Beyond the Basel Problem: Sums of Reciprocals of Figurate Numbers

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Throughout 2007, a great deal of attention was paid to the life and work of Leonhard Euler (1707–1783), and rightly so! Euler's enormous impact can certainly still be felt today. And while his work spans a great many areas of interest within mathematics, here we focus on one of his earliest pursuits – determining the sums of particular infinite series.

Summing infinite series was a hot topic in the late 17th and early 18th centuries. Indeed, Jacob Bernoulli's *Tractatus de Seriebus Infinitus* [1] was of momumental importance in the field. Bernoulli determined the sums of numerous convergent series very elegantly, making particularly good use of what we now call "telescoping". For example, he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1,$$

a fact often demonstrated to calculus students. Of course, this means that the sum of the reciprocals of the triangular numbers, the numbers $1, 3, 6, 10, 15, \ldots$, is 2.

Another series of interest at the time was

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

and finding its sum became known as the Basel Problem. Its solution eluded the best mathematical minds of the day. (See, for example, Dunham [3, Ch. 3] for more information.) It was accepted that the series converges, and many had approximated its sum with decent accuracy, but no one was able to find its exact value. Enter Leonhard Euler.

In 1737, Euler provided the first of his many proofs of the problem. In this early work, which put him on the map in the eyes of the mathematical community, Euler proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

a result often quoted, but (sadly) rarely proved, in calculus courses.

These two examples, both of which refer to sums of reciprocals of figurate numbers, serve to motivate a question that does not seem to have received much attention in the past: namely, what are the sums of the reciprocals of the other figurate numbers,

$$\sum_{n=1}^{\infty} \frac{2}{3n^2 - n}, \quad \sum_{n=1}^{\infty} \frac{2}{4n^2 - 2n}, \quad \sum_{n=1}^{\infty} \frac{2}{5n^2 - 3n}, \quad \sum_{n=1}^{\infty} \frac{2}{6n^2 - 4n}, \quad \ldots ?$$

Interestingly, such a question did arise recently. In the Spring 2007 issue of the *Pi Mu Epsilon Journal*, Problem 1147 [6] asked for the sum of the reciprocals of the pentagonal numbers. The published solution to that problem [7] follows lines similar to our work here.

Our goals for this paper are these. First, in arguably Eulerian fashion, we provide a technique, based on ideas from calculus, that reduces the sums of reciprocals of figurate numbers to the computation of an integral. We then use various tools to compute these integrals. We focus only on the sums of reciprocals of figurate numbers related to polygons with an even number of sides (squares as in Euler's original problem, hexagons, octagons, and so on). The odd case is similar, and the details are left to the reader.

It is worth noting that the recent papers of Efthimiou [4] and Lesko [5] discuss results like ours. A book that may be helpful is Davis [2], which deals with the summability of series.

We begin our analysis by noting that the n^{th} figurate number for a polygon with a sides is

$$\frac{(a-2)n^2-(a-4)n}{2}.$$

(For example, the hexagonal numbers $1, 6, 15, 28, \ldots,$ have the simple formula $n(2n-1)$.)

Thus, our object of study is

$$S_a := \sum_{n=1}^{\infty} \frac{2}{(a-2)n^2-(a-4)n}$$

for even values of $a \geq 6$. This can be thought of as twice

$$\sum_{n=1}^{\infty} \frac{1}{n((a-2)n-(a-4))} x^{(a-2)n-(a-4)} \bigg|_{x=1},$$

and since the Maclaurin series for $\ln(1-t)$ is $-\sum_{n=1}^{\infty} \frac{1}{n} t^n$, this is simply an antiderivative of

$$-\frac{1}{x^{a-3}} \ln(1-x^{a-2}).$$

Using integration by parts with $u = \ln(1-x^{a-2})$ and $dv = \frac{1}{x^{a-3}} dx$, we find that

$$\int \frac{1}{x^{a-3}} \ln(1-x^{a-2})\, dx = -\frac{\ln(1-x^{a-2})}{(a-4)x^{a-4}} - \frac{a-2}{a-4} \int \frac{x}{1-x^{a-2}}\, dx.$$

Replacing a by $2k+2$ gives

$$\int \frac{1}{x^{2k-1}} \ln(1-x^{2k})\, dx = -\frac{\ln(1-x^{2k})}{(2k-2)x^{2k-2}} - \frac{2k}{2k-2} \int \frac{x}{1-x^{2k}}\, dx.$$

Using a known result (see, for example, [8, p. 75]) for the last integral here, we find that

$$\begin{aligned}\int \frac{1}{x^{2k-1}}\ln(1-x^{2k})\,dx &= -\frac{\ln(1-x^{2k})}{(2k-2)x^{2k-2}}\\ &\quad +\frac{1}{2k-2}\sum_{j=1}^{k-1}\cos\left(\frac{2j\pi}{k}\right)\ln\left(x^2-2x\cos\left(\frac{j\pi}{k}\right)+1\right)\\ &\quad -\frac{2}{2k-2}\sum_{j=1}^{k-1}\sin\left(\frac{2j\pi}{k}\right)\tan^{-1}\left(\frac{x-\cos\left(\frac{j\pi}{k}\right)}{\sin\left(\frac{j\pi}{k}\right)}\right)\\ &\quad +\frac{1}{2k-2}\ln(1-x^2)+C,\end{aligned}$$

which we call $F_{2k+2}(x)$ after setting $C=0$. Then

$$S_{2k+2}=-2\left(\lim_{x\to 1^-}F_{2k+2}(x)-\lim_{x\to 0^+}F_{2k+2}(x)\right).$$

