a)} \\ &\quad \times \exp\left(-\int_u^{u+y-a} \frac{W_{q,1}(v,v-a)}{W_q(v,v-a)} dv\right) \mathbb{E}_{u+y-a}\{\exp(-qB_{u+y-a}^a)\} du, \end{aligned}$$

where the probability is given in (9), and the expectation in the last line is given in Proposition 4.

Proof. Note that $D_y^a = \sigma_a, \mathbb{P}_x$ -a.s. on the event $\{\sigma_a < \hat{\sigma}_y\}$. On the other hand, on the event $\{\hat{\sigma}_y < \sigma_a\}$, we have $X_{\hat{\sigma}_y} = X_{\hat{\sigma}_y} - y \leq X_{\hat{\sigma}_y} - a < \bar{X}_{\sigma_a} - a, \mathbb{P}_x$ -a.s. Thus,

$$D_y^a = \hat{\sigma}_y + B_{X_{\hat{\sigma}_y}}^a \circ \theta(\hat{\sigma}_y).$$

Using the strong Markov property we have,

$$\begin{aligned} & \mathbb{E}_x\{\exp(-qD_y^a); \sigma_a < \infty\} \\ &= \mathbb{E}_x\{\exp(-q\sigma_a); \sigma_a < \hat{\sigma}_y\} + \mathbb{E}_x\{\exp(-q\hat{\sigma}_y) \mathbf{1}_{\{\hat{\sigma}_y < \sigma_a\}} \mathbb{E}_{X_{\hat{\sigma}_y}}\{\exp(-qB_{X_{\hat{\sigma}_y}}^a)\}\} \\ &= \mathbb{P}_x(\sigma_a < \hat{\sigma}_y \wedge \mathbf{e}_q) \\ &+ \int_x^{x+a} \mathbb{P}_x(\hat{\sigma}_y < \sigma_a \wedge \mathbf{e}_q, X_{\hat{\sigma}_y} \in y - a + du) \mathbb{E}_{u-a+y}\{\exp(-qB_{u-a+y}^a)\}. \end{aligned}$$

The result now follows from Theorem 1.

4.5. Occupation time of the drawdown process at an independent exponential time

For any $y \in (0, \infty)$ and $x - y \in I$, the occupation time of the drawdown process Y above y before an independent exponential time \mathbf{e}_q is denoted by

$$E_y^q := \int_0^{\mathbf{e}_q} \mathbf{1}_{\{Y_t > y\}} dt.$$

The occupation time E_y^q can be identified with the Laplace transform of the occupation time in a finite time horizon $T > 0: \int_0^T \mathbf{1}_{\{Y_t > y\}} dt$, which is closely related to the maximum drawdown $\sup_{t \in [0, T]} Y_t$ and other quantiles of Y in the finite time-horizon T . The law of E_y^q is summarized in the following result, the proof of which can be found in Appendix A.

Theorem 5. For all $q, p > 0$ and $y > 0$,

$$\begin{aligned} & \mathbb{E}_x\{\exp(-pE_y^q)\} \\ &= 1 - \exp\left(-\int_x^\infty \frac{W_{q,2}(u-y,u) + W_{q,1}(u,u-y)(\phi_{q+p}^+(u-y)/\phi_{q+p}^+(u-y))}{W_{q,1}(u-y,u) + W_q(u,u-y)(\phi_{q+p}^+(u-y)/\phi_{q+p}^+(u-y))} du\right) \\ &- \int_x^\infty \exp\left(-\int_x^m \frac{W_{q,2}(u-y,u) + W_{q,1}(u,u-y)(\phi_{q+p}^+(u-y)/\phi_{q+p}^+(u-y))}{W_{q,1}(u-y,u) + W_q(u,u-y)(\phi_{q+p}^+(u-y)/\phi_{q+p}^+(u-y))} du\right) \\ &\quad \times \frac{(p/(q+p))s'(m)(\phi_{q+p}^+(m-y)/\phi_{q+p}^+(m-y))}{W_{q,1}(m-y,m) + W_q(m,m-y)(\phi_{q+p}^+(m-y)/\phi_{q+p}^+(m-y))} dm. \end{aligned}$$

Proof. The proof can be found in Appendix A.

