EAST JOURNAL ON APPROXIMATIONS
Volume 17, Number 1 (2011), 1-108

ZEROS OF ENTIRE FOURIER TRANSFORMS *

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Dedicated to the memory of Professor Borislav Bojanov

We survey old and new results concerning the behaviour of zeros of entire functions which are defined as Fourier transforms of certain positive kernels. The emphasis is given to those kernels whose Fourier transform is an entire function with real zeros.

Mathematics Subject Classification (2010): 42A16, 26C10, 30C15, 33C10, 33C45.

Key words and phrases: Entire function, Fourier transform, Laguerre-Pólya class, hyperbolic polynomial, Turán inequalities, zeros.

0. Prime numbers, the Riemann hypothesis and Fourier transforms

In the present paper we survey result on the zero distribution of classes of entire functions defined as Fourier transforms of the form

$$\mathcal{F}_{a,b}(K;z) = \int_a^b K(t)\exp izt\,dt, \quad -\infty \le a < b \le \infty, \tag{0.1}$$

under the assumption that $K(t)$ is integrable in (a,b).

The following two cases have been studied systematically in the literature: (i) $a=-\infty$, $b=\infty$ and (ii) $-\infty < a < b < \infty$. In the the case (i) the entire function (0.1) takes the form

$$\mathcal{E}(K;z) = \int_{-\infty}^{\infty} K(t)\exp izt\,dt. \tag{0.2}$$

*The research of the first author is supported by the Brazilian Science Foundations CNPq under Grant 305622/2009-9 and FAPESP under Grant 2009/13832-9 and of the second author by the Bulgarian National Science Fund - Ministry of Education, Youth and Science under Project D ID 02/25/2009 (ITSFA).

Particular cases of (0.2) are

$$\mathcal{U}(K;z) = \int_0^\infty K(t)\cos zt\,dt \tag{0.3}$$

and

$$\mathcal{V}(K;z) = \int_0^\infty K(t)\sin zt\,dt, \tag{0.4}$$

which correspond to the assumptions that the function $K(t)$ in (0.2) is either even or odd.

The interest in the problem of characterizing those kernels $K(t)$ for which the corresponding Fourier transform has only real zeros is motivated by one of the most fascinating results in the history of mathematics, The Prime Number Theorem, and the most celebrated open problem, The Riemann Hypothesis.

The ancient Greeks were already interested in the numbers which do not have proper divisors, the prime numbers. The sieve of Erathostenes provides an algorithm for obtaining the primes and, in the third century BC, Euclid proved the existence of infinitely many prime numbers. The latter proof is so simple and beautiful that it might be used for a test if a child could have a talent for maths; those who understand and appreciate it, certainly may develop a taste for mathematics.

The most natural question arises of how many the prime numbers are among all naturals. A result of Euler gives the first hint about the answer despite that he himself does not declare explicitly any interest on the distribution of the prime numbers. This fact is explained nicely in the introduction of Eduard's book [Edwards 1974] "Riemann's Zeta Function". In 1737 Euler proved that the sum of the reciprocals of primes $\sum_p 1/p$, where the sum is extended over all prime numbers, diverges and wrote that $1/2+1/3+1/5+1/7+\cdots = \log(\log\infty)$, most probably, having in mind that

$$\sum_{p<x}\frac{1}{p} \sim \log(\log x) \quad \text{as} \quad x\to\infty,$$

where the sum is over the primes less than x. Since $\log(\log x) = \int_e^x dt/(t\log t)$, if we consider the measure $d\mu(t) = dt/\log t$, the last function may be rewritten in the form

$$\log(\log x) = \int_e^x \frac{1}{t}\,d\mu(t)$$

For any positive x, $\pi(x)$ denotes, as usual, the number of primes less than x. It is a step function which increases by one at the prime numbers. Then, obviously

$$\sum_{p<x}\frac{1}{p} = \int_e^x \frac{1}{t}\,d\pi(t).$$

Thus, Euler's theorem already suggests that $\pi(x) \sim x/\log x$, or equivalently

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{as} \quad x \to \infty, \tag{0.5}$$

.

According to Gauss, in 1792, at the age of fifteen, he observed that the density of primes behaves like $1/\log x$. He does not mention Euler's formula and it seems his belief was supported by numerical evidences. He provides a table where compares $\pi(x)$ with the function $\mathrm{Li}(x) = \int_2^x dt/\log t$ which behaves in the same way as $x/\log x$ at infinity. Thus, he claims that

$$\pi(x) \sim \int_2^x \frac{dt}{\log t}, \quad x \to \infty, \tag{0.6}$$

The figures below show the graphs of the functions $\mathrm{Li}(x)$, $\pi(x)$ and $x/\log x$.

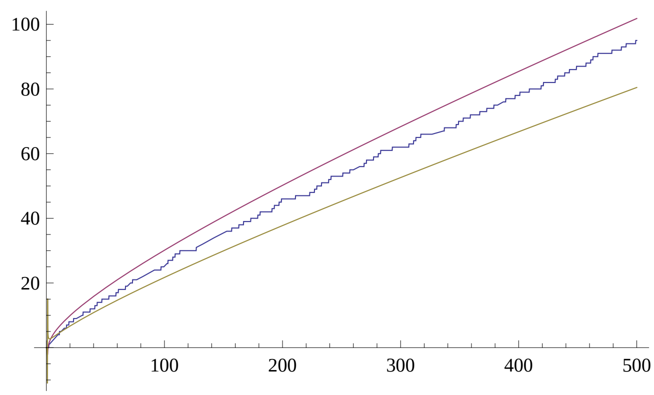

Figure 1: The graphs of $\mathrm{Li}(x)$, $\pi(x)$ and $x/\log x$ in the range $x \in [2, 500]$.

Surprisingly enough, Gauss had not published these observations and reports them only in a letter written in 1849. Meanwhile, in 1800 Legendre published similar observations. The first substantial contribution to this conjecture is due to Chebyshev. In 1952 he proved that, if the quantity $\pi(x)\log x/x$ converges as x goes to infinity, its limit must be one. However, Chebyshev was not able to prove the convergence of this quotient. He established the limits

$$0.921\,\frac{x}{\log x} \le \pi(x) \le 1.106\,\frac{x}{\log x} \quad \text{as} \quad x \to \infty.$$

Each one of limit relation (1.13) and (1.14) is called *asymptotic law of prime numbers distribution* or, briefly, *asymptotic law* (*Prime Number Theorem* as

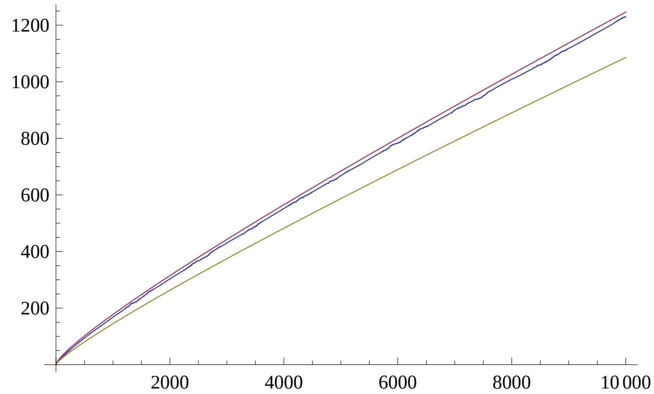

Figure 2: The graphs of $\mathrm{Li}(x)$, $\pi(x)$ and $x/\log x$ in the range $x \in [2, 10000]$.

well as *Primzahlsatz* are also used). The fact that these hold is the celebrated Prime Number Theorem which was finally proved independently by J. HADAMARD and J. DE LA VALLÉE POISSIN in 1896. Their proofs were published in [Hadamard 1896] and [de la Vallée Poussin 1896] and are based on ideas of Riemann described his short but fascinating paper **Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse**, *Monatsb. der Königl. Preuss. Akad. der Wissen. zu Berlin aus dem Jahr* 1859 (1860), 671–680.

In E. BOMBIERI'S survey **Problem of the Millenium: the Riemann Hypothesis**, this memoir is qualified as "epoch-making" as well as "really astonishing for the novelty of ideas included". M. DRMOTA in **Sieben Milleniums-Probleme**, Internat. Math. Nachr., 184 (2000) 29–36, characterizes it as "innovative for the analytic number theory" thus repeating JÖRG BRÜDERN, **Primzahlverteilung**, Vorlesung im Wintersemester 1991/92, Mathematisches Institut Göttingen. These "estimates" and many similar ones confirm the words of E. TITCHMARSH from the beginning of Chapter X of his **The theory of Riemann zeta-function**, Oxford 1951: "The memoir, in which Riemann considered the zeta-function became famous thanks to the great number of ideas included in it. Many of them has been worked afterwards, and some of them are not exhausted even till now".

RIEMANN'S memoir is devoted to the function $\pi(x)$ and, according to him, he takes as a starting point Euler's observation that

$$\prod \frac{1}{1-\dfrac{1}{p^s}} = \sum \frac{1}{n^s}, \tag{0.7}$$

where p ranges over all prime numbers and n over all natural numbers. He denotes by $\zeta(s)$ the function of the complex variable s, which these two expressions define when they converge. Obviously, the expression

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{0.8}$$

defines a holomorphic function of the complex variable $s = \sigma + it$ provided $\sigma > 1$. An easy computation leads to an integral representation of $\zeta(s)$ by means of the function $\pi(x)$. Indeed, from (0.7) it follows that $\zeta(s) \neq 0$ in the half-plane $\Re s > 1$ and

$$\begin{aligned}
\log \zeta(s) &= -\sum_p \log\left(1 - \frac{1}{p^s}\right) \\
&= -\sum_{n=2}^{\infty} \{\pi(n) - \pi(n-1)\} \log\left(1 - \frac{1}{n^s}\right) \\
&= -\sum_{n=2}^{\infty} \pi(n) \left\{\log\left(1 - \frac{1}{n^s}\right) - \log\left(1 - \frac{1}{(n+1)^s}\right)\right\} \\
&= \sum_{n=2}^{\infty} \pi(n) \int_n^{n+1} \frac{s\,dx}{x(x^s-1)},
\end{aligned}$$

which implies

$$\log \zeta(s) = s \int_2^{\infty} \frac{\pi(x)\,dx}{x(x^s-1)}, \quad s = \sigma + it, \quad \sigma > 1.$$

There is a much deeper formula expressing $\pi(x)$ in terms of the so-called nontrivial zeros of the meromorphic function, obtained by the analytical continuation of the function $\zeta(s)$ to the whole complex plane. The discovery of this formula is one of the main achievements of Riemann included in his memoir. There are many ways to prove that $\zeta(s)$ can be analytically continued to the left of the line $\Re s = 1$. Here we sketch the idea of Riemann. It is based on the equalities

$$\int_0^{\infty} \exp(-\pi n^2 x) x^{s/2-1}\,dx = \frac{\Gamma(s/2)}{\pi^{s/2} n^s}, \quad s = \sigma + it,\ \sigma > 0, \quad n = 1, 2, 3, \ldots,$$

which follow from the well-known integral representation of $\Gamma(s)$ in the half-plane $\sigma > 0$,

$$\Gamma(s) = \int_0^{\infty} x^{s-1} \exp(-x)\,dx.$$

If $\sigma > 1$, then

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^{\infty} \psi(x) x^{s/2-1}\,dx, \tag{0.9}$$

where

$$\psi(x) = \sum_{n=1}^{\infty} \exp(-\pi n^2 x), \quad x > 0. \tag{0.10}$$

RIEMANN establishes (0.9) by some properties of the Jacobi theta function. Let

$$\theta(x) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 x), \quad x > 0. \tag{0.11}$$

It follows from the inequalities $\exp(-\pi n^2 x) \leq \exp(-\pi n x)$, $x > 0$, $n \in \mathbb{N}$, that $\psi(x) = O(\exp(-\pi x))$ when $x \to \infty$. Since $\theta(x) = 2\psi(x) + 1$, then $\theta(x) = O(1)$ when $x \to \infty$. Moreover, $\psi(x) = (1/2)(\theta(x) - 1)$ and the functional relation

$$\theta(x) = x^{-1/2}\theta(1/x) \tag{0.12}$$

imply that $\psi(x) = O(x^{-1/2})$ when $x \to 0$. Since $\sigma > 1$, the integral on the right-hand side of (0.9) is absolutely convergent. The series which defines the function (0.10) is uniformly convergent on every compact subset of $(0, \infty)$. Hence,

$$\int_0^{\infty} \psi(x) x^{s/2-1}\, dx = \int_0^{\infty} \left(\sum_{n=1}^{\infty} \exp(-\pi n^2 x) \right) x^{s/2-1}\, dx$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} \exp(-\pi n^2 x) x^{-s/2-1}\, dx = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Further, after some computations, using (0.9) and

$$2\psi(x) + 1 = x^{-1/2}(2\psi(x^{-1}) + 1),$$

RIEMANN obtained the representation

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})\, dx,\ \Re s > 1. \tag{0.13}$$

The latter integral is uniformly convergent on every compact subset of the complex plane. This means that the left-hand side of (0.13) admits an analytic continuation in the whole complex plane except for the points 0 and 1. The origin is a pole of $\Gamma(s/2)$, while 1 is a simple pole of $\zeta(s)$ with residuum equal to one. Recall that the other poles of $\Gamma(s/2)$ are at the points $-2, -4, -6, \ldots$ and all they are simple. Equality (0.13) shows that all they are regular points for the already continuous function $\zeta(s)$. More precisely, these points are simple zeros of $\zeta(s)$ and they are called trivial zeros. Moreover, $\zeta(s)$ does not vanish at other points in the half-plane $\Re s < 0$. Since $\zeta(s) \neq 0$ when $\Re s > 1$, all other possible zeros of $\zeta(s)$ are in the closed strip $0 \leq \Re s \leq 1$ called the critical strip. The zeros of $\zeta(s)$ in the critical strip are called non-trivial.

The right-hand side of (0.13) does not change if we replace s by $1-s$. This leads immediately to the relation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s), \tag{0.14}$$

usually referred as a functional equation for $\zeta(s)$.

Let us set $s = 1/2 + iz$ and define the function $\xi(z)$ by the equality

$$\xi(z) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{0.15}$$

It is clear that $\xi(z)$ is an entire function. It takes real values for real z and

$$\xi(z) = \xi(-z), \tag{0.16}$$

which means that the functional equation for $\zeta(s)$ is equivalent to the fact that $\xi(z)$ is an even function. Thus, if ρ is a zero of $\xi(z)$, then $-\rho$, $\overline{\rho}$, and $-\overline{\rho}$ are also zeros of $\xi(z)$. Furthurmore, it is clear that all the possible zeros of $\xi(z)$ are in the strip $|\Im z| \leq 1/2$.

Observe that (0.13) and (0.15) imply

$$\xi(z) = \frac{1}{2} - \left(z^2 + \frac{1}{4}\right) \int_1^\infty \psi(x) x^{-3/4} \cos\left(\frac{z}{2}\log x\right)\, dx.$$

Integrating by parts and using the equality $4\psi'(1) + \psi(1) = -1/2$, RIEMANN obtained

$$\xi(z) = 4\int_1^\infty \left\{x^{3/2}\psi'(x)\right\}' x^{-1/4} \cos\left(\frac{z}{2}\log x\right)\, dx. \tag{0.17}$$

In his memoir RIEMANN claimed that the function $\zeta(s)$ has infinitely many zeros in the critical strip and that the following "explicit" formula

$$\begin{aligned} \pi(x) \quad &= \quad \mathrm{Li}(x) + \sum_{\rho \in \mathcal{N}, \Im\rho > 0} (\mathrm{Li}(x^\rho) + \mathrm{Li}(x^{1-\rho})) \\ &\qquad + \int_x^\infty \frac{dt}{(t^2-1)\log t} - \log 2, \quad x \geq 2. \end{aligned} \tag{0.18}$$

holds for the function $\pi(x)$, where Li is the integral logarithm and $\mathcal{N}$ is the set of zeros of $\zeta(s)$ in the critical strip. The relation (0.18) was established formally by H. VON MANGOLDT in 1895, in the paper **Zu Riemann's Abhandlung 'Uber die Anzahl der Primzahlen unter einer gegebenen Grösse'**, *J. Reine Angew. Math.* **114** (1895), 255–305.

Based on the last representation, RIEMANN made some comments which are astonishing indeed. He claims that the number of zeros of $\xi(z)$ whose real parts lie between 0 and T is about

$$\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}$$

Then he writes: "One finds in fact about this many real roots within these bounds and it is likely that all the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the search for a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation."

RIEMANN'S conjecture, that all the zeros of the function $\xi(z)$ are real, is equivalent to the conjecture that all the non-trivial zeros of the function $\zeta(s)$ are on the line $\Re s = 1/2$. The last one is called *The Rieman's Hypothesis*. In spite of the efforts made in the last 150 years, it is neither proved nor disproved and is considered as the most important open problem in mathematics.

One of the important relations between the zeros of the Riemann zeta-functions and The Prime Number Theorem is the fact that the latter is equivalent to the fact that $\zeta(1+it)\neq 0$ for every real $t\neq 0$, i.e, that the zeta function does not vanish on the boundary of the critical strip. This was realized by J. HADAMARD and J. DE LA VALLÉE POISSIN and served as a basis in their proofs of The Prime Number Theorem.

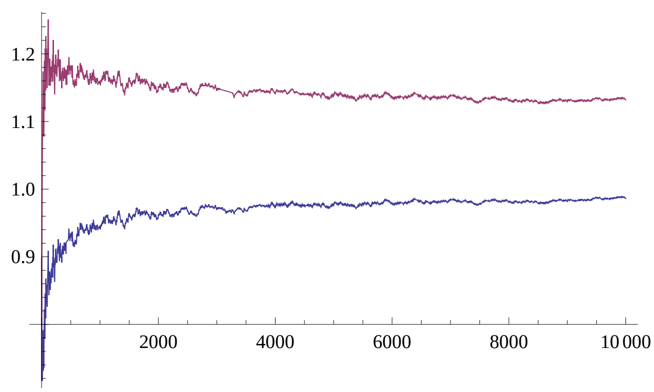

Figure 3: The graphs of $\pi(x)\log x/x$ and $\pi(x)/\mathrm{Li}(x)$ in the range $x\in[2,10000]$.

The asymptotics of the function $\pi^*(x)=\pi(x)-\mathrm{Li}(x)$ as $x\to\infty$ is still an open problem. RIEMANN'S explicit formula shows that its behaviour is strongly connected with the zero-distribution of $\zeta(s)$ in the critical strip. Indeed, all the results known till now confirm that the asymptotics of the function $\pi^*(x)$ depends on the absence of zeros of $\zeta(s)$ in subregions of the critical strip. It is quite evident from the above figures that $\mathrm{Li}(x)$ approximates $\pi(x)$ better than $x/\log x$. Moreover, it seems $\mathrm{Li}(x) > \pi(x)$, so one might believe that this inequality holds for all positive values of x. In fact Riemann himself, in his memoir, mentions that $\pi(x) \sim \mathrm{Li}(x)$ up to terms of order $x^{1/2}$ and gives a value which is slightly too large. In Figure 3 we show the graphs of the the functions $\pi(x)\log x/x$ and $\pi(x)/\mathrm{Li}(x)$ in the range $x \in [2, 10000]$, where Riemann's observations is quite clear. This means that the function $\pi(x)/\mathrm{Li}(x)$ approaches one from below. However, Littlewood proves in **Sur la distribution des nombres premiers**, *C. R. Acad. Sci. Paris* 158 (1914), 341–356, that the inequality $\mathrm{Li}(x) > \pi(x)$ fails. In fact Littlewwod proves that, for every $\varepsilon > 0$, there exist values of x for which $\pi(x) > \mathrm{Li}(x) + x^{(1/2)-\varepsilon}$.

In 1899 J. DE LA VALLÉE POUSIN showed that $\zeta(s)$ has no zeros in the region defined by the inequality $\sigma > 1 - A(\log(|t| + 2))^{-1}$ and as a corollary he obtained that $\pi^*(x) = O(x\exp(-a(\log x)^{1/2}))$ as $x \to \infty$. In 1922 J.E. LITLEWOOD proved that $\zeta(s) \neq 0$ if $\sigma > 1 - A\log(\log t)(\log t)^{-1}$, for every $t \geq t_0 > 0$ and thus concluded that $\pi^*(x) = O(x\exp(-a(\log x \log\log x)^{1/2}))$ (here and below A and a denote positive constants different in different cases). A sharpening of LITTLEWOOD'S results is given in 1936 by N.G. TCHUDAKOV. He proved that $\zeta(s) \neq 0$ when $\sigma > 1 - A(\log t)^{-3/4}(\log\log t)^{-3/4}$, provided t is sufficiently large, and this helped him to establish the limit relation $\pi^*(x) = O(x\exp(-a(\log x)^{-4/7}(\log\log x)^{-3/7}))$.

In 1958 I.M. VINOGRADOV and N.M. KOROBOV proved independently that $\zeta(s) \neq 0$ when $\sigma > 1 - A(\log(|t| + 3))^{-1/3}(\log\log(|t| + 3))^{-2/3}$. A corollary of this result is that

$$\pi^*(x) = O(x\exp(-a(\log x)^{3/5})(\log\log x)^{-1/5})$$

as $x \to \infty$. It seems the last asymptotic estimate is the best one known till now.

In 1901 H. VON KOCH proves that if RIEMANN'S hypothesis is true, then $\pi^*(x) = O(x^{1/2}\log x)$ as $x \to \infty$. Moreover, if $\pi^*(x) = O(x^{\theta+\varepsilon})$ for some fixed $\theta \in [1/2, 1)$ and arbitrary positive ε when $x \to \infty$, then $\zeta(s) \neq 0$ for $\sigma > \theta$. Since $\log x = O(x^\varepsilon)$ for every positive ε when $x \to \infty$, it follows that the validity of the estimate $\pi^*(x) = O(x^{1/2}\log x)$, $x \to \infty$ implies that the RIEMANN hypothesis is true. In 1976 L. Schoenfeld [Schoenfeld 1976] gives a quantitative version of von Koch's result proving that the Riemann hypothesis is equivalent to $|\pi(x) - \mathrm{Li}(x)| \leq x^{1/2}\log x/(8\pi)$ for $x > 2$. The figures below illustrate the result of von Koch and Schoenfeld.

Observe on Figure 4 how sharp Shoenfeld's estimate is.

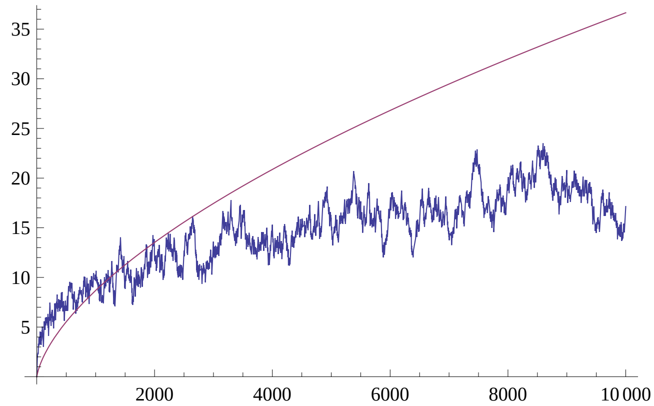

Figure 4: The graphs of $\mathrm{Li}(x) - \pi(x)$ and $x^{1/2}\log x/(8\pi)$ in the range $x \in [2, 10000]$.

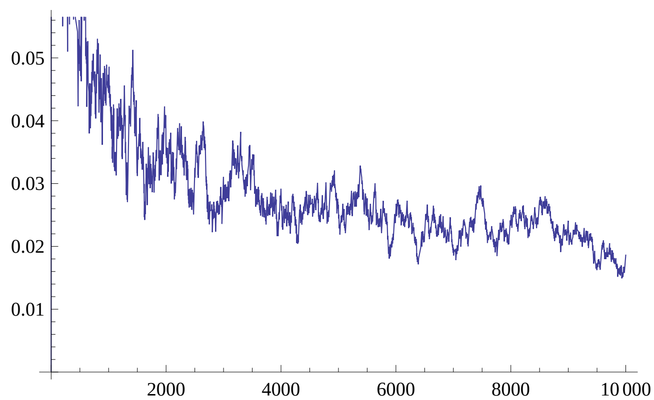

Figure 5: The graph of $(\mathrm{Li}(x) - \pi(x))/(x^{1/2}\log x)$ in the range $x \in [2, 10000]$.