A straightforward application of l'Hospital's rule shows that

$$\lim_{x\to 0^+}\frac{\ln(1-x^{2k})}{x^{2k-2}}=0,$$

so the second limit simplifies rather quickly to

$$\lim_{x\to 0^+}F_{2k+2}(x)=-\frac{2}{2k-2}\sum_{j=1}^{k-1}\sin\left(\frac{2j\pi}{k}\right)\tan^{-1}\left(\frac{-\cos\left(\frac{j\pi}{k}\right)}{\sin\left(\frac{j\pi}{k}\right)}\right).$$

The first limit can also be evaluated using l'Hospital's rule, but it is somewhat more involved. We focus our attention on

$$\lim_{x\to 1^-}\left(-\frac{\ln(1-x^{2k})}{(2k-2)x^{2k-2}}+\frac{1}{2k-2}\ln(1-x^2)\right).$$

Since $1-x^{2k}=(1-x^2)(1+x^2+x^4+\cdots+x^{2k-2})$, we see that this reduces to

$$\frac{-\ln(k)}{2k-2}-\frac{1}{2k-2}\lim_{x\to 1^-}\ln(1-x^2)\left(\frac{1}{x^{2k-2}}-1\right).$$

One more application of l'Hospital's rule shows that the limit here is 0, so

$$\begin{aligned}\lim_{x\to 1^-} F_{2k+2}(x) &= -\frac{\ln(k)}{2k-2} + \frac{1}{2k-2}\sum_{j=1}^{k-1}\cos\left(\frac{2j\pi}{k}\right)\ln\left(2-2\cos\left(\frac{j\pi}{k}\right)\right) \\ &\quad -\frac{2}{2k-2}\sum_{j=1}^{k-1}\sin\left(\frac{2j\pi}{k}\right)\tan^{-1}\left(\frac{1-\cos\left(\frac{j\pi}{k}\right)}{\sin\left(\frac{j\pi}{k}\right)}\right).\end{aligned}$$

Therefore, for every integer $k \geq 2$,

$$\begin{aligned}S_{2k+2} &= \frac{\ln(k)}{k-1} - \frac{1}{k-1}\sum_{j=1}^{k-1}\cos\left(\frac{2j\pi}{k}\right)\ln\left(2-2\cos\left(\frac{j\pi}{k}\right)\right) \\ &\quad +\frac{2}{k-1}\sum_{j=1}^{k-1}\sin\left(\frac{2j\pi}{k}\right)\tan^{-1}\left(\frac{1-\cos\left(\frac{j\pi}{k}\right)}{\sin\left(\frac{j\pi}{k}\right)}\right) \\ &\quad -\frac{2}{k-1}\sum_{j=1}^{k-1}\sin\left(\frac{2j\pi}{k}\right)\tan^{-1}\left(\frac{-\cos\left(\frac{j\pi}{k}\right)}{\sin\left(\frac{j\pi}{k}\right)}\right).\end{aligned}$$

It is now straightforward to calculate the values of these series, at least for certain figurate numbers, and some examples are given in the table.

From a computational perspective, this formula for S_{2k+2} is quite satisfactory, especially since it can be evaluated rather handily with a computer algebra package. Moreover, it is obvious that the difficulty in using this formula lies in obtaining certain values of the sine and cosine functions. Again, for wise choices of a, this is not a problem.

As frequent instructors of calculus, we are truly heartened to see that such nice results can be obtained using only elementary topics such as Maclaurin series, partial fractions, integration by parts, and l'Hospital's rule, along with straightforward evaluation of certain trigonometric and logarithmic functions.

Table 1: Values of Sums of Reciprocals of Figurate Numbers

Number of Sides	Name of Polygons	Sum of Series
6	Hexagonal	$2\ln 2$
8	Octagonal	$\dfrac{3\ln 3}{4} + \dfrac{\sqrt{3}\pi}{12}$
10	Decagonal	$\ln 2 + \dfrac{\pi}{6}$
14	Tetrakaidecagonal	$\dfrac{2\ln 2}{5} + \dfrac{3\ln 3}{10} + \dfrac{\sqrt{3}\pi}{10}$

References

[1] J. Bernoulli, *Tractatus de seriebus infinitis*, 1689.

[2] H. T. Davis, *The Summation of Series*, Principia Press, 1962.

[3] W. Dunham, *Euler: The Master of Us All*, MAA, 1999.

[4] C. J. Efthimiou, Finding Exact Values for Infinite Series, *Math. Mag.* **72** (1999) 45–51.

[5] J. Lesko, Sums of Harmonic–Type Sums, *College Math. J.* **35** (2004) 171–182.

[6] C. Rousseau, Problem 1147, *Pi Mu Epsilon J.* **12** (2007) 559.

[7] H. Chen, Solution to Problem 1147, *Pi Mu Epsilon J.* **12** (2007) 433–434.

[8] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, Schaum's Outline Series, McGraw–Hill, 1968.