5. Application

5.1. Probabilities regarding drawdowns and defaults

A realization of a large drawdown is usually considered to be a sign of market recession. In this section we use a reduced form model for default and compute the probabilities of drawdowns and default. In particular, we consider an asset process X , which is given by a time-homogeneous diffusion process with initial value x and lifetime ζ :

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t, \quad t < \zeta.$$

Here ζ is an independent positive random variable which models the ‘default time’ of asset process X . If we assume that the ‘default time’ $\zeta = e_q$ for a, $q > 0$, then the probability that there is a drawdown of a units before a drawup of b units by the default time is given by $\mathbb{P}_x(\sigma_a < \hat{\sigma}_b \wedge e_q)$, which is readily available from Theorems 1 and 2.

Moreover, we can consider a more realistic model for the default time ζ , to reflect the fact that realizations of drawdowns of the asset process are very likely to be followed by a default. In particular, we adopt the Omega model studied in [1], [8], [13] to model the hazard rate of ζ at time $t > 0$ as $q \mathbf{1}_{\{Y_t > y\}}$:

$$\mathbb{P}_x(\zeta \in t + dt \mid \zeta > t) = q \mathbf{1}_{\{Y_t > y\}} dt,$$

for some $y \in (0, a)$. Here a is a large number that characterizes a critical level of a large drawdown. Then the probability of default before the drawdown of a units is given by

$$\mathbb{P}_x(\zeta < \sigma_a) = 1 - \mathbb{P}_x(\zeta \geq \sigma_a) = 1 - \mathbb{E}_x\{\exp(-qC_y^a)\}. \tag{18}$$

The Laplace transform in (18) can be found in Theorem 3.

The hazard rate of similar form can be considered. For example, we can model the hazard rate of the default time ζ as $q \mathbf{1}_{\{X_t < x\}}$. Here the initial value x is a critical benchmark level which may trigger a default through default intensity. We can also model the hazard rate of the default time ζ as $q \mathbf{1}_{\{\hat{Y}_t < y\}}$. This is the case in which the default tends to occur when there is not enough upside momentum for the asset process. In both cases, the probability of default before a drawdown of a units can be found using Proposition 4 and Theorem 4.

5.2. Option pricing for the drawdown process

Options on maximum drawdown and drawdown processes have drawn lots of attention in recent years (see [4], [5], [25], [32], [34], [39]). In this section we use a semi-analytic approach to price a large class of options on the drawdown process. In particular, we assume that the market is complete, \mathbb{P} is the risk-neutral measure, and $r \geq 0$ is the risk-free interest rate. We model the underlying process as the time-homogeneous diffusion X defined in (1). Then a Parisian-like (see [6] for a definition of standard Parisian option) digital call on the drawdown process with barrier $y > 0$, maturity $T > 0$, and strike $K \in (0, T)$ is worth

$$P_0(x, y, K, T) = \exp(-rT) \mathbb{P}_x \left(\int_0^T \mathbf{1}_{\{Y_t > y\}} dt > K \right), \tag{19}$$

at its inception. Using double randomizations: $T = e_q$ and $K = e_p$, then the randomized option price is given by

$$\hat{P}_0(x, y, q, p) = \exp(-rT) - \exp(-rT) \mathbb{E}_x\{\exp(-pE_y^q)\}. \tag{20}$$

(Note that the underlying process is not necessarily an asset price process. It can be, for example, the logarithm of an asset price process.) The Laplace transform in (20) is readily available in Theorem 5. Hence, the price (19) can be computed via double Laplace inversion:

$$P_0(x, y, K, T) = \exp(-rT) - \exp(-rT) \mathcal{I}_p \left(\frac{1}{p} \mathcal{I}_q \left(\frac{1}{q} \mathbb{E}_x \{ \exp(-pE_y^q) \} \right) \right) \Big|_T \Big|_K,$$

where \mathcal{I}_p and \mathcal{I}_q are Laplace inversion operators.