Back to RIEMANN'S paper, recall that (0.17) is equivalent to

$$\xi(z) = 2\int_1^\infty \Psi(x)\cos((z/2)\log x)\,dx,$$

where

$$\Psi(x) = \{3\psi'(x) + 2x\psi''(x)\}x^{1/4}. \tag{0.19}$$

Then the change of x by $\exp 2u$, $-\infty < u < \infty$, yields

$$\xi(z) = 2\int_0^\infty \Phi(u)\cos zu\,du, \tag{0.20}$$

where

$$\Phi(u) = 2\Psi(\exp 2u)\exp 2u, \quad -\infty < u < \infty, \tag{0.21}$$

that is,

$$\Phi(u) = 2\sum_{n=1}^{\infty}(2\pi^2 n^4 \exp(9u/2) - 3\pi n^2 \exp(5u/2))\exp(-\pi n^2 \exp 2u). \tag{0.22}$$

It is not quite evident that the function $\Phi(u)$ is even, but this is really the fact. Indeed, by (0.19), (0.21) and the relation $x = \exp 2u$, we obtain

$$\Phi(-u) = 2\{3\psi'(x^{-1}) + x^{-1}\psi''(x^{-1})\}x^{-5/4}.$$

Further, the relation $2\psi(x)+1 = x^{-1/2}\{2\psi(x^{-1})+1\}$, $0 < x < \infty$, implies that

$$2\{3\psi'(x^{-1}) + x^{-1}\psi''(x^{-1})\}x^{-5/4} = 2\{s\psi'(x) + x\psi''(x)\}x^{5/4}, \quad 0 < x < \infty,$$

i.e. $\Phi(-u) = \Phi(u)$, $0 < u < \infty$. Then (0.20) can be written as

$$\xi(z) = \int_{-\infty}^{\infty} \Phi(u)\exp(izu)\,du, \tag{0.23}$$

or, equivalently,

$$\xi(z) = \int_{0}^{\infty} \Phi(t)\cos zt\,dt, \tag{0.24}$$

where the kernel $\Phi(t)$ is defined by (0.22).

It is know that infinitely many zeros of the function $\zeta(s)$ are located on the critical line $\Re s = 1/2$ which is equivalent to the existing of infinitely many real zeros of Rieman's ξ-function. The first proof of this fact was given by H. G. Hardy, **Sur les zéros de la fonction $\xi(s)$ de Riemann**, *C. R.* 153 (1914), 1012-1014. The Riemann hypothesis remains the most famous open problem in mathematics. However, the above considerations show that function $\xi(z)$ is an entire function which can be represented by either a Fourier transform or a cosine transform of the kernel $\Phi(t)$, $0 \le t < \infty$. Then a natural approach to the hypothesis is to establish criteria for an entire function, or more specifically, a Fourier transform of a kernel, to possess only real zeros and to apply them to the Riemann ξ function. There is no doubt this was the main reason that so many celebrated mathematicians have been interested in the zero distribution of entire functions and, in particular, of Fourier transforms. Among them are such distinguished masters of the Classical Analysis as A. Hurwitz, J.L.W. V. Jensen, G. Pólya, H.G. Hardy, E. Tichmarsh, W. de Bruin, N. Obrechkoff, L. Tchakaloff etc.

The purpose of this paper is to survey some old and new results concerning entire Fourier transforms with only real zeros .

In Section 1, entitled "The Laguerre-Pólya class", we recall some of the first ideas of Laguerre who introduced the class of entire functions of order at most two with only real zeros. A fundamental paper of Schur and Pólya, as well as further contributions, are discussed.

The paper of J.L.W.V. Jensen **Recherches sur la théorie des équations**, *Acta Math.*, 36 (1912/1913), 181-195 is discussed in Section 2 "Jensen's dream". We point out his contributions on the topic, especially the polynomials he introduced and bearing his name and the result where he reduces the problem of reality of the zeros of Riemann's ξ-function to the reality of zeros of corresponding sequences of algebraic polynomials as well as the idea to look for necessary and sufficient conditions on a kernel $K(t)$, defined for $t \in [0, \infty)$ in order that its cosine transform possesses only real zeros.

An essential part of this survey is Section 3 entitled "The great contributor" which aims to review some of Pólya's papers published during the period from 1918 to 1927. We begin the section with a discussion on the paper **Über die Nullstellen gewisser ganzer Funktionen**, *Math. Z.*, **2** (1918), 352–383. It is, in our opinion, the first systematic study on the distribution of zeros of entire functions defined by finite Fourier transforms of the form (1.13). The paper **Über trigonometrische Integrale mit nur reellen Nullstellen**, *J. r. angew. Math.*, **158** (1927), 6–16, is considered as containing the most remarkable Pólya's results concerning zeros of entire functions of the form (0.2). An extended review on his paper **Über die algebraish-funktionentheoretischen Untersuchungen von J.L.W.V Jensen**, *Kgl. Danske Vid. Sel. Math.-Fys. Medd.*, **7 (17)** 1927, 3-33, is included because there Pólya provides comprehensive information about Jensen's scientific heritage on this topic.

The paper of E.C. Titchmarsh, **The zeros of certain integral functions**, *Proc. London Math. Soc.*, **25** (1926), 283-302, is discussed in Section 4 "A knight of the classical analysis". It is the first contribution where the zero distribution of entire Laplace transforms of the form (1.16) is studied.

The paper **The roots of trigonometric integrals**, *Duke Math. J.*, **17** (1950), 197-226 of N.G. de Bruin is considered in Section 5 "The Dutch master". There one finds all essential generalizations of well-known Pólya's examples of entire Fourier transforms with only real zeros.

In Section 6 "The Bulgarian trace", we survey the most significant results of investigations of at least three generations of Bulgarian mathematicians. We discuss results of L. Tschakaloff, N. Obrechkoff, L. Ilieff and of their students and successors. It would be not so magnified to say that all they gave specific approaches to the zero distribution of entire Fourier transforms and, thus, justifying the term *Bulgarian School* in this field.

In Section 7 "The Hawaii school and the Hungarian connection" we survey some joint papers of George Csordas with his colleague Thomas Craven as well as with Richard Varga and István Vincze.

Some recent results on the theme are discussed in the last Section 8 "Variations on classical themes".

Comments and references

1. Euler's identity (0.7), considered by him only when s is a real number, is equivalent to the Fundamental Theorem of Arithmetic which states that if $p_1 < p_2 < p_3 < \dots$ are the prime numbers, then every natural number $n \geq 2$ has a unique representation of the form $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$, where α_j, $1 \leq j \leq k$, are nonnegative integers.

2. The complex function

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi n^2 \tau + 2i\pi n z), \quad z \in \mathbb{C}, \quad \Im\tau > 0,$$

is one of the theta-functions introduced by C.G.J. Jacobi in his paper **Fundamenta nova theoriae functionum ellipticarum**, Regiomonty, Sumtibus Fratorum Borntraeger, 1829. The function $\theta(0,\tau)$ is holomorphic in the upper half-plane and satisfies there the relations $\theta(0,\tau+2) = \theta(0,\tau)$ and $\theta(0,-1/\tau) = (-i\tau)^{1/2}\theta(0,\tau)$, where $(-i\tau)^{1/2} := \exp((1/2)\log(-i\tau))$. We refer to Chapter 21 of the classical book of E. T. Whittaker and G. N. Watson, **A Course of Modern Analysis**, Cambridge University Press, Cambridge, 1902, and to p. 17 of Joseph Lehner's, **Discontinuous Groups and Automorphic Functions**, Amer. Math. Soc., Providence, Rhode Island, 1964, where the function $\theta(z,\tau)$ is denoted by $\theta_3(z|\tau)$ and $\theta_3(0|\tau)$ is denoted by $\theta_3(\tau)$. It is clear that the function $\theta(x)$, defined by (0.11) is just $\theta(0,ix)$, $x > 0$. Then, the second of the above relations yields that $\theta(x) = x^{-1/2}\theta(1/x)$. A proof of the last equality can be found also in H. Davenport, **Multiplicative Number Theory**, Markham Publishing Company, Chicago, 1967.

3. The series on the right-hand sides of the equalities

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad L(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s},$$

as Dirichlet series are uniformly convergent on every compact subset of the half-plane $\Re s > 0$. The holomorphic functions defined by them satisfy the functional equations

$$(2^{s-1}-1)\eta(1-s) = -(2^s-1)\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\eta(s)$$

and

$$L(1-s) = 2^s\pi^{-s}\sin\frac{\pi s}{2}\Gamma(s)L(s).$$

The first of them is equivalent to the functional equation for $\zeta(s)$. The second is contained in a paper of L. Euler published in 1749 and verified by him only for some real values of s.

In fact, the functions $\eta(s)$ and $L(s)$ are analytically continuable as entire functions. Moreover, $\eta(s) = (1-2^{1-s})\zeta(s)$ (see Chapter II, 2.2 in G.H. HARDY, **Divergent Series**, Oxford Univ. Press, Oxford, 1949).

4. If $0 < x < 1$, the li-function is defined by

$$li(x) = \int_0^x \frac{dt}{\log t}.$$

If $x > 1$, then

$$li(x) := \lim_{\delta \to +0} \left\{ \int_0^{1-\delta} \frac{dt}{\log t} + \int_{1+\delta}^{x} \frac{dt}{\log t} \right\}.$$

It is obvious then that, when $x > 2$, we have

$$li(x) = li(2) + \mathrm{Li}(x).$$

It is well-known that

$$li(x) = \gamma + \log(-\log x) + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n}$$

for $0 < x < 1$, where γ is EULER'S constant.

If $z \in G = \mathbb{C} \setminus \{(-\infty, 0] \cup [1, \infty)\}$, then by definition

$$li(z) = \int_0^z \frac{dw}{\log w}$$

provided the path of integration is a rectifiable curve joining the points 0 and z in $G \cup \{0\}$. The equality

$$li(z) = \gamma + \log(-\log z) + \sum_{n=1}^{\infty} \frac{(\log z)^n}{n!n}$$

holds for $z \in G$. It turns out that $li(z)$ is a holomorphic extension of $li(x)$, $0 < x < 1$ in the region G.

1. The Laguerre-Pólya class

The first natural question which arises when one realizes the connection between the Riemann hypothesis and the representation of the ξ function as an entire one, is to to characterize those functions whose zeros are all real. We do not know if LAGUERRE was interested in the work of Riemann and his conjecture, but we certainly know that he was the first to study and obtain a representation in terms of WEIERSTRASS infinite product of the entire functions which are local uniform limits of sequences with either real zeros with one and the same sign or of polynomials with real zeros only (see pp. 174–177 of Laguerre's **Oeuvres** [Laguerre 1898]). Here, by local uniform convergence we mean uniform convergence in every compact subset of $\mathbb{C}$. G. PÓLYA, **Über die Annährung durch Polynome mit lauter reellen Wurzeln**, *Rend. Circ. Math. Palermo*, 36 (1913), 279-295, made the simple observation that LAGUERRE'S representation remains true if we consider uniform convergence of sequences of polynomials with real zeros in a fixed disk centered at the origin. We refer to N. OBRESHKOFF, **Zeros of polynomials**, *Bulgarian Academic Monographs* (7), Marin Drinov Academic Publishing House, Sofia, 2003, for nice proofs of LAGUERRE'S theorems.

G. PÓLYA and J. SCHUR published the very comprehensive paper **Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen**, *J. r. angew. Math.*, **144**, 2 (1914), 89–113, where they offer their own proofs and introduced a new terminology. Nowadays this terminology has changed slightly and we use the contemporary one.

According to G. PÓLYA and J. SCHUR, an entire function is called of I-st type if it is a uniform limit of real polynomials with only real zeros of one and the same sign, and it is said to be of II-nd type if it is uniform limit of real polynomials with only real zeros. Then, LAGUERRE'S theorems can be formulated as follows:

The entire function $f(z)$ is of I-*st type if and only if*

$$f(z) = az^m \exp(\mu z) \prod_{k=1}^{\omega} \left(1 - \frac{z}{a_k}\right), \quad 0 \le \omega \le \infty, \tag{1.1}$$

where $a, \mu \in \mathbb{R}$, m is a non-negative integer, $a_k \in \mathbb{R}$, $a_k \neq 0$ have the same sign and the series $\sum_{k=1}^{\omega} a_k^{-1}$ is convergent.

The entire function $f(z)$ is of II-*nd type if and only if*

$$f(z) = bz^n \exp(-\lambda z^2 + \mu z) \prod_{k=1}^{\omega} \left(1 - \frac{z}{b_k}\right) \exp\left(\frac{z}{b_k}\right), \quad 0 \le \omega \le \infty, \tag{1.2}$$

where $b, \mu \in \mathbb{R}$, n is a non-negative integer, $\lambda \ge 0$, $b_k \in \mathbb{R}$, $b_k \neq 0$, and the series $\sum_{k=1}^{\omega} b_k^{-2}$ is convergent.

When $\omega = 0$, then the products on the right-hand sides of (1.1) and (1.2) are assumed to be equal to one.

Nowadays the functions of type II are said to belong to the Laguerre-Pólya class and in this case we write $f \in \mathcal{LP}$. Similarly, if either $f(z)$ or $f(-z)$ is of type I, we write $f \in \mathcal{LP}I$. In this paper we use an additional denotation. The class of functions $f \in \mathcal{LP}I$ whose Maclaurin coefficients are all positive and its zeros, except for the one at the origin, are negative, are denoted by $f \in \mathcal{LP}I^+$.

Moreover, observe that if ω is a positive integer and the exponents do not appear, the above functions reduce to polynomials. The polynomials $p(z)$ with real coefficients with only real zeros are called *hyperbolic polynomials* and we write $p \in H$ if it is so. Similarly, if $p(z)$ has positive coefficients and only real negative zeros, we denote this fact by $p \in H^+$.

We point out that the above results of Laguerre say that $\mathcal{LP}$ and $\mathcal{LP}I^+$ are complements, in the sense of the local uniform convergence of H and H^+, respectively.

PÓLYA and SCHUR provided the following important criterion for a convergent power series to define an entire function of I-st or of II-nd type (see [Pólya Schur 1914, p. 110]):

A power series

$$f(z) = \gamma_0 + \frac{\gamma_1}{1!}z + \frac{\gamma_2}{2!}z^2 + \dots \tag{1.3}$$

is an entire function of II-*nd, respectively* I-*st type, if and only if the algebraic equations*

$$\gamma_0 z^n + \binom{n}{1}\gamma_1 z^{n-1} + \binom{n}{2}\gamma_2 z^{n-2} + \dots + \gamma_n = 0, \quad n = 0, 1, 2, \dots \tag{1.4}$$

have only real zeros respectively only real zeros with one and the same sign.

It is worth mentioning that this result was established by Jensen and we shall discuss this in the section devoted to his work.

PÓLYA and SCHUR defined two types of sequences and established their tight relation to the above classes of entire functions. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called by a *multiplier sequence* of I-st (II-nd) type if, for every polynomial $\sum_{k=0}^{n} a_k z^k$, which belongs to H (H^+), the composite polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ is hyperbolic. They proved the following result:

The sequence $\{\gamma_k\}_{k=0}^{\infty}$ *is of* I-*st* (II-*nd*) *type if and only if the entire function* (1.3) *is of* I-*st* (II-*nd*) *type.*

They called this an transcendental criterion for a sequence of real number to be of I-st (II-nd) type. Another criterion, called algebraic, is an immediate corollary of the transcendental one:

The sequence $\{\gamma_k\}_{k=0}^{\infty}$ *is of* I-*st* (II-*nd*) *type if and only if the polynomials on the left-hand sides of the equations* (1.4) *are of* I-*st* (II-*nd*) *type.*

As we shall see, this latter result was proved first by JENSEN. Observe that, in the contemporary terms, a polynomial $p(z)$ is of second type if and only if it is hyperbolic while it is if the first type if either $p(z)$ or $\pm p(-z)$ belongs to H^+.

On p. 90 of [Pólya Schur 1914] the authors point out that the first examples of multiplier sequences are due to E. LAGUERRE (**Oeuvres, I**, 31-35, 199-206). More precisely, the sequences

$$1, \frac{1}{\omega}, \frac{1}{\omega(\omega+1)}, \frac{1}{\omega(\omega+1)(\omega+2)}, \dots, \omega > 0,$$

$$1, q, q^4, q^9, \dots, |q| \leq 1,$$

are of I-st type, while

$$\cos\lambda, \cos(\lambda+\theta), \cos(\lambda+2\theta), \dots, \lambda, \quad \theta \in \mathbb{R}$$

is of II-nd type.

It is obvious that each sequence of I-st type is also of II-nd type, but the converse is not true. Indeed, as it is noted on the same p. 90, if θ is not an integral multiple of π, then the third of the above sequences is not of I-st type.

Despite that the main objective of this paper is to survey results on functions in the subclass of $\mathcal{LP}$ which are Fourier transforms of certain kernels, we shall recall briefly some other contributions which provide general necessary and/or sufficient conditions for a function to belong to the Laguerre-Pólya class. Let us begin with the simplest necessary conditions. Suppose the entire function (1.3) is in the $\mathcal{LP}$ class. This means that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is either of I-st or of II-nd type. Since the polynomial $x^{k-1}+2x^k+x^{k+1}$, $k=1,2,3,\dots$ has only real roots and its non-zero root is negative, the polynomial $\gamma_{k-1}x^{k-1}+2\gamma_k x^k+\gamma_{k+1}x^{k+1}$ has only real zeros for every $k \in \mathbb{N}$. Hence,

$$\gamma_k^2 \geq \gamma_{k-1}\gamma_{k+1}, \quad k=1,2,3,\dots \tag{1.5}$$

Thus, *A necessary condition that the power series* (1.3) *to defines an entire function belonging to the class* $\mathcal{LP}$ *is that the inequalities* (1.5) *hold.*

These inequalities are usually called the Turán inequalities though they appear first in [Pólya Schur 1914]. We shall discuss later the contributions of Turán, Szegő and Pólya as well as the history of the Turán inequalities for the coefficients of the Riemann ξ function.

Laguerre proved that if the entire function (1.3) is in $\mathcal{LP}I$, then the inequality

$$(f^{(n)}(x))^2 \geq f^{(n-1)}(x)f^{n+1}(x), \quad x \in \mathbb{R} \tag{1.6}$$

holds for every $n \in \mathbb{N}$. It is clear that (1.6) reduces to (1.5) if we set $x=0$, that is, for entire functions of the class $\mathcal{LP}I$, Turán's inequality (1.5) is a consequence of Laguerre's inequality (1.6).

It is proved in the paper of J.L.W.V. Jensen, Recherches sur la théorie des équations, *Acta Math.*, **36** (1912/1913), 181–195, that if $f \in \mathcal{LP}$, then

$$\sum_{j=0}^{2m}(-1)^{j+m}\binom{2m}{j} f^{(j)}(x)f^{(2m-j)}(x) \geq 0, \quad x \in \mathbb{R}. \tag{1.7}$$

Jensen's result was rediscovered and extended by Patrik, **Extension of inequalities of the Laguerre and Turán type**, *Pacific J. Math.* **44** (1973), 675–682, who proved that that if $f \in \mathcal{LP}$, then the inequality of LAGUERRE type

$$\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} f^{(n+j)}(x) f^{(n+2k-j)}(x) \geq 0$$

holds for $x \in \mathbb{R}$, $n \geq 1$ and $k \geq 1$. An immediate corollary is an inequality of PÓLYA-SCHUR type for each system of real functions having generating function in the $\mathcal{LP}$ class. However, it is worth mentioning that the above inequalities of Jensen and Patrik follow immediately from some nice results for hyperbolic polynomials, obtained by G. Nikolov and R. Uluchev in their paper [Nikolov Uluchev 2004]. They prove that, if $f(x)$ is a hyperbolic polynomial of degree n and $0 \leq 2m \leq n$, then

$$\sum_{j=0}^{2m} (-1)^{j+m} \binom{2m}{j} \frac{(n-j)!(n-2m-j)!}{(n-m)!(n-2m)!} f^{(j)}(x) f^{(2m-j)}(x) \geq 0 \tag{1.8}$$

for every $x \in \mathbb{R}$. It is easily seen that if we let n to go to infinity in (1.8) we obtain Jensen's inequalities (1.7). Nikolov and Uluchev provide two independent proofs of (1.8). The second one is based on an interesting result of Obrechkoff, obtained in his last published paper [Obrechkoff 1963]. If $f_1(z)$ is a polynomial of degree p and $f_2(z)$ is a polynomial of degree q and m is a natural number not exceeding p and q, Obrechkoff considers the polynomial

$$g_m(z) = \sum_{j=0}^{m} (-1)^j \binom{p-j}{m-j} \binom{q-m+j}{j} f_1^{(j)}(z) f_2^{(m-j)}(z)$$

Though the degree of $g_m(z)$ it seems to be $p+q-m$, it does not exceed $p+q-2m$. The Obrechkoff's result states that if the zeros of $f_1(z)$ belong to the circular domain K_1 and those of $f_2(z)$ are in the circular domain K_2, where K_1 and K_2 do not have common points, then the polynomial $g_m(z)$ has exactly $p-m$ zeros on in K_1, exactly $q-m$ zeros in K_2, and there are no zeros of $g_m(z)$ outside K_1 and K_2.

An interesting result is given in the paper of THOMAS CRAVEN and GEORGE CSORDAS, **Jensen polynomials and the Turán and Laguerre inequalities**, *Pacific J. Math.* **136**, No 2 (1989) 241–260. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k, \tag{1.9}$$

be a real entire function. Denote by $g_{n,p}(z)$, $n, p = 0, 1, 2, \ldots$ the JENSEN polynomials of the p-th derivative of $f(z)$,

$$g_{n,p}(z) = g_{n,p}(f; z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_{p+k} z^k.$$

It is clear that $g_{n,p}(f;z) = g_{n,0}(f^{(p)};z)$. Then the authors prove:

THEOREM 2.3 *Suppose that $\gamma_k > 0, k = 0,1,2,\ldots$. Then the following are equivalent:*

$$T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \tag{1.10}$$

for $k = 1,2,3,\ldots$,

$$\Delta_{n,p}(t) := g_{n,p}^2(t) - g_{n-1,p}(t)g_{n+1,p}(t) \geq 0 \tag{1.11}$$

for all $t \geq 0$ and $n = 1,2,3\ldots, p = 0,1,2,\ldots$,

$$L_p(f(t)) := (f^{(p+1)}(t))^2 - f^{(p)}(t)f^{(p+2)}(t) \geq 0 \tag{1.12}$$

for all $t \geq 0$ and $p = 0,1,2,\ldots$

The following necessary and sufficient conditions the real entire function (1.9) to be in the LAGUERRE-PÓLYA class are established in Theorem 2.7 of [Craven Csordas 1989]: If $\gamma_0 \neq 0$ and $\gamma_{k-1}\gamma_{k+1} < 0$ whenever $\gamma_k = 0$, $k \in \mathbb{N}$, then $f \in \mathcal{LP}$ if and only if $\Delta_n(t) := \Delta_{n,0}(t) > 0$ for $t \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$.

An important characterization of the functions in the Laguerre-Pólya class $\mathcal{LP}^+$ is a result due to Aisen, Edrei, Schoenberg and Witney. Recall that a matrix is said to be totally positive if all its minors are nonnegative. The main result in [Aissen Edrei Schoenberg Whitney 1951] reads as follows:

Let the series

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k \geq 0,$$

represents an entire function. Then it has only real zeros if and only if the infinite upper triangular matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & \ddots \\ & a_0 & a_1 & \cdots & a_{n-1} & \ddots \\ & & a_0 & \cdots & & \ddots \\ & & & \ddots & \vdots & \\ & \mathbf{0} & & & a_0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

is totaly positive.

Briefly, $\varphi \in \mathcal{LPI}^+$ if and only if the matrix is totally positive.

Now, it is evident that the truth of the RIEMANN Hypothesis is equivalent to the statement that either the entire function $\xi(z)$, defined by (0.24) is in the class $\mathcal{LP}$, or the entire function $\xi(z^{1/2})$ is in $\mathcal{LPI}$. That is why, there was a strong hope that a proper characterization of the functions in the LAGUERRE-PÓLYA class would lead to a proof of RIEMANN'S Hypothesis. Despite that many important and nice necessary and/or sufficient conditions for a function to belong to $\mathcal{LP}$ were obtained, the results turned out to be so involved that it was impossible to apply them to RIEMANN'S ξ-function. Thus, the great expectations were not justified and the research on this interesting topic was slowly abandoned around the middle of the last century.