Corridor options such as the α -quantile option have been studied in [19], [20]. Below we consider pricing of α -quantile options on the drawdown process. To this end, for an $\alpha \in (0, 1]$, we define the α -quantile of the drawdown process during $[0, T]$ by

$$Y_T^\alpha := \inf \left\{ y > 0 : \int_0^T \mathbf{1}_{\{Y_t > y\}} dt \leq (1 - \alpha)T \right\}.$$

In particular, $Y_T^1 = \sup_{t \in [0, T]} Y_t$ is the maximum drawdown at time T . An option on the α -quantile with maturity $T > 0$ and an absolute continuous, bounded payoff function $f(\cdot)$ such that $f(0) = 0$ is worth

$$A_0(x, f, T) = \exp(-rT) \mathbb{E}_x \{ f(Y_T^\alpha) \}$$

at its inception. We note that

$$\begin{aligned} \mathbb{E}_x \{ f(Y_T^\alpha) \} &= \mathbb{E}_x \left\{ \int_0^\infty \mathbf{1}_{\{Y_T^\alpha \geq u\}} f'(u) du \right\} \\ &= \int_0^\infty f'(u) \mathbb{P}_x (Y_T^\alpha > u) du \\ &= \int_0^\infty f'(u) \mathbb{P}_x \left(\int_0^T \mathbf{1}_{\{Y_t > u\}} dt \geq (1 - \alpha)T \right) du. \end{aligned}$$

It follows from (19) that

$$A_0(x, f, T) = \int_0^\infty f'(u) P_0(x, u, (1 - \alpha)T, T) du. \tag{21}$$

Again, by double Laplace inversion and Theorem 5 we can compute the price in (21).

6. Examples

6.1. Brownian motion with drift

In this section we derive a group of explicit equations for a Brownian motion with drift. In particular, we consider a Brownian motion with drift $\mu \neq 0$ and diffusion coefficient $\sigma > 0$:

$$dX_t = \mu dt + \sigma dB_t, \quad I = (-\infty, \infty).$$

Let us denote by

$$\delta := \frac{\mu}{\sigma^2}, \quad \gamma := \sqrt{\delta^2 + \frac{2q}{\sigma^2}}.$$

Then the increasing and the decreasing eigenfunctions of X can be chosen as [2]

$$\phi_q^+(x) = \exp((\gamma - \delta)x), \quad \phi_q^-(x) = \exp(-(\gamma + \delta)x).$$

Fix the scale function $s(x) = (1/\delta)(1 - \exp(-2\delta x))$, we have that

$$w_q = \gamma, \quad W_q(x, y) = 2 \exp(-\delta(x + y)) \frac{\sinh[\gamma(x - y)]}{\gamma},$$

$$\frac{W_{q,1}(x, y)}{W_q(x, y)} = \gamma \cosh[\gamma(x - y)] - \delta.$$

From Theorems 1 and 2 we have the following corollary.

Corollary 2. *It holds that*

$$\mathbb{P}_0(\sigma_a < \hat{\sigma}_b \wedge e_q)$$

$$= \frac{\sigma^2 \gamma}{2q} \left(\frac{\exp(-\delta b)(\gamma \coth[\gamma b] + \delta)}{\sinh[\gamma b]} - \frac{\gamma}{\sinh^2[\gamma b]} \right) \exp(-(a - b)(\delta + \gamma \coth[\gamma b])),$$

$a \geq b > 0;$

$$\mathbb{P}_0(\sigma_a < \hat{\sigma}_b \wedge e_q)$$

$$= \frac{1 - (\sigma^2 \gamma / 2q)((\exp(\delta a)(\gamma \coth[\gamma a] - \delta) / \sinh[\gamma a]) - (\gamma / \sinh^2[\gamma a])) \exp(-(b - a)(-\delta + \gamma \coth[\gamma a]))}{\gamma \cosh[\gamma a] - \delta \sinh[\gamma a]}$$

$$\times \gamma \exp(-\delta a), \quad b > a > 0.$$

For occupation time of the drawdown process, from Theorem 3 we have the following corollary.

Corollary 3. *For any $y \in (0, a)$, the occupation time C_y^a has the same law as the drawdown time σ_{a-y} .*

Proof. Straightforward calculation using Theorem 3 yields that, for all $q \geq 0$,

$$\mathbb{E}_0\{\exp(-qC_y^a)\} = \frac{\gamma \exp(-\delta(a - y))}{\gamma \cosh[\gamma(a - y)] - \delta \sinh[\gamma(a - y)]} = \mathbb{E}_0\{\exp(-q\sigma_{a-y})\},$$

where the last equality follows from Proposition 2 or Corollary 2 as $b \rightarrow \infty$. It follows that the occupation time C_y^a has the same distribution as σ_{a-y} under \mathbb{P}_0 .

6.2. Three-dimensional Bessel process (BES(3))

In this section we study the case of the three-dimensional Bessel process. In particular, we consider

$$dX_t = \frac{1}{X_t} dt + dB_t, \quad I = (0, \infty).$$

Let us denote by

$$v := \sqrt{2q}.$$

Then the increasing and the decreasing eigenfunctions of X can be chosen as [2]:

$$\phi_q^+(x) = \frac{1 \sinh[vx]}{x \sinh[v]}, \quad \phi_q^-(x) = \frac{\exp(-v(x - 1))}{x}.$$

Fix the scale function $s(x) = -1/x$, we have that

$$w_q = \frac{v \exp(v)}{\sinh(v)}, \quad W_q(x, y) = \frac{1}{vx y} \sinh[v(x - y)],$$

$$\frac{W_{q,1}(x, y)}{W_q(x, y)} = -\frac{1}{x} + v \coth[v(x - y)].$$

Using Theorem 3 we have the following corollary.

Corollary 4. For $x > a > y > 0$, the law of the occupation time C_y^a is the same as the drawdown time σ_{a-y} .

Proof. Straightforward calculation using Theorem 3 yields that, for all $q \geq 0$,

$$\begin{aligned} \mathbb{E}_x\{\exp(-qC_y^a)\} &= \frac{1}{\cosh[v(a-y)]} \left(\frac{x-(a-y)}{x} + \frac{\tanh[v(a-y)]}{vx} \right) \\ &= \mathbb{E}_x\{\exp(-q\sigma_{a-y})\}, \end{aligned} \tag{22}$$

where the last equality is obtained by substitutions $a \rightarrow a - y$ and $y \downarrow 0$ in (22). It follows that the law of C_y^a is the same as that of σ_{a-y} .

Remark 3. The results of Theorems 3 and 4 show a nontrivial fact: if X is a drifted Brownian motion or a three-dimensional Bessel process, then for a fixed $y > 0$, the law of σ_y is the same as C_a^{y+a} for any $a > 0$. That is, the amount of time the drawdown process Y spends in $[a, a + y]$ until the drawdown time σ_{a+y} is the same as the drawdown time σ_y .

Appendix A

In this appendix we provide proofs that have been skipped in the main text.

Proof of Lemma 1. Most equations are straightforward and we omit the proofs for them. In the sequel we only prove

$$\lim_{x \downarrow l} \phi_q^-(x) = \infty, \quad \text{for all } q > 0; \quad \lim_{q \downarrow 0} W_q(x, y) = W_0(x, y), \quad \text{for all } x, y \in I.$$

First, for $x \in (l, \kappa)$, using (2) and the monotone convergence theorem, and inaccessibility of l after time 0, we have

$$\lim_{x \downarrow l} \phi_q^-(x) = \lim_{x \downarrow l} \frac{1}{\mathbb{E}_\kappa\{\exp(-q\tau_x)\}} = \frac{1}{0} = \infty.$$

Secondly, from (3) we have that for $x \geq y, x, y \in I$,

$$\frac{\partial}{\partial x} \left(\frac{\phi_q^-(x)}{\phi_q^+(x)} \right) = -w_q \frac{s'(x)}{(\phi_q^+(x))^2} \implies W_q(x, y) = \phi_q^+(x)\phi_q^+(y) \int_y^x \frac{s'(u)}{(\phi_q^+(u))^2} du.$$

We observe from (2) and regularity of X that, $\phi_q^+(u), u \in [y, x]$ is uniformly bounded (away from 0) for all $q \in [0, q_0]$ for any fixed $q_0 > 0$:

$$0 < \mathbb{E}_y\{\exp(-q_0\tau_\kappa)\} \leq \phi_q^+(u) \leq \frac{1}{\mathbb{E}_\kappa\{\exp(-q_0\tau_x)\}} < \infty, \quad \text{for all } u \in [y, x].$$

Moreover, from (2) we obtain, for $u \in [y, x] \subsetneq I$.

$$\lim_{q \rightarrow 0^+} \phi_q^+(u) = \begin{cases} \mathbb{P}_u\{\tau_\kappa < \infty\} & \text{if } u \leq \kappa \\ \frac{1}{\mathbb{P}_\kappa\{\tau_u < \infty\}} & \text{if } u > \kappa \end{cases} = \lim_{y \downarrow l} \frac{s(u) - s(y)}{s(\kappa) - s(y)} = \beta_1 s(u) + \beta_2,$$

for some constant β_1, β_2 depending on the behavior of limit $\lim_{y \downarrow l} s(y)$. By the dominated convergence theorem, as $q \downarrow 0$,