Recall that

$$\mathcal{F}_{a,b}(K;z) = \int_a^b K(t) \exp izt\, dt, \quad -\infty \le a < b \le \infty,$$

$$\mathcal{E}(K;z) = \int_{-\infty}^{\infty} K(t) \exp izt\, dt,$$

$$\mathcal{U}(K;z) = \int_0^{\infty} K(t) \cos zt\, dt$$

and

$$\mathcal{V}(K;z) = \int_0^{\infty} K(t) \sin zt\, dt,$$

If $-\infty < a < b < \infty$, then setting $t \longrightarrow t + (a+b)/2$, we obtain $\mathcal{F}_{a,b}(K;z) = \exp(i(a+b)zt/2) E_\sigma(\kappa;z)$, where $\sigma = (b-a)/2, \kappa(t) = K(t + (a+b)/2)$ and

$$E_\sigma(\kappa;z) = \mathcal{F}_{-\sigma,\sigma}(\kappa;t) = \int_{-\sigma}^{\sigma} \kappa(t) \exp(izt)\, dt \tag{1.13}$$

Particular cases of (1.13) are

$$U_\sigma(\kappa;z) = \int_0^{\sigma} \kappa(t) \cos zt\, dt \tag{1.14}$$

and

$$V_\sigma(\kappa;z) = \int_0^{\sigma} \kappa(t) \sin zt\, dt \tag{1.15}$$

corresponding again to the cases when $\kappa(t)$ is either even or odd in $(-\sigma,\sigma)$.

Another straightforward observation is that the study of zero-distribution of the entire functions of the form

$$\mathcal{F}_{a,\infty}(K;z) = \int_a^{\infty} K(t) \exp(izt)\, dt, \quad -\infty < a < \infty$$

and

$$\mathcal{F}_{-\infty,b}(K;z) = \int_{-\infty}^{b} K(t) \exp(izt)\, dt, \quad -\infty < b < \infty$$

can be reduced, by a simple linear changes of the variables t and z to that of the entire functions of the form

$$\mathcal{F}_{0,\infty}(\kappa; z) = \int_0^\infty \kappa(t) \exp(izt)\, dt.$$

Another object of this survey, closely related to the principal one, is the zero distribution of entire functions defined by Laplace-type transforms of the form

$$\mathcal{L}_{a,b}(K; z) = \int_a^b K(t) \exp zt\, dt, \quad -\infty \le a < b \le \infty. \tag{1.16}$$

Since $\mathcal{F}_{a,b}(K; z) = \mathcal{L}_{a,b}(K; iz)$, then any information about the zeros of a function of the form (1.16) can be carried immediately to a result about the zeros of the corresponding function of the form (0.1). Of course, the study of the entire functions (1.16) offers some technical advantages, while the investigation of the location of the zeros of the entire functions (0.1) requires rather specific tools. Again, if $-\infty < a < b < \infty$, a linear change of the variable t reduces the study of zero distribution of the entire function (1.16) to that of the entire function $L_\sigma(\kappa; z) = \mathcal{L}_{0,\sigma}(\kappa; z)$, $0 < \sigma < \infty$.

Another approach to the zero distribution of entire Fourier transforms, that would have helped attacking Rieman's hypothesis, arose in the twenties of the last century and it is exactly the idea for looking for necessary and sufficient conditions for a Fourier transform of a kernel to belong to $\mathcal{LP}$. Though this idea is rather clear, it seems that the first who emphasized the importance of this problem was G. Pólya. His short paper **On the zeros of certain trigonometric integrals**, *J. London Math. Soc*, 1(1926), 98–99, begins as follows:

"What properties of the Function $F(u)$ are sufficient to secure that the integral

$$2\int_0^\infty F(u) \cos zu\, du = G(z) \tag{1}$$

has only real zeros? The origin of this rather artificial question is the Riemann hypothesis concerning the Zeta-function. If we put

$$F(u) = \sum_{n=1}^\infty (4\pi^2 n^4 e^{\frac{9}{2}u} - 6\pi n^2 e^{\frac{5}{2}u}) e^{-\pi n^2 e^{2u}}, \tag{2}$$

$G(z)$ becomes Riemann's function $\xi(z)$".

We hasten to remark that $\Phi(t) = 2F(t)$, where $\Phi(t)$ is defined by (0.24) and $F(t)$ is given in (2).

G. Pólya, as well as many other mathematicians, provided such sufficient conditions. Unfortunately, the problem of verifying these conditions for the function $\Phi(t)$ remained intractable. Nevertheless, there are many interesting and challenging problems concerning zeros of Fourier transforms, so that the interest on the topic has revived and there are various recent publications on the topic.

Comments and references

Nowadays it becomes customary the inequalities (1.5) as well as their numerous generalizations and extensions to be called TURÁN'S inequalities as well as inequalities of TURÁN type. Nevertheless, it seems that they appear still in the work of Newton and have been obtained in different ways for hyperbolic polynomials and entire functions in $\mathcal{LP}$ by many mathematicians. For example, they follow immediately from LAGUERRE'S inequalities and appear in an explicit form in the the paper of PÓLYA and SCHUR.

The inequalities

$$(P_n(x))^2 \geq P_{n-1}(x)P_{n+1}(x), \quad -1 \leq x \leq 1, \quad n = 1, 2, 3, \ldots, \tag{1.17}$$

where $\{P_n(x)\}_{n=0}^{\infty}$ are LEGENDRE'S polynomials, were established by P. TURÁN'S somewhere in the middle of the forties of the twentieth century but he published his proof later, in an appendix of his paper **On the zeros of the polynomials of Legendre**, *Čas. pest. mat. fys.* **75** (1950), 112–122. The first who called the attention to (1.17) was G. SZEGŐ, **On an inequality of P. Turán concerning Legendre polynomials**, *Bul. Amer. Math. Soc.* **54** (1948) 401–405. The third proof of this inequality, given is SZEGO'S paper, is attributed to PÓLYA and is based on the generating function

$$\exp(xz)J_0(z\sqrt{1-x^2}) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n, \quad -1 \leq x \leq 1, \quad z \in \mathbb{C}.$$

Since the entire function on the left is in $\mathcal{LP}$, inequality (1.17) follows from the fact that the Maclaurin coefficients γ_k of every entire function from $\mathcal{LP}$ satisfy the Turán inequalities $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$, $k \in \mathbb{N}$. Analogues of (1.17) hold for the ultraspherical, Laguerre and Hermite polynomials.

The method of generating functions for obtaining extensions of (1.17) was applied in [Skowgard 1954] and [Dimitrov 1998].

2. Jensen's dream

The paper of J.L.W.V. JENSEN, **Recherches sur la théorie des équations**, *Acta Math.* 36 (1912/1913), 181-195, contains the results he reported at The Second Congress of Scandinavian Mathematicians, held in Kopenhagen in 1911.

JENSEN'S idea, proposed in this paper, is to reduce the study of zero distribution of an entire function to that of a sequence of polynomials generated by the function itself.

Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \ldots$$

be an entire function. JENSEN introduced the polynomials

$$\begin{aligned} (2.1)\quad A_n(f;z) &= f(D)z^n = \sum_{k=0}^{n} n(n-1)(n-2)\ldots(n-k+1)a_k z^{n-k} \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!} a_k z^{n-k}, \quad n = 0,1,2,\ldots, \end{aligned}$$

where

$$f(D) = \sum_{k=0}^{\infty} a_k D^k$$

is the differential operator defined by the function f.

Nowadays the polynomials

$$(2.2) \qquad g_n(f;z) = z^n A_n(f;1/z) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} a_k z^k, \quad n = 0,1,2,\ldots$$

are called the JENSEN polynomials while $A_n(f;z)$ are called the APPELL polynomials, associated with the function f. Observe that, if

$$\varphi(z) = \sum_{j=0}^{\infty} \gamma_j \frac{z^j}{j!},$$

then

$$g_n(\varphi;z) = \sum_{k=0}^{n} \binom{n}{j} \gamma_j \, z^j$$

and

$$A_n(\varphi;z) = \sum_{k=0}^{n} \binom{n}{j} \gamma_j \, z^{n-j}.$$

Very important sequences of polynomials are the generalized, or shifted, JENSEN and APPELL polynomials and we define them here though JENSEN himself does not do it in his paper. If φ is defined as above and $n, k \in \mathbb{N}$, then

$$g_{n,k}(\varphi;z) = \sum_{k=0}^{n} \binom{n}{j} \gamma_{k+j} \, z^j$$

and

$$A_{n,k}(\varphi; z) = \sum_{k=0}^{n} \binom{n}{j} \gamma_{k+j}\, z^{n-j}.$$

As JENSEN points out, it is an easy exercise on the basis of CAUCHY'S inequalities for TAYLOR'S coefficients of a holomorphic function to prove that

$$\lim_{n\to\infty} g_n(f; z/n) = f(z) \tag{2.3}$$

uniformly on every bounded subset of $\mathbb{C}$.

Let ν_n be the number of the non-real zeros of $g_n(z)$. If $\nu_n \leq \nu$, $\nu \in \mathbb{N}_0$ when n is sufficiently large, then (2.3) and a classical theorem due to A. HURWITZ lead to the conclusion that f has at most ν non-real zeros. In particular, if there is $n_0 \in \mathbb{N}$ such that $g_n(z)$ have only real zeros for all $n > n_0$, then f has only real zeros.

An interesting property of Appell's polynomials is the relation

$$A_n'(f; z) = nA_{n-1}(f; z), \quad n = 1, 2, 3. \dots. \tag{2.4}$$

It is worth mentioning that the term Appell polynomials is used for sequences of polynomials, such that the sequence is invariant under differentiation, that is, when the polynomials satisfy $A_n'(z) = C_n A_{n-1}(z)$, where C_n are nonzero constants. Very important sequence of Appell polynomials are those composed by the Hermite and Bernoulli polynomials.

Suppose that f is a real entire function, i.e. all the coefficients of the power series in (2.1) are real. If $A_n(f; z)$ has only real zeros for some $n \geq 1$, then (2.4) and ROLLE'S theorem yield that the same holds for the roots of polynomials $A_k(f; z), k = 1, 2, 3, \dots, n-1$. Moreover, if the non-zero roots of $A_n(f; z)$ are of one and the same sign, then the non-zero roots of all $A_k(f; z), k = 1, 2, 3, \dots, n-1$, are also of the same sign.

Further, the problem whether all JENSEN'S polynomials of an entire function may have only real zeros is discussed. Before quoting the corresponding assertion let us consider some examples:

1) Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a real polynomial having only real zeros. Then, by a Theorem of E. MALO, **Note sur les équationes algébriques dont touts les racines sont réelles**, *J. Math. Spec.* 4 (1895), 7-10, the polynomial

$$P_M(z) = \sum_{k=0}^{n} \frac{a_k}{k!} z^k$$

has only real zeros too. Thus, its reverse

$$P_M^*(z) = \sum_{k=0}^{n} \frac{a_{n-k}}{(n-k)!} z^k$$

is also hyperbolic. Applying Malo's result to it, we conclude that the polynomials

$$\sum_{k=0}^{n}\binom{n}{k}a_{n-k}z^k \quad \text{and} \quad \sum_{k=0}^{n}\binom{n}{k}a_k z^k$$

possess only real zeros. Similar arguments, combining Malo's result and Rolle's theorem to a hyperbolic polynomial $P(z)$ or to its reverse polynomial, yield that all Appell and Jensen polynomials associated with $P(z)$ are also hyperbolic.

2) As we have already mentioned, if $\varphi(z)$ is an entire function whose Maclaurin expansion is

$$\varphi(z) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}z^k, \tag{2.5}$$

where and $\gamma_k = f^{(k)}(0), k = 0, 1, 2, \ldots$, then its JENSEN polynomials are

$$g_n(f;z) = \sum_{k=0}^{n}\binom{n}{k}\gamma_k z^k, \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

In particular, for the function $f(z) = \exp(\mu z)$, $g_n(f;z) = (1 + \mu z)^n$ and $A_n(f;z) = (z + \mu)^n$, $n \in \mathbb{N}_0$. Thus, if μ is a nonzero real number, then all the JENSEN and APPELL polynomials of the function $\exp(\mu z)$ have only real zeros.

3) Let now $f(z) = \exp(-\lambda z^2), \lambda \in \mathbb{R}$, then

$$g_n(f;z) = n!\sum_{k=0}^{[n/2]}\frac{(-1)^k\lambda^k z^{2k}}{k!(n-2k)!} = n!\lambda^{n/2}\sum_{k=0}^{[n/2]}\frac{(-1)^k(\lambda^{-1/2}z)^{n-2k}}{k!(n-2k)!}.$$

If $H_n(z)$ is the n-th HERMITE polynomial (see (5.5.4) in G. SZEGŐ'S classical book [Szegő 1975], then

$$H_n(z) = n!\sum_{k=0}^{[n/2]}\frac{(-1)^k(2z)^{n-2k}}{k!(n-2k)!}, \quad n = 0, 1, 2, \ldots.$$

Hence, $g_n(f;z) = \lambda^{n/2}H_n(\lambda^{-1/2}z/2)$ for every $n \in \mathbb{N}$. It is clear that if λ is positive, then $g_n(f;z)$ has only real zeros for every $n = 1, 2, 3, \ldots$. If λ is negative, then $\lambda^{-1/2} = i(-\lambda)^{-1/2}$, hence, $g_{2n}(f;z)$ has no real zero for every $n \in \mathbb{N}$ and the only real zero of $g_{2n+1}(f;z)$, $n \in \mathbb{N}_0$ is at the origin.

4). The BESSEL function of the first kind $J_\alpha(z)$ with parameter $\alpha > -1$ has the series expansion

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(k+\alpha+1)} \tag{2.7}$$

in the region $\mathbb{C} \setminus (-\infty, 0)]$. Therefore, the series

$$B_\alpha(z) = z^{-\alpha/2} J_\alpha(2z^{1/2}) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\Gamma(k+\alpha+1)}$$

represents an entire function. Then

$$\begin{aligned} g_n(B_\alpha; z) &= \sum_{k=0}^{n} \frac{(-1)^k n(n-1)(n-2)\dots(n-k+1)z^k}{k!\Gamma(k+\alpha+1)} \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \sum_{k=0}^{n} \frac{(-1^k)\Gamma(n+\alpha+1)z^k}{k!(n-k)!\Gamma(k+\alpha+1)} \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} z^k, \end{aligned}$$

that is,

$$g_n(B_\alpha; z) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(z), \quad n = 0, 1, 2, \dots,$$

where $\{L_n^\alpha(z)\}_{n=0}^\infty$ are LAGUERRE'S polynomials (see (5.1.6) in G. SZEGÖ'S book). These polynomials have only real and positive zeros provided $\alpha > -1$ and the same holds for the polynomials $g_n(B_\alpha; z), n = 0, 1, 2, \dots$. Since $\lim_{n\to\infty} g_n(B_\alpha; z/n) = B_\alpha(z)$ uniformly on every compact subset of $\mathbb{C}$, then $B_\alpha(z)$ has also only real and positive zeros. As a corollary we obtain the well-known fact that the Bessel function $J_\alpha(z)$ has only real zeros when $\alpha > -1$.

On page 187 of the paper JENSEN formulates a statement which he called the "théorème fondamental", which says:

A real entire function $F(z)$ of order less than two has at most 2ν non-real zeros if and only if the same holds for the polynomials $F(D)z^p$, $p = 1, 2, 3, \dots$.

Further, as an application of his fundamental theorem, JENSEN proved a generalization of a classical composition theorem due to SCHUR and MALO. See the above observations concerning their results about polynomials. JENSEN showed that if $F(z)$ and $G(z)$ are real entire functions of order less than two and such that $F(z)$ has only real zeros and either $G(z)$ or $G(-z)$ has only real and positive zeros, then the entire function

$$\sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \frac{G^{(n)}(0)}{n!} z^n$$

has only real zeros.

In the short preface of the paper JENSEN promises to publish five memoirs containing the results of his studies in "région intermédiair" of complex analysis and algebra. The titles of these memoirs were found in the written heritage of

JENSEN and the last one was supposed to be "V. Sur une classe de fonctions de genre un et en particuller sur une fonction de RIEMANN". Unfortunately, they have never been published. However, on pages 188 and 189 of [Jensen 1912] the author provides a kind of a summary of the memoir he had planned. The fifth part was supposed to deal with entire functions of the form

$$F(z) = \int_0^\infty \Psi(t) \cos zt \, dt \tag{2.8}$$

where the function Ψ is assumed to be in the the space $\mathcal{C}^\infty$ and, moreover, that for every $\nu = 0, 1, 2, \ldots$, $\lim_{t\to\infty} \Psi^{(\nu)}(t) \cos zt = 0$ uniformly with respect to z on every bounded subset of $\mathbb{C}$. As JENSEN pointed out, the importance of this class is due to the fact that RIEMAN'S ξ-function has a representation of the form (2.9). Applying his fundamental theorem to the entire function (2.9), JENSEN obtained the following result:

The entire function $F(z)$ has only real zeros if and only if the polynomials

$$\begin{aligned} F(D)z^p &= \int_0^\infty \Psi(t) \sum_{\nu=0}^{[p/2]} (-1)^\nu \binom{p}{2\nu} z^{p-2\nu} t^{2\nu} \, dt \\ &= \frac{1}{2} \int_0^\infty \Psi(t) \{(z+it)^p + (z-it)^p\} \, dt, \quad p = 1, 2, 3, \ldots \end{aligned}$$

have only real zeros.

It seems that the above result was the reason JENSEN to wrote that he reduced "le problème de RIEMANN de transcendant qu'il était à un probléme algébrique".

At the end of the first part of [Jensen 1912], the author formulates a proposition which can be regarded as a composition theorem:

Suppose that $F(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ is a real entire function of order less than two and having 2ν non real zeros, $H(z)$ is a real entire function of order less than two, having q real and positive zeros, $K(z)$ is a real entire function of order less than one having only real and negative zeros. Then the entire function

$$a_0 H(0) + a_1 \frac{H(1)}{K(0)} z + a_2 \frac{H(2)}{K(0)K(1)} z^2 + a_3 \frac{H(3)}{K(0)K(1)K(2)} z^3 + \ldots$$

has $2\nu + q$ non-real zeros.

In the second part of his paper JENSEN givee necessary and sufficient conditions for a real entire function $F(z)$ of order not greater than one to have only real zeros. For such a function he uses the WEIERSTRASS representation

$$F(z) = z^\mu \exp(c_0 + c_1 z) \prod_\alpha \left(1 - \frac{z}{\alpha}\right) \exp\left(\frac{z}{\alpha}\right),$$

where μ is a nonnegative integer and c_0, c_1 are real numbers. Since F is real, $|F(x+iy)|^2 = F(x+iy)\overline{F(x+iy)} = F(x+iy)F(x-iy)$ is an even function of y. Let $\sum_{k=0}^{\infty} A_{2k} y^{2k}$ be the series expansion of $|F(x+iy)|^2$ as a function of y. Then

$$(2.9) \qquad (2k)!\, A_{2k} = \binom{2k}{k} (F^{(k)}(x))^2 + 2 \sum_{\nu=1}^{k} (-1)^{\nu} F^{(k-\nu)}(x) F^{(k+\nu)}(x).$$

The first conclusion which JENSEN makes is that the assumption that $F(z)$ has only real zeros implies that A_{2k}, $k \in \mathbb{N}_0$, considered as functions of x, must be nonnegative for each $x \in \mathbb{R}$. He proves that the converse is also true, i.e. the non-negativity of A_{2k}, $k \in \mathbb{N}_0$, is sufficient to secure that $F(z)$ possesses only real zeros and this is his first result for the class of real entire functions of order not greater that one. But as JENSEN notes "these necessary and sufficient conditions, obtained in such simple manner, are often rather difficult to apply in practice". That is why, he replaces the infinitely many conditions $A_{2k} \geq 0$, $k \in \mathbb{N}_0$ by a single one, which reads as

$$(2.10) \qquad \frac{\partial^2}{\partial y^2} |F(x+iy)|^2 \geq 0, \quad x, y \in \mathbb{R}$$

and this is the second of his results concerning the class of entire functions he considers.

In a footnote on p. 191, it is pointed out that, as a corollary of (2.8) and the assumption that $F(z)$ has only real zeros, it follows that the inequality $(F'(x))^2 - F(x)F''(x) \geq 0$ holds for each $x \in \mathbb{R}$, which is a well-known result of LAGUERRE.

Comments and references

1. The operator of differentiation is used in the study of zero distribution of algebraic polynomials long before JENSEN, e.g. by ROLLE, GAUSS, HERMITE, POULIN and LAGUERRE. But it is really an idea of JENSEN to use it for defining the polynomials bearing now his name. Most probably PÓLYA and SCHUR were not familiar with JENSEN'S paper when writing [Pólya Schur 1914]. If they were, they would certainly had formulated their criterion in terms of JENSEN polynomials.

2. One of the versions of the theorem of J. SCHUR and MALO says that if $\sum_{k=0}^{n} a_k z^k$ is a hyperbolic polynomial, then the same hold for the polynomial $\sum_{k=0}^{n} a_k z^k/k!$ and this was used to prove that (2.6) has also only real zeros. Proofs of the theorems of SCHUR and MALO can be found on pages 146 and 147 in N. OBRECHKOFF'S book **Zeros of Polynomials**, Bulgarian Academic Monographs (7), Marin Drinov Academic Publishing House, Sofia (2003).

3. The great contributor

There is no doubt that G. Pólya is the mathematician with the most contribution to the systematic study of the distribution of the zeros of entire Fourier transforms. One of his early papers on the subject is **Über die Nullstellen gewisser ganzer Funktionen**, *Math. Z.* **2**(1918), 352–383, where the author studies the zero distribution of the entire functions

$$U(f;z) = \int_0^1 f(t)\cos zt\,dt \tag{3.1}$$

and

$$V(f;z) = \int_0^1 f(t)\sin zt\,dt. \tag{3.2}$$

Observe that all cosine and sine transforms of the form (1.14) and (1.15), with $0 < \sigma < \infty$, can be reduced, by a simple change of the variables t and z to the latter ones. In the preface Pólya provides the following examples of entire functions having integral representations of the form (3.1) and (3.2) with only real zeros:

$$\frac{2}{\pi}\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = J_0(z), \tag{3.3}$$

$$\int_0^1 t\sin zt\,dt = \frac{\cos z}{z^2}(\tan z - z) \tag{3.4}$$

and

$$\frac{2}{\pi}\int_0^1 \frac{t\sin zt}{\sqrt{1-t^2}}\,dt = J_1(z), \tag{3.5}$$

where $J_\nu(z)$ denotes, as usual, the Bessel function of order ν. It is pointed out that each of the intervals $((2k-1)\pi/2, k\pi), k = 1,2,3,\ldots$ contains exactly one zero of (3.3), while the intervals $(k\pi, (2k+1)\pi/2), k = 1,2,3,\ldots$ contain only one zero of (3.4) each. The author is motivated by these facts to develop methods for studying kernels $f(t)$, such that the entire functions (3.1) and (3.2) possess only real zeros and shows that the regular distribution of the positive zeros of the entire functions (3.3) and (3.4) is typical for the real positive zeros of a wide class of functions of the form (3.1) and (3.2).

The first author's approach is described in the first three sections of the paper under consideration. The idea is very nice and very clear and we shall describe it briefly. It is based on an approximation of the above integrals by quadrature formulae and on two classical results. The first one, the so-called

ENESTRÖM-KAKEYA theorem, concerns zeros of polynomials. It was proved first by ENESTRÖM, **Härledning af en allmän formel för antalet pensionärer, som vid en godtycklig tidpunkt förefinnas inom en sluten pensionskassa**, *Öfversigt af Svenska Vetenskaps-Akademiens Förhandlingar*, (Stockholm) **50** (1893), 405–415, and rediscovered by S. KAKEYA, **On the limits of the roots of an algebraic equation with positive coefficients**, *The Tôhoku Math. J.* **2** (1912), 140–142, and states that if $\{a_k\}_{k=0}^n$ are real numbers with $0 < a_0 < a_1 < a_2 < \cdots < a_n$, then the zeros of the polynomial

$$p(z) = \sum_{k=0}^{n} a_k z^k \tag{3.6}$$

are in the unit disk $D = \{z : |z| < 1\}$.

The second result is the trigonometric version of the HERMITE-BIEHLER theorem. It states that if the zeros of the algebraic polynomial with complex coefficients

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

belong to D and if we set $z = \cos\theta + i\sin\theta$ and separate the real and the imaginary parts, $f(z) = A(\theta) + iB(\theta)$, then the trigonometric polynomials $A(\theta)$ and $B(\theta)$ have only real zeros and they interlace.