1. if $\beta_1 \neq 0$,

$$\begin{aligned} W_q(x, y) &\rightarrow (\beta_1 s(x) + \beta_2)(\beta_1 s(y) + \beta_2) \int_y^x \frac{s'(u) du}{(\beta_1 s(u) + \beta_2)^2} \\ &= \frac{1}{\beta_1} [(\beta_1 s(x) + \beta_2) - (\beta_1 s(y) + \beta_2)] \\ &= s(x) - s(y) \\ &= W_0(x, y); \end{aligned}$$

2. if $\beta_1 = 0$, then $\beta_2 \geq \mathbb{E}_y\{\exp(-q_0 \tau_\kappa)\} > 0$, and

$$W_q(x, y) \rightarrow \beta_2^2 \int_y^x \frac{s'(u) du}{\beta_2^2} = s(x) - s(y) = W_0(x, y).$$

This completes the proof.

Proof of Proposition 1. First we note that

$$\mathbb{P}_m(\tau_n^- < \hat{\sigma}_b \wedge e_q) = \mathbb{E}_m\{\exp(-q \tau_n^-); \tau_n^- < \hat{\sigma}_b\}.$$

To compute the above expectation on the right-hand side, we follow the idea of [14], and partition the interval $[n, m]$ into N equal length subintervals with length $\varepsilon = (m - n)/N$. In particular, using the fact that $\mathbb{P}_m(\tau_m^- = 0) = 1$ and continuity of X , we have

$$\begin{aligned} &\exp\left(-q \sum_{i=0}^{N-1} (\tau_{m-(i+1)\varepsilon}^- - \tau_{m-i\varepsilon}^-)\right) \mathbf{1}_{\{\tau_{m-(j+1)\varepsilon}^- < \tau_{m-j\varepsilon+b}^+ \text{ for all } 0 \leq j \leq N-1\}} \\ &\rightarrow \exp(-q \tau_n^-) \mathbf{1}_{\{\tau_n^- < \hat{\sigma}_b\}}, \quad \mathbb{P}_m\text{-a.s.} \end{aligned}$$

as $N \rightarrow \infty$. Applying the Lebesgue dominated convergence theorem, the strong Markov property, and continuity of X , we obtain

$$\begin{aligned} &\mathbb{E}_m\{\exp(-q \tau_n^-); \tau_n^- < \hat{\sigma}_b\} \\ &= \lim_{N \rightarrow \infty} \mathbb{E}_m\left\{\exp\left(-q \sum_{i=0}^{N-1} (\tau_{m-(i+1)\varepsilon}^- - \tau_{m-i\varepsilon}^-)\right); \tau_{m-(j+1)\varepsilon}^- < \tau_{m-j\varepsilon+b}^+ \right. \\ &\quad \left. \text{for all } 0 \leq j \leq N-1\right\} \\ &= \lim_{N \rightarrow \infty} \prod_{i=0}^{N-1} \mathbb{E}_{m-i\varepsilon}\{\exp(-q \tau_{m-(i+1)\varepsilon}^-); \tau_{m-(i+1)\varepsilon}^- < \tau_{m-i\varepsilon+b}^+\}. \end{aligned} \tag{23}$$

To compute the limit in (23), we use Lemma 2 to obtain

$$\begin{aligned} &\mathbb{E}_m\{\exp(-q \tau_n^-); \tau_n^- < \hat{\sigma}_b\} \\ &= \lim_{N \rightarrow \infty} \prod_{i=0}^{N-1} \frac{W_q(m - i\varepsilon, m - i\varepsilon + b)}{W_q(m - (i + 1)\varepsilon, m - i\varepsilon + b)} \end{aligned}$$

$$\begin{aligned}
 &= \exp\left(\lim_{N \rightarrow \infty} \left[\sum_{i=0}^{N-1} \frac{W_{q,1}(m - (i + 1)\varepsilon, m - i\varepsilon + b)}{W_q(m - (i + 1)\varepsilon, m - i\varepsilon + b)} \cdot \varepsilon + O(\varepsilon) \right]\right) \\
 &= \exp\left(\int_n^m \frac{W_{q,1}(v, v + b)}{W_q(v, v + b)} dv\right),
 \end{aligned}$$

which completes the proof of (7). Equation (8) can be proved using a similar argument.

Proof of Proposition 3. We follow the perturbation method in [12], [13], [15]. For $\varepsilon > 0$ such that $y + \varepsilon < b$, we approximate $A_y^{a,b}$ by $A_{y,\varepsilon}^{a,b}$:

$$A_{y,\varepsilon}^{a,b} := \sum_{n=1}^{\infty} (\tau_{y+\varepsilon}^{+,n} \wedge \tau_{a,b} - \tau_y^{-,n} \wedge \tau_{a,b}),$$

where $\tau_y^{-,1} := \tau_y^-$, and for $n \geq 1$,

$$\tau_{y+\varepsilon}^{+,n} := \inf\{t \geq \tau_{y+\varepsilon}^{-,n} : X_t \geq y + \varepsilon\}, \quad \tau_y^{-,n+1} := \inf\{t \geq \tau_{y+\varepsilon}^{+,n} : X_t \leq y\}.$$

Using the strong Markov property and the continuity of X , we have that

$$\begin{aligned}
 &\mathbb{E}_y\{\exp(-qA_{y,\varepsilon}^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^-\} \\
 &= \mathbb{E}_y\{\exp(-(q + p)\tau_{y+\varepsilon}^+); \tau_{y+\varepsilon}^+ < \tau_a^-\} \mathbb{E}_{y+\varepsilon}\{\exp(-qA_{y,\varepsilon}^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^-\} \\
 &= \mathbb{E}_y\{\exp(-(q + p)\tau_{y+\varepsilon}^+); \tau_{y+\varepsilon}^+ < \tau_a^-\} (\mathbb{E}_{y+\varepsilon}\{\exp(-p\tau_b^+); \tau_b^+ < \tau_y^-\} \\
 &\quad + \mathbb{E}_{y+\varepsilon}\{\exp(-p\tau_y^-); \tau_y^- < \tau_b^+\} \mathbb{E}_y\{\exp(-qA_{y,\varepsilon}^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^-\}),
 \end{aligned}$$

from which we obtain

$$\begin{aligned}
 &\mathbb{E}_y\{\exp(-qA_{y,\varepsilon}^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^-\} \\
 &= \frac{\mathbb{E}_{y+\varepsilon}\{\exp(-p\tau_b^+); \tau_b^+ < \tau_y^-\} \mathbb{E}_y\{\exp(-(q + p)\tau_{y+\varepsilon}^+); \tau_{y+\varepsilon}^+ < \tau_a^-\}}{1 - \mathbb{E}_{y+\varepsilon}\{\exp(-p\tau_y^-); \tau_y^- < \tau_b^+\} \mathbb{E}_y\{\exp(-(q + p)\tau_{y+\varepsilon}^+); \tau_{y+\varepsilon}^+ < \tau_a^-\}} \\
 &= \frac{W_p(y + \varepsilon, y)}{W_p(b, y)} \frac{(W_{q+p}(y, a) / W_{q+p}(y + \varepsilon, a))}{1 - (W_p(b, y + \varepsilon) / W_p(b, y))(W_{q+p,1}(y, a) / W_{q+p}(y, a))}. \tag{24}
 \end{aligned}$$

The quantity $A_{y,\varepsilon}^{a,b}$ measures the time for X to spend below level y and the time to move from y to $y + \varepsilon$, but not from $y + \varepsilon$ to y , until X exits from (a, b) . As $\varepsilon \rightarrow 0+$, by continuity of X , we have $A_{y,\varepsilon}^{a,b} \rightarrow A_y^{a,b}$, \mathbb{P}_x -a.s. Using the Lebesgue dominated convergence theorem and the continuity of X , letting $\varepsilon \downarrow 0$ in (24) we have

$$\mathbb{E}_y\{\exp(-qA_y^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^-\} = \frac{s'(y)}{W_{p,1}(y, b) + W_p(b, y)(W_{q+p,1}(y, a) / W_{q+p}(y, a))}.$$

It follows that, for $x \in (a, y)$, using the strong Markov property of X , we have

$$\begin{aligned}
 &\mathbb{E}_x\{\exp(-qA_y^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^-\} \\
 &= \mathbb{E}_x\{\exp(-(q + p)\tau_y^+); \tau_y^+ < \tau_a^-\} \mathbb{E}_y\{\exp(-qA_y^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^-\} \\
 &= \frac{W_{q+p}(x, a)}{W_{q+p}(y, a)} \frac{s'(y)}{W_{p,1}(y, b) + W_p(b, y)(W_{q+p,1}(y, a) / W_{q+p}(y, a))}.
 \end{aligned}$$