These two results already imply that when the coefficients of the trigonometric polynomials

$$A(\theta) = \sum_{k=0}^{n} a_k \cos k\theta \quad \text{and} \quad B(\theta) = \sum_{k=1}^{n} a_k \sin k\theta$$

are real and form an increasing sequence, then their zeros are real and interlace. Then one approximates the integrals defined by (3.1) and (3.2) by a Riemann sum, which may a considered as a quadrature formula, to obtain

$$\int_0^1 f(t)\cos zt\, dt \approx \frac{1}{n+1}\sum_{k=0}^{n} f(k/(n+1))\cos(kz/(n+1))$$

and

$$\int_0^1 f(t)\sin zt\, dt \approx \frac{1}{n+1}\sum_{k=1}^{n} f(k/(n+1))\sin(kz/(n+1)).$$

The functions on the right-hand sides of the latter formulae converge locally uniformly to the corresponding integrals. They have only real zeros if and only if the trigonometric polynomials

$$C_n(\theta) = \sum_{k=0}^{n} f(k/(n+1))\cos kz \quad \text{and} \quad S_n(\theta) = \sum_{k=1}^{n} f(k/(n+1))\sin kz$$

do and this holds by the previous observations if $f(t)$ is an increasing function in $[0,1]$. Then, the Hurwitz theorem for the zeros of local uniform limits of entire functions guarantees that the functions $U(f,z)$ and $V(f,z)$ have only real and interlacing zeros. We illustrate this nice idea with a simple example. If $f(t)=1/\sqrt{1-t^2}$, then

$$\frac{2}{\pi}\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = J_0(z). \tag{3.7}$$

and

$$\frac{2}{\pi}\int_0^1 \frac{\sin zt}{\sqrt{1-t^2}}\,dt = H_0(z).$$

where $J_0(z)$ and $H_0(z)$ are the Bessel and the Struve functions. We demonstrate how the idea goes, considering the corresponding steps, constructing the complex polynomials, the trigonometric ones and their zeros. Despite that the function $f(t)$ is not defined at $t=1$, it is integrable and we can apply the above scheme. Having in mind that we divide the interval into $n+1$ subintervals and consider proper Riemann sums to approximate the integral, we consider the polynomial

$$p(z)=\sum_{k=0}^{n} f(k/(n+1))\; z^k = \sum_{k=0}^{n} \frac{n+1}{\sqrt{(n+1)^2-k^2}}\; z^k.$$

By the Eneström-Kakeya theorem its zeros are in D and this can be seen in the figure. In fact, the zeros are rather regularly distributed.

Then the trigonometric polynomials

$$C_n(\theta)=\sum_{k=0}^{n} \frac{n+1}{\sqrt{(n+1)^2-k^2}}\cos kz \quad \text{and} \quad S_n(\theta)=\sum_{k=1}^{n} \frac{n+1}{\sqrt{(n+1)^2-k^2}}\sin kz$$

have only real and interlacing zeros.

Thus, the zeros of the Bessel function $J_0(z)$ and the Struve functions $H_0(z)$ have only real and interlacing zeros and it is quite clear from their graphs shown on Figure 8.

Now we are back to Pólya's paper. In order to formulate and prove the main result of Section 1, entitled **Analogon des Kakeyaschen Satzes**, he

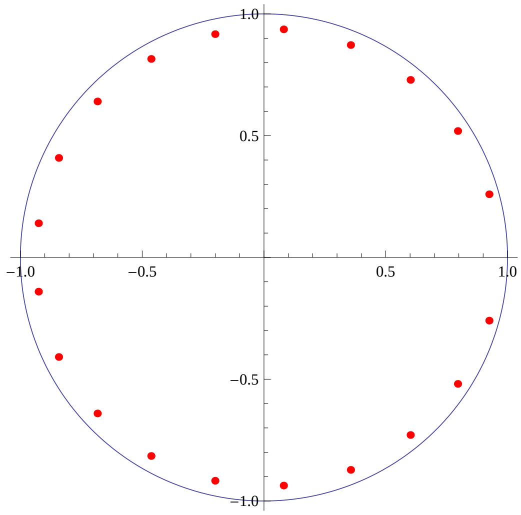

Figure 6: The zeros of $p(z)$ for $n = 20$.

introduces the class $\mathcal{P}[0,1)$ of real functions $f(t)$ defined for $t \in [0, 1)$ and such that:

1. $f(t)$ is positive, i.e. $f(t) > 0$ for every $t \in [0, 1)$;
2. $f(t)$ is increasing, i.e. if $t' < t'' \in [0, 1)$, then $f(t') \le f(t'')$;
3. There exists $\int_0^1 f(t)\,dt := \lim_{\delta\to+0} \int_0^{1-\delta} f(t)\,dt$.

A function $f \in \mathcal{P}[0,1)$ is said to be in the *exceptional case* if $f([0, 1))$ is a finite set and for each c in this set, $f^{-1}(c)$ is a subinterval of $[0, 1)$ with rational endpoints. In other words, $f(t)$ is a step function with jumps at rational points, or equivalently, a finite linear combination of characteristic functions of subintervals of $[0, 1)$ with rational endpoints. The function $f \in \mathcal{P}[0,1)$ is in the *general case*, if it is not in the exceptional case.

The main result in Section 1 is the following assertion:

If the function $f \in \mathcal{P}[0,1)$ is in the general case, then all the zeros of the entire function

$$F(z) = \int_0^1 f(t) \exp zt\,dt \tag{3.8}$$

are in the left half-plane. If $f(t)$ is in the exceptional case, then the zeros of (3.8) *are in the closed left half-plane and, moreover, infinitely many of them are on the imaginary axes. Conversely, if the function* (3.8) *has a pure imaginary zero, then $f(t)$ is in the exceptional case.*

In Section 2, **Trigonometrische Polynome mit nur reellen Nullstellen**, PÓLYA considers the trigonometric polynomials

$$u(z) = a_0 + a_1 \cos z + a_2 \cos 2z + \cdots + a_n \cos nz \tag{3.9}$$

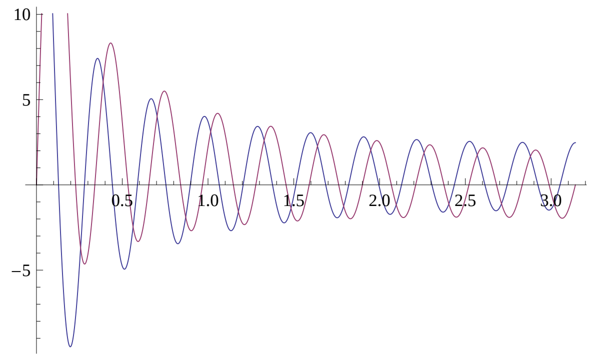

Figure 7: The graphs of $C_n(\theta)$ and $S_n(\theta)$ for $n = 20$.

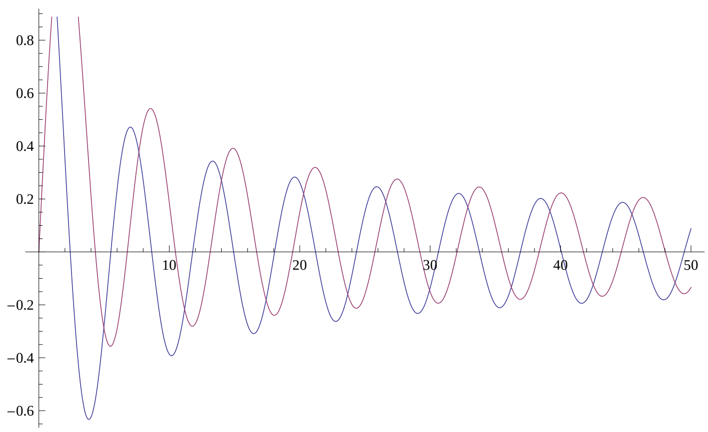

Figure 8: The graphs of $J_0(z)$ and $H_0(z)$.

and

$$v(z) = a_1 \sin z + a_2 \sin 2z + \cdots + a_n \sin nz, \tag{3.10}$$

where $\{a_k\}_{k=0}^{n}$ are real numbers, $a_n > 0$ and all the zeros of the polynomial (3.6) are in the unit disk. Under these conditions he proves that the trigonometric polynomials $\lambda u(z) - \mu v(z)$ and $\mu u(z) + \lambda v(z)$ have only real, simple and mutually interlacing zeros when $\lambda, \mu \in \mathbb{R}$ and $\lambda^2 + \mu^2 \neq 0$. In particular, this holds for $u(z)$ and $v(z)$. The proof is furnished by a classical method usually called argument principle.

Section 3, entitled **Trigonometrische Integrale mit nur reellen Nullstellen** is the central one in PÓLYA'S paper under consideration. The main results concerning the entire functions $U(f; z)$ and $V(f; z)$ can be summarized in the following assertion:

If $f \in \mathcal{P}[0,1)$ is in the general case, then the functions $U(f;z)$ and $V(f;z)$ or, more generally, the functions $\lambda U(f;z) - \mu V(f;z)$ and $\mu U(f;z) + \lambda V(f;z)$, with $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 \neq 0$, have only real, simple and mutually interlacing zeros.

If f is in the exceptional case, then the functions $U(f;z)$ and $V(f;z)$ have only real zeros, infinitely many of which are common. Each such zero is of multiplicity two for $V(f;z)$ and a simple zero of $U(f;z)$. The last function has no multiple zeros at all.

As PÓLYA points out, the assumption that the entire functions $U(f;z)$ and $V(f;z)$ have a common real zero when f is in the general case, leads to a contradiction since, if $U(f;x_0) = V(f;x_0) = 0$, then ix_0 would be a zero of the function (3.8) which is impossible.

Further, Pólya proves that the entire functions (3.1) and (3.2) obey the inequality

$$U(f;x)V'(f;x) - U'(f;x)V(f;x) > 0 \tag{3.11}$$

for every $x \in \mathbb{R}$ when $f \in \mathcal{P}[0,1)$ is in the general case. Inequality (3.11) yields both the absence of common zeros and the interlacing of the zeros of $U(f;x)$ and $V(f;x)$. Indeed, an easy computation yields that

$$\begin{aligned}
&(\lambda U(f;x) - \mu V(f;x))(\mu U'(f;x) + \lambda V'(f;x)) \\
&-(\lambda U'(f;x) - \mu V'(f;x))(\mu U(f;x) + \lambda V(f;x)) \\
&= (\lambda^2 + \mu^2)(U(f;x)V'(f;x) - U'(f;x)V(f;x)).
\end{aligned}$$

In Section 4, **Nachträge und Beispiele zu 1**, PÓLYA considers holomorphic functions of the form (3.7) or, more generally, of the form

$$\int_a^b f(t) \exp zt\, dt, \quad -\infty \leq a < b \leq \infty. \tag{3.12}$$

assuming that the real function f is positive, continuous, piecewise differentiable and its logarithmic derivative is limited as follows:

$$\alpha \leq -\frac{f'(t)}{f(t)} \leq \beta, \quad -\infty \leq \alpha < \beta \leq \infty.$$

Under these requirements, it is proved that:

If the logarithmic derivative of f is not identically constant, then all the zeros of the function (3.12) are in the strip $\alpha < x < \beta$, where $x = \Re z$.

In fact, a proof of the above assertion is given when $-\infty < a < b < \infty$ and $-\infty < \alpha < \beta < \infty$. Other cases are illustrated by means of suitable examples. The entire function

$$\int_1^\infty t^{z-1} \exp(-t)\, dt = \int_0^\infty \exp(-\exp t) \exp(zt)\, dt$$

is of the above form with $f(t) = \exp(-\exp t)$, so that $(-\log f(t))' = -\exp t$, that is $\alpha = 1$ and $\beta = \infty$. Hence, by the latter result, its zeros are located in the half-plane $x > 1$. Moreover, it is shown that the upper bound of the real parts of its zeros is ∞. Next Pólya considers functions $f(t)$, defined for $t \in (0, a)$ for which

$$\gamma = \sup_{t\in(0,a)} \left| \frac{f'(t)}{f(t)} \right|,$$

exists and proves that:

If $\gamma > 0$, then the zeros of the entire function

$$U_a(F; z) = \int_0^a f(t) \cos zt \, dt$$

are in the strip $-\gamma < y < \gamma$.

In fact, this statement is a consequence of that for the functions of the form (3.12). Indeed, setting $f(-t) = f(t)$ for $t \in (-a, 0)$, it follows immediately that

$$U_a(f; iz) = \frac{1}{2} \int_{-a}^a f(t) \exp zt \, dt.$$

The example considered by Pólya is the function

$$\int_0^a \exp(-t) \cos zt \, dt = \frac{\exp(-a)(z \sin az - \cos az) + 1}{1 + z^2}, \; z \in \mathbb{C} \setminus \{i, -i\}, \tag{3.13}$$

for which $\gamma = 1$. The zeros of this function are always in the strip $-1 < y < 1$. It is pointed out that it has finitely many imaginary, i.e. non-real zeros and that, in general, the absolute values of the imaginary parts of these zeros increase when a increases and the zeros tend to the lines $y = \pm 1$ as $a \to \infty$.

In the case $\gamma = 0$, i.e. when f is a constant function, say $f(t) = c, t \in (0, a)$, the entire function

$$\int_0^a c \cos zt \, dt = c \, \frac{\sin az}{z}$$

has only real zeros.

The theorem in Section 5, entitled **Über einen Satz von Hurwitz**, is the following:

Let $f(t)$ be a real and even function defined and integrable in Riemann's *sense (proper or improper) in $(-1, 1)$, and let*

$$f(t) \sim \frac{a_0}{2} + a_1 \cos \pi t + a_2 \cos 2\pi t + \ldots. \tag{3.14}$$

If $(-1)^k a_k > 0$ for $k = 0, 1, 2, \ldots$, then the entire function (3.1) *has only real and simple zeros. Moreover, each of the intervals*

$$\ldots, (-2\pi, -\pi), (-\pi, 0), (0, \pi), (\pi, 2\pi), \ldots \tag{3.15}$$

contains exactly one of its zeros.

Since (3.13) is equivalent to

$$a_k = \int_{-1}^{1} f(t)\cos k\pi t\, dt = 2U(f;k\pi), \quad k = 0,1,2,\ldots,$$

it follows that

$$(3.16) \qquad (-1)^k\, a_k\, U(f;k\pi) > 0, \quad k = 0,1,2,\ldots.$$

Hence, the function (3.1) has at least one zero in each of the intervals (3.15). As PÓLYA points out, HURWITZ'S proof that there is only one zero of this function in each of these intervals is based on the representation of the meromorphic function $U(f;z)/\sin z$ as a sum of elementary fractions

$$(3.17) \qquad \frac{U(f;z)}{\sin z} = \frac{U(f;0)}{z} + \sum_{k=1}^{\infty}(-1)^k U(f;k\pi)\left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi}\right),$$

$$z \neq k\pi, \quad k = 0, \pm 1, \pm 2, \ldots.$$

PÓLYA proves the validity of (3.16) supposing that the integrals $\int_0^1 f(t)\,dt$ and $\int_0^1 |f(t)|\,dt$ exist. He shows first that the series on the right-hand side of (3.17) is uniformly convergent on every compact subset of $\mathbb{C}$ not containing any of the points $k\pi$, $k \in \mathbb{Z}$ and, afterwards, that the entire function

$$(3.18) \qquad \varphi(z) = \frac{U(f;z)}{\sin z} - \frac{U(f;0)}{z} - \sum_{k=1}^{\infty} \frac{2(-1)^k U(f;k\pi)z}{z^2 - k^2\pi^2}$$

is bounded in $\mathbb{C}$. This leads to the conclusion that in fact the function $\varphi(z)$ is a constant. Since it is an odd function, it is identically zero.

Further, it is pointed out that (3.17) is equivalent to the equality

$$\frac{U(f;z)}{\sin z} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{(-1)^k U(f;k\pi)}{z-k\pi},$$

which holds uniformly on every compact subset $\mathbb{C}$ which does not contain any of the points $k\pi$, $k \in \mathbb{Z}$. Because of the inequalities (3.16), the rational function

$$R_n(z) = \sum_{k=-n}^{n} \frac{(-1)^k U(f;k\pi)}{z-k\pi}$$

has only real zeros and, moreover, each interval $(k\pi,(k+1)\pi), k = -n, -n+1, \ldots, -1, 0, 1\ldots, n-1$ contains exactly one of its zeros. Hence, by the theorem of HURWITZ, the meromorphic function $U(f;z)/\sin z$ has only real and simple zeros and each interval $(k\pi,(k+1)\pi)$, $k \in \mathbb{Z}$, contains exactly one of its zeros.

In Section 6, entitiled **Die Trennung der Nullstellen**, HURWITZ'S approach is applied to the entire functions $U(f;z)$ and $V(f;z)$. The first two of the corresponding results are the following:

I. *If $f \in \mathcal{P}[0,1)$, then the function $U(f;z)$ has only real zeros. More precisely, it has no zeros in the interval $(0,\pi/2)$, but it has exactly one zeros in each of the intervals $((2k-1)\pi/2,(2k+1)\pi/2)$, $k=1,2,3,\dots$.*

II. *If $f \in \mathcal{P}[0,1)$ is in the general case, then the only zero of $V(f;z)$ in the interval $(-\pi,\pi)$ is at the origin, and the intervals $(k\pi,(k+1)\pi), k=1,2,3,\dots$ contain exactly one zero of $V(f;z)$.*

The proofs are based on the expansions

$$\begin{aligned}\frac{U(f;z)}{z\cos z} &= \frac{U(f;0)}{z}\\ &+ \sum_{k=1}^{\infty}\frac{(-1)^k U(f;\frac{(2k-1)\pi}{2})}{(2k-1)\pi/2}\left(\frac{1}{z-(2k-1)\pi/2}+\frac{1}{z+(2k-1)\pi/2}\right),\end{aligned} \tag{3.19}$$

$$z \neq 0, (2k+1)\frac{\pi}{2}, \quad k=0,\pm1,\pm2,\dots$$

and

$$\frac{V(f;z)}{z\sin z} = \frac{V'(f;0)}{z} + \sum_{k=1}^{\infty}\frac{(-1)^k V(f;k\pi)}{k\pi}\left(\frac{1}{z-k\pi}+\frac{1}{z+k\pi}\right), \tag{3.20}$$

$$z \neq k\pi, \quad k=0,\pm1,\pm2,\dots$$

respectively, and on the inequalities $U(f;0)>0$ and

$$(-1)^k U(f;(2k-1)\pi/2)>0, V'(f;0)>0, (-1)^k V(f;k\pi)>0, \quad k\in\mathbb{N},$$

which follow from the requirements in **I**. and **II**.

The following assertions are consequences of the previous ones:

III. *Let the function $f(t)$, defined in the interval $(0,1)$, be increasing, convex and, let $\lim_{t\to+0} f(t)=0$. Then the entire function $V(f;z)$ has only real zeros. It has no zeros in the interval $(0,\pi)$, but each one of the intervals $(k\pi,(2k+1)\pi/2), k=1,2,3,\dots$ contains only one of its positive zeros.*

The entire functions

$$\int_0^1 t\sin zt\,dt = \frac{\cos z}{z}(\tan z - z)$$

and

$$\frac{2}{\pi}\int_0^1 \frac{t\sin zt}{\sqrt{1-t^2}}\,dt = J_1(z)$$

are illustrative examples of **III**. It is pointed out that the positive zeros of the first one tend asymptotically to the right endpoints of the intervals $(k\pi,(2k+$

$1)\pi/2)$, $k \in \mathbb{N}$, but the zeros of the second function tend to the midpoints of these interval.

IV. *Let the function $f(t), 0 \le t < 1$, be increasing, convex and let its right derivative be not in the exceptional case. Then the entire function $U(f; z)$ has only one zero in each of the intervals $((2k-1)\pi/2, k\pi)$, $k \in \mathbb{N}$, and has no imaginary zeros.*

An example of a function satisfying the requirements of the last statement is the Bessel function $J_0(z)$, defined by (3.7), see also (2.7).

V. *Let the function $f(t), 0 \le t < 1$ be positive, decreasing and, if $f'(t)$ is its right derivative, let $-f'(t)$ be in the general case. Then the entire function $U(f; z)$ has only real zeros. More precisely, $U(f; x) > 0$ when $-\pi \le x \le \pi$ and $U(f; z)$ has only one zero in each of the intervals $(k\pi, (k+1)\pi)$, $k \in \mathbb{N}$.*

VI. *Let $f(t)$, $0 < t < 1$, be increasing, convex and suppose that $f(\alpha) = 0$ for some $\alpha \in (0, 1)$. If $\int_0^1 f(t)\, dt > 0$, then the entire function $U(f; z)$ has only real zeros. If $\int_0^1 f(t)\, dt < 0$, then it has only two non-real zeros.*

In fact, since $U(f; z)$ is an even entire function, its non-real zeros must be purely imaginary. For example, the entire function

$$J_0(z) - c\frac{\sin z}{z} = \int_0^1 \left(\frac{2}{\pi\sqrt{1-t^2}} - c \right) \cos zt\, dt$$

has no imaginary zeros when $c < 1$, and only two such zeros if $c > 1$.

Entire functions of the form (0.2) and (0.3), in the case when $a = \infty$, are considered at the end of Section 6. It is proved that if the function $f(t), 0 \le t < \infty$, is positive and decreasing, then the entire function $\int_0^\infty f(t) \sin zt\, dt$ has no positive zeros and, hence, it has no real zeros except those at the origin. In addition, if $f(t)$ is convex, then the entire function $\int_0^\infty f(t) \cos zt\, dt$ has no real zeros too. PÓLYA observed that the first of these assertions follows from the inequality

$$\int_0^\infty f(t) \sin xt\, dt = \int_0^{\pi/2} \left(\sum_{k=0}^{\infty} (-1)^k f\left(t + \frac{k\pi}{x} \right) \right) \sin xt\, dt > 0, \quad x > 0$$

and the second is a corollary of the first one because

$$x \int_0^\infty f(t) \cos xt\, dt = \int_0^\infty (-f'(t)) \sin xt\, dt,$$

where $f'(t)$ is the right derivative of $f(t)$.

Another paper of PÓLYA on the zero distribution of entire functions defined by Fourier transforms is **On the zeros of an integral function represented by Fourier's integral**, *Messenger of Math.*, 52 (1923), 185–188, where he mentions: "We do not possess a general method for discussing the reality of zeros of an integral function represented by Fourier's integral (such a method would be available for RIEMANN'S ξ-function.) I present here a special case where the discussion is not quite trivial, but may be carried out with the help of known results". The author studies the location of zeros of the function

$$F_\alpha(z) = \int_0^\infty \exp(-t^\alpha) \cos zt \, dt \tag{3.21}$$

when the parameter α takes real positive values.

It is pointed out that if $0 < \alpha < 1$, then the improper integral in (3.21) exists only if z is real. If $\alpha = 1$, then it is uniformly convergent in each strip of the form $-\lambda \le y \le \lambda$, where $y = \Im z$ and $0 < \lambda < 1$, and its analytic continuation in the complex plane is the meromorphic function $1/(1+z^2)$.

If $\alpha > 1$, then (3.21) is an entire function and its Maclaurin expansion is

$$F_\alpha(z) = \frac{1}{\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{2n+1}{\alpha}\right)}{\Gamma(2n+1)} \, z^{2n}.$$

It follows from this representation that the order of $F_\alpha(z)$ is $\alpha/(\alpha-1)$. In particular,

$$F_2(z) = \frac{\sqrt{\pi}}{2} \exp(-z^2/4).$$

The main result in the paper provides a complete characterization of the zeros of the functions $F_\alpha(z)$, defined by (3.21) and reads as follows:

(I) *If $\alpha = 2$, then there are no zeros at all.*

(II) *If $\alpha = 2, 4, 6, \ldots$, then $F_\alpha(z)$ possesses an infinite number of real zeros and no complex zeros.*

(III) *If $\alpha > 1$ is not an even integer, then $F_\alpha(z)$ has an infinite number of complex zeros and a finite number, not less than $2\,[\alpha/2]$, real zeros.*

The proof is based on an assertion which is, as it is pointed out by PÓLYA, a particular case of LAGUERRE'S theorem:

If $\Phi(z)$ is an integral function of order less than 2 which takes real values along the real axis and possesses only real negative zeros, then the zeros of the integral function

$$\Phi(0) + \frac{\Phi(1)}{1!} z + \frac{\Phi(2)}{2!} z^2 + \cdots + \frac{\Phi(n)}{n!} z^n + \ldots \tag{3.22}$$

are also real and negative.