For $x \in (y, b)$, we can similarly obtain

$$\begin{aligned} & \mathbb{E}_x \{ \exp(-qA_y^{a,b} - p\tau_b^+); \tau_b^+ < \tau_a^- \} \\ &= \frac{W_p(x, y)}{W_p(b, y)} + \frac{W_p(b, x)}{W_p(b, y)} \frac{s'(y)}{W_{p,1}(y, b) + W_p(b, y)(W_{q+p,1}(y, a)/W_{q+p}(y, a))}. \end{aligned}$$

Using the similar argument as above, we obtain (12).

Proof of Proposition 4. We let $\varepsilon = a/N$ for a large integer $N > 0$. Using the Lebesgue dominated convergence theorem, continuity, and the strong Markov property of X , we have

$$\begin{aligned} & \mathbb{E}_x \{ \exp(-qB_x^a); \bar{X}_{\sigma_a} \geq x + a \} \\ &= \mathbb{E}_x \{ \exp(-qB_x^a); \tau_{x+a}^+ < \sigma_a \} \\ &= \lim_{N \rightarrow \infty} \prod_{i=0}^{N-1} \mathbb{E}_{x+i\varepsilon} \{ \exp(-qA_x^{x+i\varepsilon-a, x+(i+1)\varepsilon}); \tau_{x+(i+1)\varepsilon}^+ < \tau_{x+i\varepsilon-a}^- \} \\ &= \exp \left(\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} [\mathbb{E}_{x+i\varepsilon} \{ \exp(-qA_x^{x+i\varepsilon-a, x+(i+1)\varepsilon}); \tau_{x+(i+1)\varepsilon}^+ < \tau_{x+i\varepsilon-a}^- \} - 1] \right) \\ &= \exp \left(- \int_x^{x+a} \frac{(s'(u)/s'(x))(W_{q,1}(x, u-a)/W_q(x, u-a)) du}{1 + ((s(u) - s(x))/s'(x))(W_{q,1}(x, u-a)/W_q(x, u-a))} \right). \end{aligned}$$

Here we used Proposition 3 in the last equality. Moreover, notice that $\mathbb{E}_x \{ \exp(-qA_x^{u-a, z}); \tau_{u-a}^- < \tau_z^+ \} = \mathbb{E}_x \{ \exp(-qA_x^{u-a, \infty}); \bar{X}_{\tau_{u-a}^-} < z \}$. It follows that

$$\begin{aligned} & \mathbb{E}_x \{ \exp(-qB_x^a); \bar{X}_{\sigma_a} \in (x, x + a) \} \\ &= \int_x^{x+a} \mathbb{E}_x \{ \exp(-qA_x^{u-a, \infty}); \bar{X}_{\tau_{u-a}^-} \in du \} \\ &= \int_x^{x+a} \frac{\partial}{\partial z} \Big|_{z=u} \mathbb{E}_x \{ \exp(-qA_x^{u-a, z}); \tau_{u-a}^- < \tau_z^+ \} du \\ &= \int_x^{x+a} \frac{(s'(u)/s'(x)) du / W_q(x, u-a)}{(1 + ((s(u) - s(x))/s'(x))(W_{q,1}(x, u-a)/W_q(x, u-a)))^2}, \end{aligned}$$

where we used Proposition 3 in the last equality.

Proof of Theorem 5. Consider the random time $g := \inf \{ t \geq 0 : X_t = \bar{X}_{e_q} \}$. Then we have

$$E_y^q = \int_0^g \mathbf{1}_{\{Y_t > y\}} dt + \int_g^{e_q} \mathbf{1}_{\{Y_t > y\}} dt := E_y^{q,1} + E_y^{q,2}.$$

Below we compute the Laplace transforms of $E_y^{q,1}$ and $E_y^{q,2}$ conditioning on \bar{X}_{e_q} . More specifically, for $m > x$, we let $\varepsilon = ((m-x)/N)$ for a large $N > 0$. Then we have

$$\begin{aligned} & \frac{\mathbb{E}_x \{ \exp(-pE_y^{q,1}); \bar{X}_{e_q} \in dm \}}{dm} \\ &= - \frac{\partial}{\partial h} \Big|_{h=0} \mathbb{E}_x \left\{ \exp \left(-p \int_0^{\tau_m^+} \mathbf{1}_{\{Y_t > y\}} dt \right); \bar{X}_{e_q} \geq m + h \right\} \end{aligned}$$

$$\begin{aligned}
 &= \exp\left(\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \left[\mathbb{E}_{x+i\varepsilon} \exp\left\{-p \int_0^{\tau_{x+(i+1)\varepsilon}^+} \mathbf{1}_{\{X_t < x+i\varepsilon-y\}} dt - q\tau_{x+(i+1)\varepsilon}^+\right\} - 1\right]\right) \\
 &\quad \times \frac{\phi_q^{+'}(m)}{\phi_q^+(m)} \\
 &= \exp\left(-\int_x^m \frac{W_{q,2}(u-y, u) + W_{q,1}(u, u-y)(\phi_{q+p}^{+'}(u-y)/\phi_{q+p}(u-y))}{W_{q,1}(u-y, u) + W_q(u, u-y)(\phi_{q+p}^{+'}(u-y)/\phi_{q+p}^+(u-y))} du\right) \\
 &\quad \times \frac{\phi_q^{+'}(m)}{\phi_q^+(m)}.
 \end{aligned}$$

The fourth equality follows from Corollary 1. On the other hand,

$$\begin{aligned}
 &\mathbb{E}_x\{\exp(-pE_y^{q,2}) \mid \bar{X}_{e_q} = m\} \\
 &= \lim_{\varepsilon' \downarrow 0} \frac{\mathbb{E}_m\{\exp(-p \int_0^{e_q} \mathbf{1}_{\{X_t < m-y\}} dt); e_q < \tau_{m+\varepsilon'}^+\}}{\mathbb{P}_m(e_q < \tau_{m+\varepsilon'}^+)} \\
 &= 1 + \lim_{\varepsilon' \downarrow 0} \left[\frac{\mathbb{P}_m(\tau_{m-y}^- < e_q \wedge \tau_{m+\varepsilon'}^+)}{\mathbb{P}_m(e_q < \tau_{m+\varepsilon'}^+)} \mathbb{E}_{m-y} \right. \\
 &\quad \left. \times \left\{ \left[\exp\left(-p \int_0^{e_q} \mathbf{1}_{\{X_t < m-y\}} dt\right) - 1\right]; e_q < \tau_{m+\varepsilon'}^+ \right\} \right].
 \end{aligned}$$

To get the limit in the above equation, we use Corollary 3.4 of [15] to obtain

$$\begin{aligned}
 &\lim_{\varepsilon' \downarrow 0} \left[\frac{\mathbb{P}_m(\tau_{m-y}^- < e_q \wedge \tau_{m+\varepsilon'}^+)}{\mathbb{P}_m(e_q < \tau_{m+\varepsilon'}^+)} \mathbb{E}_{m-y} \left\{ \left[\exp\left(-p \int_0^{e_q} \mathbf{1}_{\{X_t < m-y\}} dt\right) - 1\right]; e_q < \tau_{m+\varepsilon'}^+ \right\} \right] \\
 &= -\frac{s'(m)}{\phi_q^{+'}(m)} \frac{\phi_q^{+'}(m-y) + [(p/(q+p))\phi_q^+(m) - \phi_q^+(m-y)](\phi_{q+p}^{+'}(m-y)/\phi_{q+p}^+(m-y))}{W_{q,1}(m-y, m) + W_q(m, m-y)(\phi_{q+p}^{+'}(m-y)/\phi_{q+p}^+(m-y))}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 &\mathbb{E}_x\{\exp(-pE_y^{q,2}) \mid \bar{X}_{e_q} = m\} \\
 &= \frac{W_{q,2}(m-y, m) + [W_{q,1}(m, m-y) - (p/(q+p))s'(m)](\phi_{q+p}^{+'}(m-y)/\phi_{q+p}^+(m-y))}{W_{q,1}(m-y, m) + W_q(m, m-y)(\phi_{q+p}^{+'}(m-y)/\phi_{q+p}^+(m-y))} \\
 &\quad \times \frac{\phi_q^+(m)}{\phi_q^{+'}(m)}.
 \end{aligned}$$

The proof is complete after integration with respect to m .

Acknowledgements

The author is grateful to Professor Ryozi Miura and Professor Thomas Mikosch for their helpful comments. The author thanks the Editor and anonymous referees for their useful remarks and suggestions.

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