PÓLYA applies it to the entire function

$$\Phi(z) = \frac{\Gamma((2z+1)/(2k))\,\Gamma(z+1)}{\Gamma(2z+1)} \tag{3.23}$$

and takes into account that

$$\sum_{n=0}^{\infty} \frac{\Gamma((2n+1)/(2k))\,\Gamma(n+1)}{\Gamma(2n+1)} \, \frac{z^n}{n!} = 2k\, F_{2k}(i\sqrt{z}).$$

At the end of the paper PÓLYA gives an example of an entire function of the form (3.21) with only real zeros. It is

$$\int_0^\infty \exp(-t^{2k} + at^2) \cos zt \, dz,$$

where $a \geq 0$ and k is an integer, not less than 2.

PÓLYA'S short paper **On the zeros of certain trigonometric integrals**, *J. London Math. Soc.*, 1 (1926), 98–99, is devoted to the RIEMANN hypothesis. It begins with the following comment: "What properties of the function $F(u)$ are sufficient to secure that the integral

$$2\int_0^\infty F(u) \cos zu \, du = G(z) \tag{3.24}$$

has only real zeros? The origin of this rather artificial question is the RIEMANN hypothesis concerning the Zeta-function. If we put

$$F(u) = \sum_{n=1}^{\infty} \left(4\pi^2 n^4 \exp \frac{9}{2}u - 6\pi n^2 \exp \frac{5}{2}u\right) \exp(-\pi n^2 \exp 2u), \tag{3.25}$$

$G(z)$ becomes RIEMANN'S function $\xi(z)$".

Further, the author claims that in a few cases the entire function $G(z)$ has only real zeros:

(i) $F(u) = (1 - u^{2n})^{\alpha - 1}, \quad 0 \leq u < 1, \quad \alpha > 0, \quad F(u) = 0, 1 \leq u < \infty,$
(ii) $F(u) = \exp(-u^{2n} - \alpha u^{4n}), \quad \alpha > 0,$
(iii) $F(u) = \exp(-2\alpha \cosh u), \quad \alpha > 0,$
(iv) $F(u) = 8\pi^2 \exp(-\cosh 2u) \cosh \frac{9}{2}u,$
(v) $F(u) = (8\pi^2 \cosh \frac{9}{2}u - 12\pi \cosh \frac{5}{2}u) \exp(-2\pi \cosh 2u).$

PÓLYA mentions that the case (i) is a well-known theorem that the zeros of BESSEL'S function $J_{\alpha-1/2}(z)$ are all real provided α is a real positive number. Further he writes: "Observe that the function of case (iv) differs little, and that of case (v) differs yet less, from the leading term of the series (3.25), corresponding to $n = 1$, as u tends to ∞". No proofs of the of the reality of zero of the Fourier transforms of the functions in (i) - (v) are given in the paper.

Pólya mentions that the existence of an infinity of real zeros is generally easier to establish than the non-existence of complex zeros and that the method of HARDY leads to the following very convenient criterion:

Suppose that $F(u)$ is an even function, analytic and real for real values of u, and such that $\lim_{u\to\infty} F^{(n)}(u)u^2 = 0$ for $n = 0, 1, 2, \ldots$. If the function $G(z)$ has only a finite number of real zeros, then there is an integer N such that $|F^{(n)}(it)|$ is a steadily increasing function of t if $n > N$ and $0 < t < T, iT$ being the singular point of $F(u)$ which is next to the origin [$T = \infty$ *if $F(u)$ is an integral function*].

Then Pólya states that the above criterion, applied to the cases (ii), (iii), (iv) and (v) yields that the corresponding function $G(z)$ has an infinity of real zeros. In order to apply this criterion to $\xi(z)$, he uses the function

$$\tilde{\theta}(x) = \sqrt[4]{x} \sum_{n=-\infty}^{\infty} \exp(-n^2 \pi x), \quad x > 0, \tag{3.26}$$

which is a slight modification of Jacobi's function $\theta(0, ix)$ used by Riemann in his memoir on the prime numbers. By means of (3.26) the function (3.25) can be presented in the following more convenient form:

$$F(u) = \tilde{\theta}''(x) - (1/4)\tilde{\theta}(x), \qquad x = \exp(2u).$$

Then the functional relation $\tilde{\theta}(1/x) = \tilde{\theta}(x)$ yields that $F(-u) = F(u)$ and $F^{(n)}(it) \to 0$ as $t \to \pi/4 - 0$, for $n = 0, 1, 2, \dots$ and these imply the existence of an infinitely many real zeros of $\xi(z)$. This provides a slightly different proof of the existence of infinitely many zeros of $\zeta(s)$ on the critical line, a fact which was first proved by Hardy.

In his paper **Bemerkung über die Darstellung der Riemanschen ξ-Funktion**, *Acta Math.*, 48 (1926), 305–317, Pólya discusses the question about the location of the zeros of a cosine transform, where, instead of considering the representation

$$\xi(z) = 2 \int_0^\infty \Phi(u) \cos zu \, du \tag{3.27}$$

of Riemann's ξ-function, where

$$\Phi(u) = 2\pi \exp \frac{5u}{2} \sum_{n=1}^{\infty} n^2 (2\pi n^2 \exp(2u) - 3) \exp(-\pi n^2 \exp(2u)), \quad u \in \mathbb{R},$$

one replaces the kernel $\Phi(u)$ by other kernels which are asymptotically close to it. The author assumes as an evident fact that

$$\Phi(u) \sim 4\pi^2 \exp\left(\frac{9u}{2} - \pi \exp(2u)\right) \tag{3.28}$$

as $u \to \infty$ without clarifying why he considers the above function close to $\Phi(u)$. Moreover, since $\Phi(u)$ is an even function, he claims also that if $u \to \pm\infty$, then

$$\Phi(u) \sim 4\pi^2 \left(\exp \frac{9u}{2} + \exp\left(-\frac{9u}{2}\right)\right) \exp(-\pi(\exp(2u) + \exp(-2u))). \tag{3.29}$$

Further, Pólya points out that if the function $\Phi(u)$ in (3.27) is replaced by the function in the right-hand side of (3.28), then the corresponding entire

Fourier transform would have infinitely many non-real zeros. On the other hand, he stated that the entire function

$$\xi^*(z) = 8\pi^2 \int_0^\infty \left(\exp\frac{9u}{2} + \exp\left(-\frac{9u}{2}\right)\right) \exp(-2\pi\cosh(2u)) \cos zu \, du,$$

called "verfälschte ξ- Funktion", has only real zeros.

The last assertion is the main result in the paper and it says that the entire function

$$\mathcal{G}(z; a) = \int_{-\infty}^{\infty} \exp(-a(\exp u + \exp(-u))) \exp(zu) \, du, \quad a > 0,$$

has only real zeros. Further, the reality of the zeros of $\xi^*(z)$ is established by the aid of the representation

$$\xi^*(z) = 2\pi^2 \left\{\mathcal{G}\left(\frac{iz}{2} - \frac{9}{4}; \pi\right) + \mathcal{G}\left(\frac{iz}{2} + \frac{9}{4}; \pi\right)\right\}, \tag{3.30}$$

as well as of the following statement mentioned at the end of the paper:

Let a be a positive constant and $G(z)$ be an entire function of order 0 *or* 1*, whose values are real for real z, with no imaginary zeros and at least one real zero. Then the function*

$$G(z - ia) + G(z + ia) \tag{3.31}$$

has only real zeros.

The problems considered in PÓLYA'S paper **Über trigonometrische Integrale mit nur reellen Nullstellen**, *J. r. angew. Math.* 158 (1927), 6-16, concern entire functions of the form

$$\int_{-\infty}^{\infty} F(t) \exp(izt) \, dt \tag{3.32}$$

provided the complex function $F(t)$ satisfies the following conditions:

$1^0 F(-t) = \overline{F(t)}$ for each $t \in \mathbb{R}$.

$2^0 F$ is locally integrable.

3^0 There exist positive constants A and α such that

$$|F(t)| \leq A \exp(-|t|^{2+\alpha}) \tag{3.33}$$

when $|t|$ is large enough. The last condition implies that (3.31) is indeed an entire function of the complex variable z. Moreover, it follows from 1^0 that it is real. Again the problem under what conditions on the kernel $F(t)$ the entire function defined by (3.31) possesses only real zeros is discussed. In order to formulate and prove the first of his results, PÓLYA introduces the notion of *universal factor* preserving the reality of the zeros. This is a complex function

$\varphi(t)$, $t \in \mathbb{R}$, with the property that, for any entire function (3.31) with only real zeros, the Fourier transform

$$\int_{-\infty}^{\infty} \varphi(t)F(t)\exp(izt)\,dt \tag{3.34}$$

also has all its zeros real. Pólya obtained a complete characterization of the functions φ with the above property. In fact, he proved that:

A real-analytic function $\varphi(t)$, $t \in \mathbb{R}$, is an universal factor preserving the reality of the zeros if and only if its holomorphic extension $\varphi(z)$ in $\mathbb{C}$ is such that $\varphi(iz)$ is an entire function of II-nd type.

Suppose that the real function $f(t), 0 \leq t < \infty$ is absolutely locally integrable and let there exist positive constants B and β such that

$$|f(t)| \leq B\exp(-t^{1/2+\beta}) \tag{3.35}$$

for all sufficiently large t. Suppose, in addition, that f has a holomorphic extension in a neighbourhood of the origin. Then Pólya proves that:

The complex function

$$H(z) = \int_0^{\infty} f(t)t^{z-1}\,dt, \tag{3.36}$$

which is holomorphic in the half-plane $\Re z > 0$, admits an analytic continuation as a meromorphic function in $\mathbb{C}$. If this function does not have zeros in the region $\mathbb{C} \setminus (-\infty, 0]$ and q is a positive integer, then the entire function

$$\int_{-\infty}^{\infty} f(t^{2q})\exp(izt)\,dt \tag{3.37}$$

has only real zeros.

Then Pólya applies this statement to provide another proof of his earlier result that the entire function

$$w(a;z) = \int_{-\infty}^{\infty} \exp(-a\cosh t)\exp(izt)\,dt$$

has only real zeros. As he points out, a method developed by A. Hurwitz and improved later by E. Hille, is used. It is shown first that $w(a;z)$ as a function of the real variable a satisfies the differential equation

$$(aw'(a;z))' = \left(a - \frac{z^2}{a}\right)w(a;z).$$

Further, using the fact that the entire function $\cos az$, $a \in \mathbb{R}$, is of II-nd type and applying the first of the already formulated assertions, Pólya concludes that the entire function

$$\int_{-\infty}^{\infty} \cosh(at)\exp(-a\cosh t)\exp(izt)\,dt$$

has only real zeros, as well as that if $A > B, a > b > 0$, then so does the entire function

$$\int_{-\infty}^{\infty} (A \cosh at - B \cosh bt) \exp(-a \cosh t) \exp(izt)\, dt.$$

The meromorphic functions $\Gamma(z)\Gamma(a)/\Gamma(z+a)$, $a > 0$, and $\Gamma(z)$ have no zeros in the region $\mathbb{C} \setminus (-\infty, 0]$. If $\Re z > 0$, then

$$\int_0^1 t^{z-1}(1-t)^{a-1}\, dt = \frac{\Gamma(z)\Gamma(a)}{\Gamma(z+a)}, \qquad \int_0^{\infty} t^{z-1} \exp(-t)\, dt = \Gamma(z),$$

which yields that the entire functions

$$\int_{-1}^{1} (1-t^{2q})^{a-1} \exp(izt)\, dt$$

and

$$\int_{-\infty}^{\infty} \exp(-t^{2q}) \exp(izt)\, dt = 2\int_0^{\infty} \exp(-t^{2q}) \cos zt\, dt$$

have only real zeros. Moreover, if $P(z)$ is an algebraic polynomial with only real and negative zeros and l, q are positive integers, then the the entire function

$$\int_1^1 (1-t^{2q})^{l-1} P(t^{2q}) \exp(izt)\, dt$$

possesses only real zeros too. Then, choosing $P(t) = (1+t)^{l+k-1}$, Pólya concludes that so does the entire function

$$\int_{-\infty}^{\infty} \exp(-at^{4q} + bt^{2q} + ct^2) \exp(izt)\, dt, \quad a > 0,\ b \in \mathbb{R},\ c \geq 0,$$

thus providing a new proof of this fact.

The last of Pólya's papers, where he discusses the zero distribution of entire functions defined by Fourier Transforms is **Über die algebraisch-funktionentheoretischen Untersuchungen von J.L.W.V. Jensen**, *Kgl. Danske Vid. Sel. Math.-Fys. Medd.* 7 (17) 1927, 3–33. It is a kind of a survey on Jensen's written heritage.

The essential part of the paper is Kapitel I. entitled **Über die Realität der Nullstellen gewisser trigonometrischer Integrale**, where the author studies the class of entire functions

$$F(z) = \exp(-\lambda z^2) H(z), \tag{3.38}$$

where $H(z)$ is an entire function of order less than 2, and λ is a non-negative real number, and more precisely, functions of the form

$$F(z) = 2\int_0^{\infty} \Psi(t) \cos zt\, dt, \tag{3.39}$$

where the kernel $\Psi(t)$ is supposed to obey the following requirements:

1) $\Psi(t)$ is real, nonnegative and not identically equal to zero.
2) $\Psi(t)$ is infinitely many times differentiable.
3) $\lim_{t\to\infty} t^{-1}\log|\Psi^{(n)}(t)| = -\infty$ for each $n = 0, 1, 2, \ldots$.
4) $F(z)$ is of order less than 2.

As PÓLYA observes, 3) with $n = 0$ ensures that $F(z)$ is not an algebraic polynomials, i.e. it is a transcendental entire function. It takes real values if and only if $z = x$ is real and, moreover, $F(-z) = F(z)$, i.e. it is an even function. An example of a function of the form (3.38) is the RIEMANN ξ-function. More precisely,

$$\xi\left(\frac{z}{2}\right) = 2\int_0^\infty \Phi(t)\cos zt\,dt, \tag{3.40}$$

where

$$\begin{aligned}\Phi(t) &= \omega''(t) - \omega(t),\\ \omega(t) &= (1/4)(1 + 2\psi(\exp(4t)))\exp t, \quad -\infty < t < \infty,\end{aligned}$$

and $\psi(x), 0 < x < \infty$, is the function defined by (0.10). Hence,

$$\Phi(t) = 4\sum_{n=1}^{\infty}(2n^4\pi^2\exp(9t) - 3n^2\pi\exp(5t))\exp(-\pi n^2\exp(4t)). \tag{3.41}$$

Replacing t by $u/2$ and z by $2z$, (3.40) becomes the integral representation (1.18).

Further, PÓLYA recalls some properties of the function (3.41):

I. *For each* $\varepsilon > 0$ *and every* $n \in \mathbb{N}$,

$$\lim_{t\to\infty}\Phi^{(n)}(t)\exp((\pi - \varepsilon)\exp(4t)) = 0; \tag{3.42}$$

II. $\Phi(t)$ *is an even function*;
III. $\Phi(t) > 0$ *for each* $t \in \mathbb{R}$;
IV. *If* $t \to \pm\infty$, *then*

$$\Phi(t) \sim 8\pi^2(\exp(9t) + \exp(-9t))\exp(-\pi(\exp(4t) + \exp(-4t))). \tag{3.43}$$

PÓLYA calls the last property curious since if $\tilde{\Phi}(t)$ is the right-hand side of (3.43), then the entire function

$$\int_0^\infty \tilde{\Phi}(t)\cos zt\,dt$$

has only real zeros. These properties of $\Phi(t)$ were already known and, in a footnote on p. 11, PÓLYA claims that (3.42) was verified on p. 188, 189 of JENSEN'S paper in *Acta Math.*, and II, III were in his heritage.

It is also known that the function $\Phi(t)$ admits a holomorphic extension in the strip $|\Im t| < \pi/8$. This is a simple corollary of the fact that the function (1.4) has a holomorphic extension in the half-plane $\Re z > 0$. Moreover, $|\Im t| < \pi/8$ is the widest strip where $\Phi(t)$ is holomorphic and it is a consequence of the following fact:

V. *For each fixed n, $\Phi^{(n)}(t)$ tends to zero if $\Re t = 0$ and $t \to i\pi/8$.*

In other words, $i\pi/8$ is a singular point for $\Phi(t)$ considered as a function of the complex variable t in the strip $|\Im t| < \pi/8$. Since $\Phi(t)$ is even in this strip, the same holds for the point $-i\pi/8$.

Further, PÓLYA formulates the following result about the zero distribution of the entire functions (3.38) that he found in JENSEN'S heritage (foot-note on p. 14):

I. *If $\Psi''(t) \le 0$ for $t \ge 0$, then $F(z)$ has no real zeros.*

II. *If $F(z)$ has infinitely many zeros in a strip of the form $-k \le \Im z \le k$, where k is a positive constant, then $\Psi'(0) = \Psi'''(0) = \Psi^{(5)}(0) = \cdots = 0$.*

III. *If $F(z)$ has only real zeros and*

$$F(z) = b_0 - \frac{b_1}{1!}z^2 + \frac{b_2}{2!}z^4 + \dots$$

is its Maclaurin expansion, then the real numbers $b_0, b_1, b_2, \dots$ have one and the same sign.

Then PÓLYA emphasizes that the simplest necessary conditions for reality of the zeros, namely

$$b_n^2 - b_{n-1}b_{n+1} \ge 0, \quad n = 1, 2, 3, \dots \tag{3.44}$$

have not been verified for RIEMANN'S ξ-function. Observe that the above inequalities are corollaries of the inequalities (1.5) applied to the entire function $F(z^{1/2})$.

Then PÓLYA provides some criteria for reality of the zeros of the entire functions (3.39) that he found in JENSEN'S heritage:

I. *The zeros of $F(z)$ are real if and only if*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\alpha)\Psi(\beta)\exp(i(\alpha+\beta)(x+y))(\alpha-\beta)^2\,d\alpha\,d\beta \ge 0$$

for each real x and y.

II. *The zeros of $F(z)$ are real if and only if*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\alpha)\Psi(\beta)\exp(i(\alpha+\beta)x)(\alpha-\beta)^{2n}\,d\alpha\,d\beta \ge 0$$

for every $x \in \mathbb{R}$ and $n = 0, 1, 2, \dots$.

III. *The zeros of $F(z)$ are real if and only if*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\alpha)\Psi(\beta)(x+i\alpha)^n(x+i\beta)^n(\alpha-\beta)^2\,d\alpha\,d\beta > 0$$

for every $x \in \mathbb{R}$ *and* $n = 0, 1, 2, \dots$.

Further, it is pointed out that in order to prove the criteria I, II and III, they have to be formulated in a more convenient and somewhat general form. Let $F(z)$ be a real entire function of raised genus 1. i.e. of the form 3.38. Then, necessary and sufficient conditions for its zeros to be real, are given by the following assertions:

I′. $\frac{\partial^2}{\partial y^2}|F(x+iy)|^2 \geq 0$ *for all real* $x, y \in \mathbb{R}$.

II′. *For each fixed* x *all the coefficients of the Maclaurin expansion of* $|F(x+iy)|$, *as functions of* y *to be non-negative.*

Suppose that $F(z)$ is not of the form $\exp(\alpha z)P(z)$, where α is a constant and $P(z)$ is an algebraic polynomial. Let

$$F(z) = a_0 + \frac{a_1}{1!}z + \frac{a_2}{2!}z^2 + \dots$$

and denote

$$A_n(F; z) = a_0 z^n + \binom{n}{1} a_1 z + \binom{n}{2} a_2 z^2 + \dots + a_n, \quad n = 0, 1, 2, \dots.$$

Then PÓLYA proposes the following criterion:

III′. *The function* $F(z)$ *has only real zeros if and only if*

$$A_n^2(F; x) - A_{n-1}(F; x) A_{n+1}(F; x) > 0 \tag{3.45}$$

for all real x *and* $n \in \mathbb{N}$. It is clear that $A_n(F; z) = F(D)z^n$ is the n-th Appell polynomial, associated with $F(z)$.

PÓLYA mentions that JENSEN'S proofs of the criterions I′ and II′, given in *Acta Math.* are rather elementary indeed, while the proof of III' is preceded by five auxiliary statements and requires a lot of efforts. Something more about these statements is said in the second foot-note on p. 19.

Suppose that $\Psi(t)$, $-\infty < t < \infty$, is a real and even function having a holomorphic extension in $\mathbb{C}$ and that $\lim_{t\to\infty} t^2 \Psi^{(n)}(t) = 0$ for each $n = 0, 1, 2, \dots$. The following assertion can be used as a criterion for existence of infinitely many real zeros of a holomorphic functions $F(z)$ of the form (3.38):

I. *Let* $c_0 - c_1 t^2 + c_2 t^4 - \dots$ *be the Maclaurin expansion of* $\Psi(t)$. *If the function* $F(z)$ *has only* p *real zeros and all they are with odd multiplicity, then among the numbers of the sequence*

$$c_0, c_1, c_2, \dots \tag{3.46}$$

there are no more than p *equal to zero and, moreover, this sequence has at most* p *sign changes.*

II. *Suppose that* $F(z)$ *has finitely many real zeros. Then, there exists* T, $0 < T \leq \infty$, *such that the function* $\Psi(t)$ *has no singular points in the disk*

$|t| < T$. *Moreover, there is a positive integer* N, *such that, if* $n > N$ *is fixed, then* $|\Psi^{(n)}(t)|$ *increases when* $0 < t < T$ *and* t *tends to* T.

An immediate corollary of the first statement is that if the sequence (3.45) has infinitely many sign changes, then the function $F(z)$ possesses infinitely many real zeros. At the end of his survey on JENSEN'S heritage PÓLYA gives the following interesting example of an entire function of the form (3.38):

$$\Psi(a;t) = \exp(-t^2/2)(\exp t + \exp(-t) + 2a\sqrt{e}),$$

where a is a positive constant. It is easy to verify that this function satisfies the conditions I-III . Moreover, $\Psi(t) \sim \exp(-t^2/2)(\exp t + \exp(-t))$ when $t \to \infty$. The corresponding entire function

$$F(a;z) = \int_0^\infty \Psi(a;t)\cos zt\,dt$$

has only real zeros when $0 < a \leq 1$ and no real zeros if $a > 1$.

Comments and references

1. It is clear that $\lim_{t\to 1-0} f(t) = a(f)$ exists for every function $f(t) \in \mathcal{P}[0,1)$ and that $0 < a(f) \leq \infty$. Denote by $\mathcal{P}^*[0,1)$ the set of function $f \in \mathcal{P}[0,1)$ for which $a(f) < \infty$, i.e. $\mathcal{P}^*[1,0)$ is the set of bounded functions in $\mathcal{P}[0,1)$. In fact, PÓLYA has proved that if $f \in \mathcal{P}^*[0,1)$, then the entire functions $U(f;z)$ and $V(f;z)$ have only real zeros. He applies the results from Section 2 in [Pólya 1918] to the trigonometric polynomials

$$\widetilde{U}_n(f;z) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\exp\frac{k}{n^2}\cos z\frac{k}{n}, \tag{3.47}$$

$$\widetilde{V}_n(f;z) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\exp\frac{k}{n^2}\sin z\frac{k}{n} \tag{3.48}$$

and concludes that they have only real and mutually interlacing zeros. Since

$$\lim_{n\to\infty}\widetilde{U}_n(f;z) = \int_0^1 f(t)\cos zt\,dt$$

and

$$\lim_{n\to\infty}\widetilde{V}_n(f;z) = \int_0^1 f(t)\sin zt\,dt$$

uniformly on every compact subset of $\mathbb{C}$, HURWITZ'S theorem yields that the entire functions $U(f;z)$ and $V(f;z)$ have only real zeros.

In order to prove that $U(f;z)$ and $V(f;z)$ have the same property when $f \in \mathcal{P}[0,1)$ but $a(f) = \infty$, define the functions $f_\delta(t), 0 < \delta < 1$, assuming that $f_\delta(t) = f(t)$ for $t \in [0,\delta)$ and $f_\delta(t) = f(\delta)$ for $t \in [\delta, 1)$. Further, since $a(f_\delta) = f(\delta) < \infty$, then $f_\delta(t) \in \mathcal{P}^*[0,1)$. Hence, the entire functions $U(f_\delta; z)$ and $V(f_\delta; z)$ have only real zeros for each $\delta \in (0,1)$. It remains to observe that $\lim\limits_{\delta\to 1-0} U(f_\delta; z) = U(f; z)$ and $\lim\limits_{\delta\to 1-0} V(f_\delta; z) = V(f; z)$ uniformly on every compact subset of $\mathbb{C}$.

2. The reality of the zeros of the entire functions $U(f; z)$ and $V(f; z)$ as well as their simplicity and mutual interlacement when $f \in \mathcal{P}[0,1)$ can be established by making use of Hermite-Biehler's theorem for entire functions of exponential type having non-negative defect (B.J. Levin, **Distribution of zeros of entire functions**, AMS, Providence, Rhode Island, 1964, Chapter 4.).

Recall that the defect of an entire function F of exponential type is the real number

$$d_F = \frac{1}{2}\left(h_F\left(-\frac{\pi}{2}\right) - h_F\left(\frac{\pi}{2}\right)\right),$$

where

$$h_F(\theta) = \limsup_{r\to\infty} \frac{\log|F(r\exp i\theta)|}{r}$$

is the indicator function of F.

Suppose that all the zeros of F are in the closed upper half-plane and that its defect is positive. If $F(z) = U(z) + iV(z)$ is the representation of F by means of its real and imaginary parts, then Hermite-Biehler's theorem says that $U(z) = R(z)U^*(z)$ and $V(z) = R(z)V^*(x)$, where the entire functions $R(z), U(z)$ and $V(z)$ have only real zeros and the zeros of $U^*(z)$ and $V^*(z)$ are simple and mutually interlacing. Moreover, if $F(z)$ has no zeros on the real axis, then $R(z)$ is a constant.

It is clear that the entire function

$$E(f; z) = U(f; z) + iV(f; z) = \int_0^1 f(t)\exp(izt)\,dt. \tag{3.49}$$

is of exponential type even if $f(t)$ is integrable in Lebesgues' sense. Moreover, if $f \in \mathcal{P}[0,1)$, then its zeros are in the closed upper half-plane. Indeed, by the Eneström-Kakeya theorem the zeros of the polynomials

$$\widetilde{P}_n(f_\delta; z) = \sum_{k=0}^{n-1} f_\delta\left(\frac{k}{n}\right)\exp\frac{k}{n^2}z^k, \quad 0 < \delta < 1, \tag{3.50}$$

are in the unit disk and, hence, the zeros of the entire functions $\widetilde{P}_n(f_\delta; \exp iz)$, $n = 1, 2, 3, \ldots$ are in the upper half-plane. Since

$$E(f_\delta; z) = \lim_{n\to\infty} n^{-1}\widetilde{P}_n\left(f_\delta; \exp\frac{iz}{n}\right)$$

and $E(f;z) = \lim_{\delta\to 1-0} E(f_\delta;z)$ uniformly on each compact subset of $\mathbb{C}$, it follows that the zeros of $E(f;z)$ are in the closed upper half-plane. Moreover, if $f(t)$ is in the general case, then the zeros of $E(f;z)$ are in the upper half-plane. It remains to show that the defect of the function $E(f;z)$ is positive. In fact it is equal to $1/2$. This is a corollary of a property of entire functions of exponential type of the form (0.1) provided $-\infty < a < b < \infty$, i.e. of the functions

$$\mathcal{F}_{a,b}(f;z) = \int_a^b f(t)\exp(izt)\,dt, \quad -\infty < a < b < \infty. \tag{3.51}$$

We have:

Suppose that the real function $f(t), a < t < b$ is absolutely integrable and that there exist $\delta \in (0,(b-a)/2)$ and $\lambda > 0$ such that $f(t) \geq \lambda$ for $t \in (a,a+\delta)\cup(b-\delta,b)$. Then,

$$d_{\mathcal{F}_{a,b}} = \frac{a+b}{2}\cdot \tag{3.52}$$

If $a = -\sigma$, $b = \sigma$, $0 < \sigma < \infty$, then (1.13) yields

$$|E_\sigma(f;z)| \leq \left(\int_{-\sigma}^{\sigma} |f(t)|\,dt\right)\exp(\sigma|z|).$$

It is clear that the type of $E_\sigma(f;z)$ is not greater than σ. Then $h_{E_\sigma}(\theta) \leq \sigma$ for $0 \leq \theta < 2\pi$ and, in particular, $h_{E_\sigma}(-\pi/2) \leq \sigma$ and $h_{E_\sigma}(\pi/2) \leq \sigma$. Therefore,

$$\begin{aligned} &r\exp(-\sigma r)E_\sigma(f;-ir)\\ &= r\exp(-\sigma r)\left(\int_{-\sigma}^{\sigma-\delta} f(t)\exp(rt)\,dt + \int_{\sigma-\delta}^{\sigma} f(t)\exp(rt)\,dt\right)\\ &\geq \lambda - \lambda\exp(-\delta r) - r\exp(-\delta r)\int_{-\sigma}^{\sigma} |f(t)\,dt. \end{aligned}$$

The last inequality implies that $h_{E_\sigma}(-\pi/2) \geq \sigma$. Hence, $h_{E_\sigma}(-\pi/2) = \sigma$. Since

$$E_\sigma(f;z) = \int_{-\sigma}^{\sigma} f(-t)\exp(-izt)\,dt,$$

it follows that $h_{E_\sigma}(\pi/2) = \sigma$ too. The change of t by $t + (a+b)/2$ yields

$$\mathcal{F}_{a,b}(f;z) = \exp\left(i((a+b)/2)z\right)E_\sigma(f_{a,b};z), \tag{3.53}$$

where $\sigma = (b-a)/2$ and $f_{a,b}(t) = f(t+(a+b)/2)$. Then, (3.52) is a corollary of (3.53).

HERMITE-BIEHLER'S theorem can be applied to the entire functions

$$\lambda U(f;z) - \mu V(f;z) \quad \text{and} \quad \mu U(f;z) + \lambda V(f;z)$$

when λ, μ are real and $\lambda^2 + \mu^2 \neq 0$. Indeed,

$$\lambda U(f;z) - \mu V(f;z) + i(\mu U(f;z) + \lambda V(f;z)) = (\lambda + i\mu)(U(f;z) + iV(f;z)).$$

3. Let us define $f(1) = a(f)$ for a function $f \in \mathcal{P}^*[0,1)$. Then, instead of (3.49), the polynomials

$$P_n(f;z) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) z^k, \quad k = 0, 1, 2, \ldots \tag{3.54}$$

can be used to prove that the zeros of the entire function $E(f;z)$ are in the closed upper half-plane.

4. Another approach to the zero distribution of the entire functions (3.1) and (3.2) is used in Section 5.3 in the paper of HASEO KI and YOUNG-ONE KIM, **On the number of nonreal zeros of real entire functions and the Fourier-Pólya conjecture**, *Duke Math. J.*, 104 (2000), 45-73. Under the assumption that the function $f \in \mathcal{P}[0,1)$ is in the general case, the authors prove that the inequalities

$$(\cos x + x \sin x)U(x) + x \cos x U'(x) > 0,$$

$$(\sin x - x \cos x)V(x) + x \sin x V'(x) > 0$$

hold for each $x > 0$. They imply the inequalities $(-1)^n U(\pi/2 + n\pi) > 0$, $n = 0, 1, 2, \ldots$ and $(-1)^{n+1} V(n\pi) > 0$, $n - 1, 2, 3, \ldots$. The final result that the entire functions (3.1) and (3.2) have no nonreal zeros is obtained as an application of *Theorem 4.3* on p. 63. It states that a real entire function of the form $\exp(-\alpha z^2) f(z)$, $\alpha \geq 0$ has no critical points provided the entire function $f(z)$ is of growth $(2,0)$. Moreover, if $\{b_j\}_{j=1}^{\omega}$, $0 \leq \omega \leq \infty$ are its real zeros different from zero, then $\sum_{j=0}^{\omega} b_j^{-2} < \infty$.

5. The zero distribution of entire finctions $F_\alpha(z)$ has already been treated earlier. PÓLYA points out that, following the method employed by G.H. HARDY (*C.R.*, 6 April, 1914) to prove that RIEMANN'S $\xi(t)$ has infinite number of real zeros, F. BERNSTEIN, **Über das Fourierintegral** $\int_0^\infty \exp(-x^4) \cos xt\, dt$ *Math. Ann*, 59 (1919), 265-268, proved the same thing for $F_4(z), F_6(z), F_8(z), \ldots$.

6. The entire functions of the form (3.21) are studied in the paper of JOE KAMIMOTO, HASEO KI and YOUNG-ONE KIM, **On the multiplicity of the zeros of Laguerre-Pólya functions**, *Proc. Amer. Math. Soc.*, 128 (1999), 189-194. It is proved that not only this function, but all its derivatives have only real and simple zeros. This result is applied to the completeness in the space $L(\mathbb{R}^n)$ of the translations of a function of the form

$$p(x_1, x_2, x_3, \ldots, x_n) \exp\left(\sum_{j=1}^{n} x_j^{2m_j}\right),$$

where p is an algebraic polynomials and m_j, $j \in \mathbb{N}$ are positive integers.

7. A classical theorem of LAGUERRE says that if $f(z) = \sum_{k=0}^{n} a_k z^k$ is a real polynomial and $\varphi(z)$ is a real polynomial with only real zeros located outside the interval $[0, n]$, then the polynomial

$$\sum_{k=0}^{n} a_k \varphi(k) z^k \tag{3.55}$$

has at least so many real zeros as the polynomial $f(z)$. In particular, if $f(z)$ has only real zeros, then so does (3.45). Moreover, it has so many positive, equal to zero and negative zeros as the polynomial $f(z)$. In particular, if $f(z)$ has only real zeros of one and the same sign, then the same holds for the polynomial (3.45). The last assertion was generalized by LAGUERRE for real entire functions of order less than two with only real and negative zeros (E. TITCHMARSH, **The theory of functions**, Oxford, 1939, **8.6.1, 8.6.2**).

8. The meaning of the relation (3.27) is that the representation

$$\Phi(u) = 4\pi^2 \exp\frac{9u}{2} \exp(-\pi \exp 2u)(1 + \varphi(u)) \tag{3.56}$$

holds for $u > 0$, where

$$\varphi(u) = O(\exp(-2u)) \tag{3.57}$$

when $u \to \infty$, i.e. the function $\varphi(u) \exp 2u$ is bounded when u tends to infinity. Indeed, if

$$\varphi(u) = -\frac{3}{2\pi} \exp(-2u) + \sum_{n=2}^{\infty} n^4 \left(1 - \frac{3}{\pi n^2} \exp(-2u)\right) \exp(-\pi(n^2 - 1) \exp 2u),$$

then (3.55) holds for $u > 0$ and it remains to prove the validity of (3.56). It is clear that

$$\varphi(u) = O\left(\exp(-2u) + \sum_{n=2}^{\infty} n^4 \exp(-n^2 \exp 2u)\right), \quad u \to \infty.$$

Further, if $u > 1$, then $t^4 \exp(-t^2 \exp 2u)$ as a function of the variable t is decreasing in the interval $[1, \infty)$. Hence,

$$\begin{aligned}\sum_{n=2}^{\infty} n^4 \exp(-n^2 \exp 2u) &= O\left(\int_1^{\infty} t^4 \exp(-t^2 \exp 2u)\, dt\right) \\ &= O\left(\exp(-5u) \int_1^{\infty} t^{3/2} \exp(-t)\, dt\right), \quad u \to \infty.\end{aligned}$$

Since $\Phi(u)$ is an even function, then $\Phi(u) = (\Phi(u) + \Phi(-u))/2$. Thus, (3.29) is a consequence of (3.28).

9. The assertion **III** from Section 5 of PÓLYA'S paper [Pólya 1918] concerning the entire function $V(f;z)$ is proved when $\lim_{t\to+0} f(t) = 0$. A.M. SEDLETSKI, **Addition to Pólya's theorem on zeros of Fourier sin-transform**, *Integral Trans. Spec. Funct.*, 1 (2000), 65-68, considers the general situation when this condition is omitted. He proves that if the non-constant function $f(t)$, $0 \le t \le 1$ is positive, non-decreasing and convex, then the function $V(f;z)$ has also only real and simple zeros, each of the intervals $(\pi, 2\pi), (2\pi, 5\pi/2), (3\pi, 4\pi), (4\pi, 9\pi/2), \ldots$ contains exactly one of them and it has no other positive zeros.

4. A knight of the classical analysis

The paper of E. C. TITCHMARSH **The zeros of certain integral functions**, *Proc. London Math. Soc.* 25 (1926), 283-302, is probably the first study of the zero distribution of entire functions of the form

$$F(z) = \int_a^b f(t) \exp zt\, dt, \quad -\infty < a < b < \infty, \tag{4.1}$$

under the only assumption that $f(t)$ is a real Lebesgue integrable function in $a < t < b$ or, more generally, $f(t) = f_1(t) + if_2(t)$, where $f_1(t)$ and $f_2(t)$ are real Lebesgue integrable functions in the same interval. The author points out that the related forms

$$\int_a^b f(t) \cos zt\, dt, \int_a^b f(t) \sin zt\, dt$$

may be reduced to (4.1) by simple transformations. The type includes many well known functions, such as Bessel functions. Further, TITCHMARSH recalls that some striking results as to the distributions of the zeros when $f(t)$ satisfies simple conditions have been obtained by PÓLYA in [Pólya 1918] and that his own aim is to determine properties which are common to all the function $F(z)$, without further restriction on $f(t)$ than that of integrability.

It is supposed that a and b are the effective lower and upper limits of the integral in (4.1), i.e. there is no $\varepsilon \in (0, (b-a)/2)$ such that

$$\int_a^{a+\varepsilon} |f(t)|\, dt = \int_{b-\varepsilon}^b |f(t)|\, dt = 0.$$

It is assumed also that $F(0) \ne 0$, which is no loss of generality. The zeros of $F(z)$ are denoted by $r_1 \exp i\theta_1, r_2 \exp i\theta_2, \ldots$, where $r_1 \le r_2 \le \ldots$. The first result concerning these zeros is:

Theorem I. *The function $F(z)$ has an infinity of zeros and such that the series*

$$\sum_{n=1}^{\infty} \frac{1}{r_n}$$

is divergent.

This result is proved by PÓLYA in [Pólya 1918], but TITCHMARSH'S proof is simpler than the original one. The next result is:

Theorem II. *The series*

$$\sum_{n=1}^{\infty} \frac{\cos\theta_n}{r_n}$$

is absolutely convergent.

The author mentions that it might be supposed that $\cos\theta_n$ tends to zero when $n \to \infty$ or, which is the same, that any angular domain with vertex at the origin containing the imaginary axis would contain all but finitely many zeros of the function $F(z)$, but this is not the case in general. In fact, the following statement holds:

Theorem III. *There is a function of the given class with an infinity of real zeros.*

Let $0 < \theta < 1$ and $0 < \delta < \theta(1-\theta)$. TITCHMARSH defines the function $\lambda(t)$, $0 \le t \le \theta$,

$$\lambda(t) = \begin{cases} (-1)^n \mu_n, & \theta^n - \delta^n \le t \le \theta^n, \ n = 1, 2, 3, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

provided that μ_n, $n = 1, 2, 3, \ldots$ are positive and the series $\sum_{n=1}^{\infty} \mu_n \delta^n$ is convergent. Then, under the additional assumptions that $\theta < 1/2$, $\delta < 1/a$ and $a > 2/\theta$, it is proved that the corresponding function

$$\Lambda(z) = \int_0^{\theta} \lambda(t) \exp(zt)\, dt \tag{4.2}$$

has a real zero in every interval $(-a^{n+1}, -a^n)$ when n is large enough.

According to the author, the principal result in the paper is:

Theorem IV. *If $n(r)$ is the number of zeros of $F(z)$ in the disc $|z| \le r$, then*

$$n(r) \sim \frac{b-a}{\pi} r, \quad r \to \infty \tag{4.3}$$

The validity of the statement

$$\lim_{r\to\infty} \frac{n(r)}{r} = \frac{b-a}{\pi} \tag{4.4}$$

is proved when $a = -1$, $b = 1$, which, of course, is no loss of generality. The proof is preceded by a considerable number of auxiliary statements. The main goal is to verify that

$$N(r) \sim \frac{2r}{\pi} \quad \text{as} \quad r \to \infty,$$

where

$$N(r) = \int_0^r \frac{n(t)}{t}\, dt,$$

and this is the assertion of **Lemma 4.7**. Then, the final result follows from the latter one and

Lemma 4.8. *If $N(r)/r$ tends to a limit, then $n(r)/r$ tends to the same limit.*

As TITCHMARSH pointed out, the above assertion is a form of a well-known theorem of E. LANDAU, *Rend. di Palermo*, 26 (1908). The next result in the paper is:

Theorem V. *The series*

$$\sum_{n=1}^{\infty} \frac{\sin\theta_n}{r_n} \tag{4.5}$$

is conditionally convergent.

The author not only proves **Theorem V** and **Theorem II**, but obtains the closed form expressions

$$\sum_{n=1}^{\infty} \frac{\cos\theta_n}{r_n} = \frac{1}{2}(a+b) - \Re\left\{\frac{F'(0)}{F(0)}\right\}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin\theta_n}{r_n} = \Im\left\{\frac{F'(0)}{F(0)}\right\}. \tag{4.6}$$

After proving **Theorem V**, TITCHMARSH makes interesting comments. He claims that the result is, of course, obvious if $f(t)$ is real, since then the zeros of $F(z)$ are conjugate complex numbers, and the terms of the series (4.6) cancel in pairs. He also verifies (4.6) in a less trivial case. Let $f(t) = 1$ when $-1 < t \le 0$, and $f(t) = i$ for $0 < t < 1$. Then $F(z) = (e^z - 1)(e^{-z} + i)/z$. The zeros of $F(z)$ are at $\pm 2i\pi, \pm 4i\pi, \ldots$ and $i\pi/2, -3i\pi/2, 5i\pi/2, -7i\pi/2, \ldots$. Thus,

$$\sum_{n=1}^{\infty} \frac{\sin\theta_n}{r_n} = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = \frac{1}{2}.$$

Since $F(0) = i + 1, F'(0) = (i-1)/2$, then $F'(0)/F(0) = i/2$.

The author obtains Weierstrass' factorization of the entire functions of the form (4.1). The corresponding result is

Theorem VI. *We have*

$$F(z) = F(0)\exp((a+b)z/2)\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right),$$

the product being conditionally convergent.

The most interesting result in the paper is

Theorem VII. *If $\varphi(t)$ and $\psi(t)$ are Lebesgue integrable functions, such that*

$$\int_0^t \varphi(u)\psi(t-u)\,du = 0 \tag{4.7}$$

almost everywhere in the interval $0 < t < a$, then $\varphi(t) = 0$ almost everywhere in $(0,\lambda)$ and $\psi(t) = 0$ almost everywhere in $(0,\mu)$, where $\lambda + \mu \geq a$.

It is remarkable that it is obtained as a corollary of **Theorem IV**. The latter concerns the asymptotics of the number $n(r)$ and it is quite surprising that it implies such a deep result as **Theorem VII**.

Comments and references

1. Suppose the function $f(t), -1 \leq t \leq 1$, is of bounded variation, continuous at the points ± 1 and, moreover, $f(-1) = f(1) = 1$. Then:

There exists a positive constant K such that the zeros of the entire function (4.1) *with $a = -1$ and $b = 1$ are located in the strip $|\Re z| < K$.*

If $n(r)$ is the number of the zeros of (4.1) *with $a = -1$ and $b = 1$ located in the disk $|z| \leq r$, then*

$$n(r) = \frac{2r}{\pi} + O(1).$$

Suppose, in addition, that the function $f(t)$ is continuous on the whole interval $[-1, 1]$ and let $\omega(f;\delta)$ be its modulus of continuity. Then:

The zeros of the function (4.1) *with $a = -1$ and $b = 1$ are located in the region defined by the inequality*

$$|\Re z| < K|z|\,\omega\left(f;\frac{1}{|z|}\right)$$

with a suitable constant K, and

$$n(r) = \frac{2r}{\pi} + O\left(r\,\omega\left(f;\frac{1}{r}\right)\right).$$

These results appear in the survey paper of R.E. Langer **On the zeros of exponential sums and integrals**, *Bull. Amer. Math. Soc.* **37** (1930), 213-239, where the author claims that they have been taken from the paper of M.L. Cartwright **The zeros of certain integral functions**, *Quarterly J. Math.* 1 (1930), 38-59 and are attributed by her to Hardy.

2. Titchmarsh's results about the zeros of the entire functions (4.1) can be carried over the entire functions

$$\int_a^b f(t)\exp(izt)\,dt, \quad -\infty < a < b < \infty. \tag{4.7}$$

For example, if $\{z_n = \rho_n \exp i\sigma_n\}_{n=1}^{\infty}$, $\rho_1 \leq \rho_2 \leq \rho_3 \leq \dots$ are the zeros of (4.7) in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $n(r)$ is their number in the disk $|z| \leq r$, then (4.3) holds. Further, the series

$$\sum_{n=1}^{\infty} \frac{\sin \sigma_n}{\rho_n} \tag{4.8}$$

is absolutely convergent. This property leads to the conclusion that whatever $\delta \in (0, \pi/2)$ is, there are infinitely many zeros of (4.7) located in the union $\mathcal{A}_\delta$ of the angular domains $|\arg z| < \delta$ and $|\arg(-z)| < \delta$. In particular, the last assertion holds for the zeros of the entire functions (1.14) and (1.15). However, the answer of the question whether, for each fixed $\delta \in (0, \pi/2)$, all but finitely many zeros of a function of the form (4.7) are located in $\mathcal{A}_\delta$, is negative. A counterexample is the entire function $\Lambda(iz)$.

3. Let $\mathcal{C}[0,\infty)$ be the $\mathbb{R}$-linear space of all real and continuous functions on the interval $[0,\infty)$. If $\varphi, \psi \in \mathcal{C}[0,\infty)$, then the function $\varphi * \psi$, defined by

$$(\varphi * \psi)(t) = \int_0^t \varphi(u)\psi(t-u)\,du, \quad 0 < t < \infty,$$

is called their convolution. It is an easy exercise to establish that the composition law just defined is associative and commutative. Assuming it as a rule of multiplication in $\mathcal{C}[0,\infty)$, it ia clear that $(\mathcal{C}[0,\infty), *)$ becomes an associative and commutative algebra over the field of real numbers. The most remarkable property of this algebra, considered as ring with respect to the convolution, is that it is an integral domain. This means that, if $\varphi, \psi \in \mathcal{C}[0,\infty)$ and $\varphi * \psi = 0$, that is $(\varphi * \psi)(t) = 0$ for $0 \leq t < \infty$, then either $\varphi = 0$, or $\psi = 0$. In other words, there are no divisors of the zero in the ring $(\mathcal{C}[0,\infty), *)$. The last property is a simple corollary of **Theorem VII** in Titchmarsh's paper.

It is worth mentioning that the existence of quotient field of the ring $(\mathcal{C}[0,\infty), *)$ was used by the polish mathematician J. Mikusinski for justifying the Heviside operational calculus and, thus, giving a general approach for constructing other such calculi.

5. The Dutch master

N. G. DE BRUIN begins his paper **The roots of trigonometric integrals**, *Duke Math. J.*, 17 (1950), 197-226, with results concerning the reality of zeros of entire functions of the form

$$\int_{-\infty}^{\infty} F(t) \exp izt \, dt. \tag{5.1}$$

obtained by PÓLYA under the conditions:

1. $F(-t) = \overline{F(t)}$ for each $t \in \mathbb{R}$.
2. F is locally integrable.
3. $|F(t)| = O \exp(-|t|^b)$ for $t \to \pm\infty$, $b > 2$.

First he provides concrete examples of such functions,

$$\int_{-\infty}^{\infty} \exp(-t^{2n}) \exp izt \, dt, \quad n = 1, 2, 3, \ldots \tag{5.2}$$

and

$$\int_{-\infty}^{\infty} C(t) \exp izt \, dt, \quad C(t) = \exp(-\lambda \cosh t), \quad \lambda > 0. \tag{5.3}$$

Then he formulates one of the main results in POLYA'S paper [Pólya 1927a] concerning the universal factors preserving the reality of the zeros of the functions (5.1) and, claiming that the paper is a continuation of PÓLYA'S research, announces the main results:

THEOREM 1. *Let $f(t)$ be an integral function of t such that its derivative $f'(t)$ is the limit (uniform in any bounded domain in the t-plane) of a sequence of polynomials, all of whose roots lie on the imaginary axis. Suppose furthermore that $f(t)$ is not a constant, and that $f(t) = f(-t)$, $f(t) \geq 0$ for real values of t. Then the integral*

$$\int_{-\infty}^{\infty} \exp(-f(t)) \, \exp izt \, dt \tag{5.4}$$

has real roots only.

THEOREM 2. *Let N be a positive integer and put*

$$P(t) = \sum_{n=-N}^{N} p_n \exp nt, \quad \Re p_n > 0, \quad p_{-n} = \overline{p}_n, \quad n = 1, 2, 3, \ldots, N. \tag{5.5}$$

Let the function $q(z)$ be regular in the sector $-\pi/2N - N^{-1} \arg p_N < \arg z < \pi/2N - N^{-1} \arg p_N$ and on its boundary, with possible exception of $z = 0$ and

$z = \infty$ *which may be poles (of arbitrary finite order) for* $q(z)$. *Furthermore suppose* $\overline{q(z)} = q(1/\overline{z})$ *in this sector (in other words,* $q(z)$ *is real for* $|z| = 1$*). Then all but a finite number of roots of the function*

$$(5.6) \qquad \Phi(z) = \int_{-\infty}^{\infty} \exp(-P(t))Q(t)\exp izt\,dt, \quad Q(t) = q(\exp t)$$

are real.

Then the author makes the following comments: "It may be remarked that our method fails to give any useful information concerning the number and location of the non-real roots of (5.6) in the general case, so that this very peculiar result may be of very little practical importance". This sounds a little strange because a considerable part of the paper is devoted to its proof and it plays a decisive role in proving THEOREM 1. On the other hand, the latter is an essential generalization of the fact that the entire functions (5.2) and (5.3) have only real zeros.

In Sections 2 and 3 DE BRUIN introduces special universal factors. Suppose that the kernel F in (5.1) satisfies the requirements $1, 2$ and 3 and the complex function $S(t)$, $-\infty < t < \infty$, obeys the following properties:

(α) If the roots of (5.1) lie in a strip $|\Im z| \le \Delta$, $\Delta > 0$, then those of

$$(5.7) \qquad \int_{-\infty}^{\infty} F(t)S(t)\exp izt\,dt$$

lie in a strip $|\Im z| < \widetilde{\Delta}$, where $\widetilde{\Delta} < \Delta$, and $\widetilde{\Delta}$ is independent of F.

(β) If, for any $\varepsilon > 0$, all but a finite number of roots of (5.1) lie in the strip $|\Im z| \le \varepsilon$, then the function (5.7) has only a finite number of non-real roots.

If $S(t)$ satisfies (α) and (β), it is called a *strong universal factor*. As it is pointed out, it follows from (α) that any strong universal factor is a universal factor in PÓLYA'S sense.

Let $f(z)$ be a real algebraic polynomial, $\delta_k \pm \Delta_k i$, $\delta_k \in \mathbb{R}$, $\Delta_k > 0$, $k = 1, 2, 3, \ldots, n$, be its nonreal zeros, and let $J_k(f)$ be the closed disk $|z - \delta_k| \le \Delta_k$, $k = 1, 2, 3, \ldots, n$. A well-known result of J.L.W.V. JENSEN states that the zeros of $f'(z)$ are either real or are located in $J(f) = \bigcup_{k=1}^{n} J_k(f)$. The theorem of JENSEN is a refinement of the classical result for the location of zeros of the derivative of algebraic polynomial with arbitrary complex coefficients, usually referred to as The GAUSS-LUCAS Theorem.

In the paper under consideration DE BRUIN establishes a generalization of JENSEN'S theorem by using a translation operator instead of the operator of differentiation.

Let N be a positive integer and, for $k = 1, 2, 3, \ldots, n$, denote by $B_k(f; N, \lambda)$, $\lambda > 0$, the closed set of points (x, y), defined by

$$B_k(f; N, \lambda) = \begin{cases} N^{-1}(x - \delta_k)^2 + y^2 \le \Delta_k^2 - \lambda^2 N, & \text{if } \Delta_k > \lambda N^{1/2}, \\ \emptyset & \text{if } \Delta_k \le \lambda N^{1/2}. \end{cases}$$

Then de Bruin proves:

Theorem 4. *Suppose that all the roots of the polynomial* $\varphi(u) = \sum_{k=0}^{N} a_k u^k$, $a_N \neq 0$ *lie on the unit circle* $|u| = 1$, *that* $f(z)$ *is a real polynomial, and that* $\lambda > 0$. *Then the roots of the polynomial*

$$T^{-\lambda N}\varphi(T^{2\lambda})f(z) = \sum_{k=0}^{N} a_k \; f(z + (2k - N)\lambda i) \tag{5.8}$$

are contained in $B(f; N, \lambda) = \bigcup_{k=1}^{n} B_k(b; N, \lambda)$. *Here* T^λ *represents a translation operator defined by* $T^\lambda f(z) = f(z + \lambda i)$.

The proof goes by induction with respect to N. The case $N = 1$ is established in Theorem 3 which states that if $\gamma \in \mathbb{C} \setminus \{0\}$, then the zeros of the polynomial $(\gamma T^\lambda + \overline{\gamma} T^{-\lambda})f(z) = \gamma f(z + \lambda i) + \overline{\gamma} f(z - \lambda i)$ are located in $B(f; 1, \lambda)$. As is it pointed out, Jensen's theorem is the limit case of Theorem 3 when $\lambda \to 0$. Furthermore, it is observed that if a real polynomial $g(z)$ has its zeros in $B(f; m, \lambda)$, then the roots of $\gamma g(z + \lambda i) + \overline{\gamma} g(z - \lambda i)$ are in $B(f; m + 1, \lambda)$, and this is the main property which is used in the induction step to prove Theorem 4.

These results for polynomials are generalized to entire functions in Section 3. A typical one is:

Theorem 9. *Let the real integral function* $f(z)$ *be of order* < 2 *and suppose that* $f(z)$ *has all but a finite number of roots outside the strip* $|\Im z| \leq \Delta$. *If furthermore* $\varphi(u)$ *satisfies the conditions of* Theorem 4, *then all but a finite number of roots of* $T^{-\lambda N}\varphi(T^{2\lambda})f(z)$ *satisfy* $|\Im z| \leq \{\mathrm{Max}(\Delta^2 - N\lambda^2, 0)\}^{1/2}$.

Immediately after the definition of a strong universal factor de Bruin claims that a function of the type

$$S(t) = \sum_{n=-N}^{N} a_n \exp(\lambda n t), \quad \lambda > 0, \quad a_{-n} = \overline{a_n}, \quad n = 0, 1, 2, \ldots, N, \tag{5.9}$$

is a strong universal factor if all its roots lie on the imaginary axis and, conversely, if a function of the form (5.9) is such a factor, then its roots lie on the imaginary axis.

The first part of the above assertion is a corollary of a more general result concerning the zeros of entire functions of the form (5.1) when the function $F(t)$ satisfies the requirements $1, 2$ and 3:

Theorem 11. *Let the roots of the function* (5.9) *lie on the imaginary axis. Then we have: If the roots (all but a finite number of the roots) of* (5.1) *lie in the strip* $|\Im z| \leq \Delta$, *then the roots (all but a finite number of the roots) of the real integral function* (5.7) *lie in the strip* $|\Im z| \leq \{\Delta^2 - \lambda^2 N/2\}^{1/2}$ *if* $\Delta > \lambda(N/2)^{1/2}$, *and are real if* $\Delta \leq \lambda(N/2)^{1/2}$.

Theorem 12 may be considered as a "multiplicative" version of Theorem

11, where, as an universal factor one uses the function

$$\prod_{k=1}^{N}(\gamma_k \exp\lambda_k t + \overline{\gamma}_k \exp(-\lambda_k t)),$$

where $|\gamma_k| = 1$ and $\lambda_k > 0$ for $k = 1, 2, 3, \dots, N$. Further, the author comments upon Theorem 11:

"Theorem 11 proves the statements (α) and (β) made in the introduction concerning strong universal factors. Although it will be not used in this paper, we shall prove here that also the functions $\exp((\lambda^2/2)t^2), \lambda^2 > 0$ has property (α). We do not know whether they have or have not the property (β)."

The assertion in question is:

Theorem 13. *Suppose that the function $F(t)$ satisfies the conditions $1, 2, 3$ and that the zeros of the entire function* (5.1) *lie in the strip $|\Im z| \le \Delta$. Then all the roots of the entire function*

$$\int_{-\infty}^{\infty} F(t)\exp(\lambda^2 t^2/2)\exp(izt)\,dt$$

lie in the strip $|\Im z| \le \{\mathrm{Max}(\Delta^2 - \lambda^2), 0\}^{1/2}$

A similar statement, which the author qualifies as "a slight extension of Pólya's result on universal factors" is the next one:

Theorem 14. *Let the roots of* (5.1) *lie in the strip $|\Im z| \le \Delta$. Let $\varphi(z)$ be a real integral function of genus 0 or 1 with real roots only. Then the roots of the entire function*

$$\int_{-\infty}^{\infty} F(t)\varphi(it)\exp(izt)\,dt$$

lie in the strip $|\Im z| \le \Delta$ also.

The asymptotics of the complex function

$$H(s) = \int_{0}^{\infty} u^s \exp(-g(u))\,du, \quad \Re s > 0, \tag{5.10}$$

where $g(u) = u + \alpha_1 u^{(N-1)/N} + \alpha_2 u^{(N-2)/N} + \dots + \alpha_N$, N is a positive integer and $\alpha_k \in \mathbb{C}$, for $k = 1, 2, 3, \dots, N$, as $s \to \infty$, is studied in Section 4 of de Bruin's paper. The result, obtained by a very technical application of the saddle-point method, is:

Theorem 15. *If b is a positive constant and $H(s)$ is given by* (5.10), *we have*

$$H(s) = (2\pi\xi)^{1/2}\, exp(-g(\xi))\,\xi^s\,\{1 + O(s^{-1/N})\}, \tag{5.11}$$

uniformly for $\Re s > -b$, $|s| \to \infty$. Here $\xi = s + \gamma_1 s^{1-1/N} + \gamma_2 s^{1-2/N} + \dots$ is absolutely convergent for s large and satisfies $\xi g'(\xi) = s$.

It it noticed that if $N = 1$ and $\alpha_1 = 0$, then $g'(u) = 1, \xi = s$ and (5.11) becomes the familiar STIRLING'S formula for $\Gamma(s+1)$.

The proofs of THEOREM 2 and THEOREM 1 are given in Section 5. For this purpose, an asymptotic formula for the entire function (5.6), involving the function $H(z)$, is obtained. It leads to the conclusion that, for every $\varepsilon > 0$, this entire function has a finite number of zeros outside the strip $|\Im z| \le \varepsilon$ (see THEOREM 18). Further, it is pointed out that if $P(t)$ and $Q(t)$ satisfy the requirements in THEOREM 2, then the same holds for $P(t)$ and $Q(t)(\exp t + 2 + \exp(-t))^{-1}$. Hence, the entire function

$$\int_{-\infty}^{\infty} \exp(-P(t))Q(t)(\exp t + 2 + \exp(-t))^{-1} \exp(izt)\, dt$$

has a finite number of zeros outside any strip $|\Im z| \le \varepsilon$. Then the property (β) of the strong universal factors with $S(t) = \exp t + 2 + \exp(-t)$ implies THEOREM 2.

Let the polynomial $P(t)$ is as required in THEOREM 2 and suppose that $P'(t)$ has purely imaginary roots only. Then, by virtue of THEOREM 2 the entire function

$$\mathcal{P}(z) = \int_{-\infty}^{\infty} \exp(-P(t)) \exp(izt)\, dt$$

has a finite number of non-real zeros. Denote by Δ the smallest non-negative number such that all the roots of $\mathcal{P}(z)$ lie in the strip $|\Im z| \le \Delta$ and suppose that $\Delta > 0$. The function $P'(t)$ is a strong universal factor and, hence, by property (α) of such factors, the roots of the entire function

$$\widetilde{\mathcal{P}}(z) = \int_{-\infty}^{\infty} \exp(-P(t))\, i\, P'(t)\, \exp(izt)\, dt$$

lie in a strip $|\Im z| \le \widetilde{\Delta}$, $\widetilde{\Delta} < \Delta$. Since $\mathcal{P}(z) = -z\widetilde{\mathcal{P}}(z)$, it follows that the roots of $\mathcal{P}(z)$ lie in the strip $|\Im z| \le \tilde{\Delta}$, which contradicts the minimum property of Δ. Hence, $\Delta = 0$ and all the zeros of $\mathcal{P}(z)$ are real (see THEOREM 19).

Further, the author proves that if all the roots of the derivative of the real polynomial $f_n(t) = \sum_{k=1}^{2n} a_k t^k$ are on the imaginary axis, and if $a_{2n} > 0$, then the entire function

$$\int_{-\infty}^{\infty} \exp(-f_n(t))\, \exp(izt)\, dt$$

has real zeros only (see THEOREM 20).

Suppose that the function $f(t)$ satisfies the conditions of THEOREM 1, i.e.

$$f'(t) = a\, \exp(bt^2)\, t^{2p+1} \prod_{k=1}^{\infty} (1 + \delta_k^2 t^2),$$

where $a > 0$, $b \geq 0$, $\delta_k > 0$, $k = 1, 2, 3, \dots$ and $\sum_{k=1}^{\infty} \delta_k^2 < \infty$. Let the polynomials $\{f_n(t)\}_{n=1}^{\infty}$ be defined by

$$f_n'(t) = a\left(1 + \frac{bt^2}{n}\right)^n t^{2p+1} \prod_{k=1}^{n}(1 + \delta_k^2 t^2) \quad \text{and} \quad f_n(0) = f(0), \quad n = 1, 2, 3, \dots.$$

Then the entire functions

$$F_n(z) = \int_{-\infty}^{\infty} \exp(-f_n(t)) \exp(izt)\, dt, \quad n = 1, 2, 3, \dots$$

have only real zeros. Moreover,

$$\lim_{n\to\infty} F_n(z) = \int_{-\infty}^{\infty} \exp(-f(t)) \exp(izt)\, dt$$

uniformly in any bounded domain of $\mathbb{C}$. Then the well-known theorem of HURWITZ implies that the entire function (5.4) has real zeros only.

In Section 6 DE BRUIN'S investigates the distribution of zero of entire functions of the form

$$\int_{-\infty}^{\infty} \exp(-\lambda \cosh t) \left(\sum_{n=-N}^{N} \alpha_n \exp nt\right) \exp(izt)\, dt. \tag{5.12}$$

THEOREM 2 guarantees that all the zeros of this function, except for at most a finite number, are real. In fact, it has at most N pairs of conjugate complex zeros (see THEOREM 21).

The entire functions

$$\int_{-\infty}^{\infty} \exp(-\lambda \cosh t)(\mu + \cosh t) \exp(izt)\, dt \tag{5.13}$$

and

$$\int_{-\infty}^{\infty} \exp(-\lambda \cosh t)(\mu + \cosh^2 t) \exp(izt)\, dt \tag{5.14}$$

are particular cases of (5.12). For both of them it is proved that they have only real zeros when $\mu \geq 0$ (THEOREM 22). It is pointed out that the same is true if $-1 \leq \mu < 0$, since then $\mu + \cosh t$ and $\mu + \cosh^2 t$ are universal factors, and that each of the functions (5.13) and (5.14) has a pair of purely imaginary roots if $-\mu$ is positive and large enough.

Further, suppose that the real polynomial $f(z)$ of degree N possesses negative roots only, and let $\lambda \geq N/2$. Then the entire function

$$\int_{-\infty}^{\infty} \exp(-\lambda \cosh t) f(\cosh t) \exp(izt)\, dt$$

has only real roots (see THEOREM 24).

Under the same requirement on the zeros of the real polynomial $f(z)$ it is proved that the entire function

$$\int_{-\infty}^{\infty} \exp(-\lambda \cosh^2 t) f(\cosh t) \exp(izt)\, dt$$

has only real roots when $\lambda > 0$ and $f'(1)/f(1) \le 2\lambda$ (see THEOREM 26).

Comments and references

1. The paper of CHARLES M. NEWMAN, **Fourier transforms with only real zeros**, *Proc. Amer. Math. Soc.* **61**, Number 2 (1976), 245-251 deserves a special attention. Indeed, the author is an expert in the mathematical physics and, which maybe is the most striking, his main result is in fact a corollary of his "study of certain quantum field theoretic problems".

In the beginning of the paper, using TITCMARSH'S notation for RIEMAN'S ξ-function, the author introduces the entire functions

$$\Xi_b(z) = \int_{-\infty}^{\infty} \exp(izt - bt^2)\Phi(t)\, dt, \tag{5.15}$$

when b is real and $\Phi(t)$ is the function whose cosine Fourier transform is $\Xi(z) = \Xi_0(z)$. Further, it is pointed out that THEOREM 13 of DE BRUIN (formulated as THEOREM 1 in the paper) and the fact that the zeros of $\Xi(z)$ lie in the strip $|\Im z| \le 1/2$ imply that, if $b \le -1/8$, then the function (5.15) has only real zeros. The main result in the paper is:

THEOREM 3. *There exists a real number b_0 with $-1/8 \le b_0 < \infty$ such that $\Xi_b(z)$ has only real zeros when $b \le b_0$ but has nonreal zeros when $b > b_0$.*

Then the author makes the following comment (REMARK 2 on p. 247): "The Riemann hypothesis is the statement that $b_0 \ge 0$; we make the complementary conjecture that $b_0 \le 0$. This new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so".

Motivated mainly by THEOREM 13 of DE BRUIN, the author defines the class $\mathcal{R}$ of even, nonnegative, finite measures ρ on the real line such that for any $b > 0$ the FOURIER transform of $\exp(-bt^2)\, d\rho(t)$ has only real zeros. It seems that this is motivated also by the fact that $\mathcal{R}$ contains, as it is proved by PÓLYA, the absolutely continuous measures ρ with density

$$\frac{d\rho}{dt} = Kt^{2m} \exp(-\alpha t^4 - \beta t^2) \prod_{k=1}^{\omega} \left(1 + \frac{t^2}{a_k^2}\right), \quad 0 \le \omega \le \infty, \tag{5.12}$$

where $K > 0$, m is a nonnegative integer, α and $\{a_k\}_{k=1}^{\omega}$ are positive, the series $\sum_{k=1}^{\omega}(1/a_k^2)$ is convergent and β is real (or else $\alpha = 0$ and $\beta > 0$).

It is worth reminding another comment of the author that: "the class $\mathcal{R}$ is also a natural one in quantum field theory where one is particularly interested in function $V(t)$ such that $\exp(-\lambda V(t))\,dt \in \mathcal{R}$ for all $\lambda > 0$, and the example $V(t) = \alpha t^4 + \beta t^2$ of (5.16) was in fact rediscovered by statistical mechanical methods in B. SIMON and R.B. GRIFFITHS, **The $(\phi^4)_2$ field theorie as a classical Ising model**, *Comm. Math. Phys.*, **33** (1973), 145-164".

The main contribution of the author is a complete characterization of the class $\mathcal{R}$, given in THEOREM 2 which concerns holomorphic functions defined by LAPLACE'S transforms of the form

$$Z_b(z) = \int_{-\infty}^{\infty} \exp(zt - bt^2)\,d\rho(t), \quad b > 0, \tag{5.13}$$

where ρ is an even, nonnegative, finite measure on the real line. A short version of this theorem is the following assertion:

The entire function (5.13) *has only pure imaginary zeros for every $b > 0$ if and only if either $\rho(t) = K(\delta(t - t_0) + \delta(t + t_0))$ for some $K > 0$ and $t_0 \geq 0$, where $\delta(t - t_0)$ denotes the point measure of unit mass concentrated at the point t_0, or else ρ is absolutely continuous with density of the kind* (5.12).

As the author points out, his main result is an immediate corollary of THEOREM 13 of DE BRUIN and his THEOREM 2.

2. The main result in the paper of HASEO KI and YOUNG-ONE KIM, **de Bruin's question on the zeros of Fourier transforms**, *J. Anal. Math.*, 91 (2003), 369-387, is the following:

Let $f(z)$ be a real entire function of genus 1^* and if it is of order 2, then let its type $\sigma < \infty$. Suppose that there exists $\Delta \geq 0$ such that for each $\varepsilon > 0$ all but a finite number of the zeros of $f(z)$ lie in the strip $|\Im z| \leq \varepsilon$. If $\lambda > 0$ and $\lambda\sigma < 1$, then the entire function $\exp(-\lambda^2 D^2) f(z)$ is also of genus 1^* and all but a finite number of its zeros lie in the strip $|\Im z| \leq \sqrt{\max\{\Delta^2 - 2\lambda, 0\}} + \varepsilon$. Further, if $\Delta^2 < 2\lambda$, then all but a finite number of the zeros of $\exp(-\lambda^2 D^2) f(z)$ are real and simple.

The case $\Delta = 0$ gives an affirmative answer to DE BRUIN'S question posed after the comments on THEOREM 12. This is a corollary of the easily verified equality

$$\exp(-\lambda^2 D^2) f(z) = \int_{-\infty}^{\infty} F(t) \exp(\lambda^2 t^2) \exp(izt)\,dt$$

where $f(z)$ is the entire function (5.1) with $F(t)$ satisfying PÓLYA'S conditions $1 - 3$ stated in the beginning of this section.

Remark. The entire functions of genus 1^* are, in fact, the entire functions of the form (3.38) introduced by PÓLYA in his survey paper on the written heritage of JENSEN.

6. The Bulgarian trace

The first bulgarian mathematician who was influenced by the work of G. PÓLYA on entire functions with real zeros, was L. TSCHAKALOFF. His paper **On a class of entire functions**. *Journal Acad. Bulgare Sci.* **36** (1927), 51 - 92, (Bulgarian, German Summary), is devoted to the class (A) of entire functions which are either algebraic polynomials with zeros in the upper half-plane or uniform limits of such polynomials. The main result of the paper is **Theorem IV** on p. 68 which can be considered as an extension of the HERMITE-BIHLER theorem. It states that:

If the function $g(z)$ is in the class (A), then the functions

$$G_\alpha(z) = \exp(i\alpha)g(z) + \exp(-i\alpha)\overline{g}(z), \quad \overline{g}(z) = \overline{g(\overline{z})}, \quad \alpha \in \mathbb{R}, \tag{6.1}$$

which are not identically equal to a constant, have the following properties:

1) *The function $G_\alpha(z)$ cannot have non-real zeros; indeed, it has infinitely many real zeros except when the function $g(z)$ has the special form $g(z) = P(z)\exp p(z)$, where P and p are algebraic polynomials such that P is in the class (A) and p is a real polynomial of degree at most two with non-positive leading coefficient.*

2) *Both $g(z)$ and $\overline{g}(z)$ vanish at each multiple zero of $G_\alpha(z)$.*

3) *If $g(z)$ does not vanish for real values of z, then between each two consecutive zeros of $G_\alpha(z)$ there is only one zero of $G_\beta(z)$ when the difference $\alpha - \beta$ is not a multiple of π.*

It is well known that the order of every functions in the class (A) is at most two (EGON LINDWART AND GEORG PÓLYA, **Über einen Zusammenhang zwischen der Konwergenz von Polynomfolgen und der Verteilung ihrer Wurzeln**. *Rend. Circ. Mat. Palermo*, **37** (1914) 1-8). On the other hand, TSCHAKALOFF points out that an entire function of order not greater than two with zeros in the closed upper half-plan does not need to be in the class (A). He proves this with suitable examples, one of them being the entire function

$$F(z) = \frac{1}{z^2}(\exp(-iz) - 1 + iz). \tag{6.2}$$

It is clear that it is of order one and of normal exponential type. This means that it is of exponential type and its type is neither zero nor infinity. In fact, the type of $F(z)$ is equal to one. The zeros of this function are

$$z_k = \pm\left((2k+1)\frac{\pi}{2} - \varepsilon_k\right) + i(\log(2k\pi) + \eta_k), \quad k = 1, 2, 3, \ldots,$$

where $\{\varepsilon_k\}_{k=1}^\infty$ and $\{\eta_k\}_{k=1}^\infty$ are positive numbers such that $\lim_{k\to\infty} \varepsilon_k = \lim_{k\to\infty} \eta_k = 0$. The function (6.2) is not in the class (A). Otherwise, since this class is invariant under differentiation, the function $z^3 f(z)$ as well as all its derivatives would be in the same class. However, it is impossible because $(z^3F(z))^{(n)}$ vanishes at the point $-ni$ for $n > 2$. One may provide another proof having in mind that

$$\frac{1}{2i}(F(z) - \overline{F}(z)) = \frac{1}{z^2}(z - \sin z) = \int_0^1 (1-t)\sin zt\, dt, \quad \overline{F}(z) = \overline{F(\overline{z})}.$$

Indeed, HERMITE-BIEHLER'S theorem holds for the functions in the class (A). Hence, the entire function $(F(z) - \overline{F}(z))/(2i)$ has only real zeros. On the other hand, it is a general property of the entire Fourier transforms $\int_0^\infty \varphi(t)\sin zt\, dt$ with non-negative and decreasing kernels φ that they have only one real zero. TSCHAKALOFF gives a proof of the latter statement in the case when the support of the function φ is in the interval $[0,1]$. In fact, he reproduces the proof given by G. PÓLYA in *Math. Z.* **2** (1918), 342-383.

We recall that the main ingredient in the proofs of the Prime Number Theorem given by HADAMARD and J. DE LA VALLÉE POUSSIN is the fact that the Riemann zeta function does not vanish on the boundary of the critical strip, which means that $\zeta(s) \neq 0$ when $\Re s \geq 1$. This, together with the WEIERSTRASS factorization of the entire function $\xi(s)$, shows that if $\sigma \geq 1/2$, then the entire function $\xi(-i\sigma + z)$ is in the class (A). Thus TSCHAKALOFF comes to the following conclusion:

The entire function

$$\exp(i\alpha)\xi(-i\sigma + z) + \exp(-i\alpha)\xi(i\sigma + z), \quad \alpha \in \mathbb{R},$$

has infinitely many zeros, all real and simple. Moreover, to every two α's whose difference is not a multiple of π, there correspond functions with mutually interlacing zeros.

As another application of the results about the zeros of the functions in the class (A) TSCHAKALOFF obtains the following assertion:

Let $f(t)$ be a real, non-negative, non-decreasing and bounded function on the interval $(-1,1)$. If α is real, then the entire function

$$F_\alpha(f;z) = \int_{-1}^1 f(t)\cos(zt + \alpha)\, dt \tag{6.3}$$

has infinitely many zeros all of which are real. Moreover, if $f(t)$ is not exceptional in the sense of G. PÓLYA, *then all the zeros of $F_\alpha(z)$ are simple and between each two consecutive zeros of $F_\alpha(f;z)$ there is only one zero of $F_\beta(f;z)$ when $\alpha - \beta$ is not a multiple of π.*

Using ENESTRÖM-KAKEYA'S theorem, TSCHAKALOFF proves that the entire function

$$g(f;z) = \int_{-1}^1 f(t)\exp(izt)\, dt \tag{6.4}$$

is in the class (A) and then applies (6.1).

Let $\varphi(t)$ be a positive, increasing and bounded function in the interval $(0,1)$ and define the function $f(t)$ in the interval $(-1,1)$ assuming that $f(t)=\varphi(t+1)$ if $-1\le t<0$, and $f(t)=\lambda\varphi(t)$ if $0\le t<1$, where λ is a real constant such that $|\lambda|\ge 1$. Then TSCHAKALOFF proves that the entire function $g(f;z)$ is in the class (A). Hence, the entire functions

$$\frac{1}{2}\{\exp(i\alpha)g(f;z)+\exp(-i\alpha)g(f;z)\}$$

$$=\int_0^1\{\varphi(1-t)\cos(zt-\alpha)+\lambda\varphi(t)\cos(zt+\alpha)\}\,dt,\quad \alpha\in\mathbb{R},$$

possess only real zeros. In particular, by choosing $\varphi(t)=t^a$, $a>0$, $\lambda=\pm 1$, $\alpha=0$ or $\pi/2$, one concludes that the entire functions

$$\int_0^1\{(1-t)^a\pm t^a\}\cos zt\,dt\quad\text{and}\quad\int_0^1\{(1-t)^a\pm t^a\}\sin zt\,dt$$

have only real zeros.

Let now $\varphi(t)$ be a real, nonnegative and convex function defined on the interval $[0,1]$ with $\varphi(0)=0$. Define $f(t)=\varphi(t+1)$ for $-1\le t<0$ and $f(t)=\lambda-\varphi(t)$ for $0\le t\le 1$ where $\lambda\ge\varphi(1)$. Then the entire function $g(f;z)$ is in the class (A). Hence, the entire functions

$$\int_0^1\varphi(1-t)\cos(zt-\alpha)\,dt+\int_0^1(\lambda-\varphi(t))\cos(zt+\alpha)\,dt,\quad\alpha\in\mathbb{R},$$

have only real zeros. In particular, if $\varphi(t)=t^a, a\ge 1$ and $\lambda=1$, then the entire functions

$$\int_0^1\{(1-t)^a+1-t^a\}\cos zt\,dt\quad\text{and}\quad\int_0^1\{(1-t)^a-(1-t^a)\}\sin zt\,dt$$

have only real zeros.

In the last part of his paper TSCHAKALOFF studies the zero distribution of the entire function $F_\alpha(f;z)$ by applying HURWITZ'S method, i.e. by using MITTAG-LEFFLER'S decomposition of the meromorphic function

$$\frac{F_\alpha(f;z)}{\cos(z-\beta)}.$$

The result obtained by him is the following:

Let the real function $f(t)$ have a continuous derivative in the interval $[-1,1]$. If at least one of the numbers $(f(1)+f(-1))\cos\alpha, (f(1)-f(-1))\sin\alpha$ is different from zero, then the entire function $F_\alpha(f;z)$ has infinitely many real and only a finite number of non-real zeros.

Further, it is claimed that under the same assumption for the function $f(t)$, $F_\alpha(f;z)$ has finitely many multiple zeros and that the differences between its consecutive real zeros converges to π at infinity.

In his paper **Sur le nombre des zéros non-réels d'une classe de fonctions entière**. *C. R. Acad. Bulgare Sci.* **2** (1949), 9-12, TSCHAKALOFF answers the question: How many non-real zeros do the polynomials

$$P_n(z) = \int_{-1}^{1} \{p(t)(z+it)^n + \overline{p}(t)(z-it)^n\}\,dt, \quad n = 1,2,3,\dots, \tag{6.5}$$

have, provided $p(z)$ is an algebraic polynomial with arbitrary complex coefficients? It is remarkable that the upper bound obtained by the author depends only on the degree of p. More precisely, the following result holds:

Let $p(z)$ be an algebraic polynomial of degree m. If the polynomial (6.5) *is not identically zero, then the number of its non-real zeros is at most equal to* $2\,[m/2]$. *Moreover, to each positive number m there corresponds a polynomial $p(z)$ of degree m (not depending on n) such that the polynomial* (6.5) *has exactly* $2\,[m/2]$ *non-real zeros for every* $n \ge 2[m/2]$.

Further, the equalities

$$\left(\frac{z}{n}\right)^n P_n\left(\frac{n}{z}\right) = \int_{-1}^{1} \left\{p(t)\left(1+\frac{izt}{n}\right)^n + \overline{p}(t)\left(1-\frac{izt}{n}\right)^n\right\} dt,\ n = 1,2,3,\dots$$

and the theorem of HURWITZ imply that the entire function

$$\int_{-1}^{1} \{p(t)\,\exp(izt) + \overline{p}(t)\,\exp(-izt)\}\,dt$$

has at most m non-real zeros provided $p(z)$ is of degree m.

In particular, if $p(z)$ is a real polynomial of degree m, then the entire function

$$\int_{-1}^{1} p(t)\cos zt\,dt$$

may have at most m non-real zeros. If m is odd, then, since the above entire function is real, i.e. it assumes real values when z is real, it has at most $m-1$ non-real zeros.

In N. OBRECHKOFF'S paper **On the zeros of the polynomials and of some entire functions**. *Annuaire Univ. Sofia, Phys.-Math. Fac.* **37** (1940/41) No 1, 1-115 (Bulgarian, French Summary), the following complex version of a classical theorem of LAGUERRE for polynomials with real coefficients is obtained:

Let the zeros of the polynomial

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0,$$

be in the closed unit disk $|z| \leq 1$ *and let* $p(z)$ *be an arbitrary polynomial with zeros in the half-plane* $\Re z \leq n/2$. *Then the zeros of the polynomial*

$$P(z) = a_0 p(0) + a_1 p(1)\, z + a_2 p(2)\, z^2 + \cdots + a_n p(n)\, z^n$$

are in the disk $|z| \leq 1$ *too. If the zeros of* $A(z)$ *are in* $|z| \geq 1$ *and those of* $p(z)$ *are in the half-plane* $\Re z \geq n/2$, *then the zeros of* $P(z)$ *are in* $|z| \geq 1$. *If* $A(z)$ *has its zeros on* $|z| = 1$ *and* $p(z)$ *has its zeros on the line* $\Re z = n/2$, *then* $P(z)$ *has its zeros on* $|z| = 1$.

Let $h(z)$ be a real polynomial whose zeros are in the half-plane $\Re z \leq 1/2$. If f is a non-negative and non-decreasing function in the interval $[0, 1]$, then the above result and ENESTRÖM-KAKEYA'S theorem imply that the polynomials

$$\sum_{k=0}^{n} h\left(\frac{k}{n}\right) f\left(\frac{k}{n}\right) z^k, \quad n = 0, 1, 2, \ldots$$

have their zeros in the unit disk. Applying the method of variation of the argument, OBRECHKOFF obtains the following remarkable result:

Let $f(t)$ *be a positive and non-decreasing function in the interval* $(0, 1)$ *and let the zeros of the real polynomial* $h(z)$ *be in the half-plane* $\Re z \leq 1/2$. *Then the entire functions*

$$\int_0^1 h(t) f(t) \cos zt \, dt \quad \text{and} \quad \int_0^1 h(t) f(t) \sin zt \, dt$$

have only real zeros.

This beautiful result, which is sometimes called OBRECHKOFF'S h-theorem, appeared in the above paper, published in Bulgarian, so it is not very well-known to the the experts on the topic.

Another theorem about the zeros of algebraic polynomials, proved in the same paper, is the following:

Let the zeros of the polynomial $\sum_{k=0}^{n} a_k z^k$ *be in the unit disk. If* δ *is an arbitrary positive real number and* $p(z)$ *is a polynomial with zeros in the half-plane* $\Im z \geq 0$, *then the zeros of the polynomial*

$$\sum_{k=0}^{n} a_k p(z + (n - 2k)\delta i)$$

lie in the same half-plane.

Again, the above mentioned classical results of ENESTRÖM-KAKEYA and HURWITZ, combined with the latter statement, yield:

Let $\varphi(t)$ *be a positive and nondecreasing function in the interval* $(-a, a)$, $0 < a < \infty$. *If the zeros of the polynomial* $p(z)$ *are in the (closed) upper half-plane, then the same holds for the zeros of the polynomial*

$$\int_{-a}^{a} \varphi(t)\, p(z - it)\, dt.$$

It is clear that both Tschakaloff and Obrechkoff were inspired by the results and methods of G. Pólya. If we compare their results, it is quite evident that Tschakaloff's approach is rather analytical while that of Obrechkoff is more algebraic. Of course, both obtain results for the zero distribution of entire functions defined as finite Fourier transforms. However, Tschakaloff uses mainly the techniques developed by Pólya and Hurwitz, while Obrechkoff prefers to establish first results about zeros of algebraic polynomials. Another example of a result obtained by the latter approach is:

Let $\varphi(t)$ and $\psi(t)$ be non-negative functions in the interval $(0, \lambda)$, $\lambda > 0$, let $\varphi(t)$ be non-increasing, $\psi(t)$ be non-decreasing, with $\varphi(0) \le \psi(0)$. Let us define $f(t) = \varphi(t) + \psi(t)$ for $0 \le t \le \lambda$ and $f(t) = f(-t)$ for $-\lambda \le t \le 0$. Consider three polynomials $P(z), Q(z)$ and $R(z)$, such that $P(z)$ is real, $Q(z)$ has only real and negative zeros, and $R(z)$ has only real zeros. Then the polynomial

$$\int_{-\lambda}^{\lambda} f(t)P(z+it)\,dt$$

has at least as many real zeros as $P(z)$. Moreover, the polynomial

$$\int_{-\lambda}^{\lambda} f(t)Q(izt)\,dt$$

and the entire functions

$$\int_{-\lambda}^{\lambda} f(t)R(it)\exp(izt)\,dt$$

have only real zeros.

We hasten to remark that the idea to use functions like φ and ψ is due to Tschakaloff.

The above result is inspired by a Problem stated by G. Pólya in *Jahresber. Deutsch. Math. Ver.* **35** (1926), and solved by Obrechkoff (*Jahresber. Deutsch. Math. Ver.* **36** (1927). The Problem is the following:

Let $P(z), Q(z), R(z)$ and $S(z) = \sum_{k=0}^{n} a_k z^k$ be polynomials such that:

- *$P(z)$ is real,*
- *$Q(z)$ has only real and negative zeros,*
- *$R(z)$ has only real zeros, and*
- *$S(z)$ has its zeros on the unit circle.*

Then

- *the number of non-real zeros of the polynomial $\sum_{k=0}^{n} a_k P(z-(n-2k)i)$ is less or equal to those of $P(z)$;*

- *the polynomial* $\displaystyle\sum_{k=0}^{n} a_k Q((2k-n)iz)$ *has only real zeros;*
- *the zeros of the polynomial* $\displaystyle\sum_{k=0}^{n} R((2k-n)i)z^k$ *are on the unit circle.*

OBRECHKOFF obtains some interesting results which may be considered as necessary conditions that an entire Fourier transform is in $\mathcal{LP}$.

Let F be a Lebesgue-integrable function in the interval $(-a,a)$, $0 < a \leq \infty$ and let the entire function

$$\int_{-a}^{a} F(t)\exp(izt)\,dt$$

be in the class $\mathcal{LP}$. If the zeros of the polynomial $p(z)$ are in a strip parallel to the imaginary axis, then the zeros of the polynomial

$$\int_{-a}^{a} F(t)p(z+t)\,dt$$

are also in the same strip.

As a corollary of the above theorem, OBRECHKOFF obtains the following result:

Let $f(t)$ be an Lebesgue-integrable function in $(-1,1)$, such that $f(-t) = \overline{f(t)}$ for $t \in (-1,0)$ and let the entire function

$$\int_{-1}^{1} f(t)\,\exp(izt)\,dt$$

have only real zeros. If $P(z)$ is an arbitrary real polynomial and $Q(z)$ is a polynomial with only real and negative zeros, then the polynomial

$$\int_{-1}^{1} f(t)\,P(z+it)\,dt$$

has at least as many real zeros as $P(z)$, and the polynomial

$$\int_{-1}^{1} f(t)\,Q(izt)\,dt$$

has only real zeros.

In the beginning of his dissertation **Über die Nullstelle gewisser Klassen von Polynomen und ganzen Funktionen**, *Inagural Dissertation zur Erlangung der Doktorwürde der mathematischen Wissenschaften an der Physikalisch-Mathematischen Fakultät der Universität Sofia*, Sofia, 1940 (Bulgarian, German Summary), L. ILIEFF proposes an elementary proof of the well-known result of PÓLYA for reality of the zeros of the entire function

$$F_q(z) = \int_0^\infty \exp(-t^{2q}) \cos zt \, dt, \quad q \in \mathbb{N}.$$

Using only ROLLE'S theorem the author proves by induction that the polynomials

$$P_{m,n}(z) = \int_{-1}^{1} (1-t^{2q})^m (1+itz)^n \, dt, \quad k, n = 0, 1, 2, \dots$$

have only real zeros. Further, replacing z by z/n and letting n to converge to ∞, one concludes that the entire functions

$$\int_0^1 (1-t^{2q})^m \cos zt \, dt, \quad m = 1, 2, 3. \dots,$$

have only real zeros. Hence, the entire functions

$$E_m(z) = \int_0^{m^{1/2q}} \left(1 - \frac{t^{2q}}{m}\right)^m \cos zt \, dt, \quad m = 1, 2, 3, \dots$$

have also only real zeros. Since $\lim_{m\to\infty} E_m(z) = F_q(z)$ uniformly on every bounded domain, the assertion for the zeros of the function $F_q(z)$ is a consequence of HURWITZ'S theorem.

A very fruitful idea of ILIEFF is the synthesis of two algebraic theorems. The first of them is due to OBRECHKOFF (**Sur les racines des équations algébriques**, *The Tôhoku Mathem. J.* **38** (1933) 93-100):

Let λ be an arbitrary non-zero complex number with amplitude θ and let S be the closed strip between two parallel lines crossing the positive real axis at an angle $\theta + \pi/2$. If the zeros of the polynomial $p(z)$ are in S and the zeros of the polynomial $a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ are on $|z| = 1$, then the zeros of the polynomial

$$\sum_{k=0}^{n} a_k \, p(z + (n-2k)\lambda) \tag{6.6}$$

are also in S.

The second assertion, which plays the role of a "key lemma", is formulated as an auxiliary theorem on p. 15 of ILIEFF'S Dissertation:

Let $P(z)$ be a polynomial of n-th degree with zeros in the region $|z| \geq 1$ and let us define $P^*(z) = z^n \overline{P}(1/z)$. Then the zeros of the polynomials

$$P(z) + \gamma z^k P^*(z), \quad |\gamma| = 1, \quad k = 0, 1, 2, \dots \tag{6.7}$$

are on $|z| = 1$.

The following statement is more general than Theorem IV on p. 15 in the Dissertation. We provide a sketch of the proof where we follow the idea of Ilieff.

If $f(t)$ is a real, positive and non-decreasing function in the interval $(0,a)$, $a>0$ and the zeros of the algebraic polynomial $p(z)$ are in the strip $\alpha \le \Re z \le \beta$, then each of the polynomials

$$\int_0^a f(t)\{p(z+t)+\gamma p(z-t)\}\,dt, \quad |\gamma|=1, \tag{6.8}$$

has its zeros in the same strip.

It follows from the ENESTRÖM-KAKEYA theorem and the key lemma that, for every $n \in \mathbb{N}$ and $|\gamma|=1$, the zeros of the polynomial

$$\sum_{k=0}^{n-1} f\left(a\left(1-\frac{k}{n}\right)\right) z^k + (1+\gamma) f(0) z^n + \gamma \sum_{k=1}^{n} f\left(a\frac{k}{n}\right) z^{n+k},$$

are on the unit circle. If $\lambda = a/(2n)$, then OBRECHKOFF'S theorem implies that the zeros of the polynomials

$$Q_n(f,p;z) = \frac{a}{n}\sum_{k=0}^{n-1} f\left(a\left(1-\frac{k}{n}\right)\right) p\left(z+a\left(1-\frac{k}{n}\right)\right)$$

$$+\frac{a}{n}(1+\gamma)f(0)z^n + \frac{a}{n}\sum_{k=1}^{n} f\left(a\frac{k}{n}\right) p\left(z-a\frac{k}{n}\right), \quad n=1,2,3,\dots$$

are in the strip $\alpha \le \Re z \le \beta$. Since

$$\begin{aligned}\lim_{n\to\infty} Q_n(f,p;z) &= \int_0^1 f(a(1-t))p(z+a(1-t))\,dt + \gamma\int_0^1 f(at)p(z-at)\,dt \\ &= \frac{1}{a}\int_0^a f(t)\{p(z+t)+\gamma p(z-t)\}\,dt\end{aligned}$$

uniformly on every compact subset of the complex plane, HURWITZ'S theorem yields that the zeros of the polynomial (6.8), with $|\gamma|=1$, are in the strip $\alpha \le \Re z \le \beta$.

If $p(z)=z^n$, $\alpha=\beta=0$ and $a=1$, then the zeros of the polynomials

$$\int_0^1 f(t)\{(z+t)^n + \gamma(z-t)^n\}\,dt, \quad |\gamma|=1, \quad n=1,2,3,\dots$$

are on the imaginary axis and, hence, the polynomials

$$\int_0^1 f(t)\left\{\left(1+\frac{izt}{n}\right)^n + \gamma\left(1-\frac{izt}{n}\right)^n\right\} dt, \quad n=1,2,3,\dots$$

have only real zeros. Therefore, the entire function

$$\int_0^1 f(t)\{\exp(izt) + \gamma \exp(-izt)\}\, dt$$

has only real zeros too.

Classical results of Pólya are particular cases for $\gamma = 1$ and $\gamma = -1$, respectively.

In the paper **On the distribution of the zeros of a class of entire functions**. *Annuaire Univ. Sofia, Faculté des Sci.* **44** (1948) 143-174 (Bulgarian, German Summary), Ilieff proposes a technique which generates classes of entire functions, defined as finite cosine transforms, with only real zeros. Some of the results in this paper are published also in the note **Ganze Funktionen mit lauter reellen Nullstellen**. *C. R. Acad Bulgare Sci.* **2** (1949) 17-20. The first one is the following:

Let the function $f_0(t)$ be non-negative, increasing and integrable in the interval $(0,1)$. Define $x = x(t) = 1 - t^\alpha, \alpha \geq 1, 0 \leq t \leq 1, \varphi_0(x) = f_0((1-x)^{1/\alpha}), \varphi_1(x) = \int_0^x \varphi_0(u)\, du, 0 \leq x \leq 1$, and $f_1(t) = \varphi_1(1-t^\alpha), 0 \leq t \leq 1$. Then the entire function

$$\int_0^1 f_1(t)\, \cos zt\, dt$$

has only real zeros. If $\alpha > 1$, then its zeros are simple.

Examples to the above assertion are the entire functions

$$\int_0^1 (1-t^\alpha)^\lambda \cos zt\, dt, \quad \int_0^1 \sin^\lambda (1-t^\alpha) \cos zt\, dt,$$

$$\int_0^1 (1-t^\alpha)(1 - \log(1-t^\alpha)) \cos zt\, dt,$$

where $\alpha \geq 1$, $0 < \lambda \leq 1$.

The second result in the paper is more general:

Let the function $\psi(t)$ be positive and integrable in the interval $(0,1)$. Define $x = \omega(t) = \int_t^1 \psi(u)\, du, 0 \leq t \leq 1$. Suppose that the function $f_0(t)$ is nonnegative, increasing and integrable in $(0,1)$. Define $\varphi_0(x) = f_0(\omega^{-1}(x))$, $\varphi_1(x) = \int_0^x \varphi_0(u)\, du$ and $f_1(t) = \varphi_1((\omega(t))$. If the entire function

$$\int_0^1 \psi(t) f_0(t) \sin zt\, dt$$

has either only real zeros or only real and simple zeros, then the entire function

$$\int_0^1 f_1(t) \cos zt\, dt \tag{6.9}$$

has either only real zeros or only real and simple zeros too.

There is no doubt that ILIEFF'S paper **Über trgonometrische Integrale, welche ganze Funktionen mit nur reellen Nulstellen darstellen**. *Acta Math. Acad. Sci. Hungar.* **6** (1955), 191-195, is one of his main contributions to the zero distribution of entire Fourier transforms.

Let $p(z)$ be either a real and even polynomial or a real and even entire function such that: a) $p(a) = 0$, $a > 0$; b) $p'(iz)$ is in the class $\mathcal{LP}$. Denote by $A(a)$ the set of the real and even function $p(z), z \in \mathbb{C}$ satisfying the conditions a) and b). The first result in the paper of ILIEFF concerns cosine transforms. It is the corollary on p. 193:

If $p(z) \in A(a)$, $p(0) > 0$ and $\lambda > -1$, then the entire function

$$\int_0^a p^\lambda(t)\, \cos zt\, dt$$

has only real zeros.

It is pointed out that PÓLYA'S classical result that the entire function

$$\int_0^1 (1 - t^{2q})^\lambda \cos zt\, dt$$

has only real zeros, when q is a positive integer and $\lambda > -1$, is a consequence of the fact that the real and even function $1 - z^{2q}$ is in $A(1)$.

Let $p(z)$ be real, positive and even polynomial, such that the polynomial $p'(iz)$ has only real zeros. It is proved that if the positive integer $n > p(0)$, then there exists a unique sequence of real positive numbers a_n, such that $p(a_n) = n$. Moreover, $\lim_{n\to\infty} a_n = \infty$. It turns out that the function $1 - p(z)/n \in A(a_n)$ for every sufficiently large n. Hence, the entire function

$$\int_0^{a_n} \left(1 - \frac{p(t)}{n}\right)^n \cos zt\, dt$$

has only real zeros if n is large enough. Since $0 < p(t) < n$ and

$$0 < \left(1 - \frac{p(t)}{n}\right)^n < \exp(-p(t)) \quad \text{for} \quad n > p(0) \quad \text{and} \quad t \in (0, a_n),$$

it follows that the entire function $\int_0^\infty \exp(-p(t)) \cos zt\, dt$ has only real zeros. Further, HURWITZ'S theorem yields:

Let $f(z)$ be a nonconstant, real and even entire function such that $f'(iz)$ is in the class $\mathcal{LP}$. If $f(t) \geq 0$ for $t \in (0, \infty)$, then the entire function

$$\int_0^\infty \exp(-f(t)) \cos zt\, dt \tag{6.10}$$

has only real zeros.

The classical statement of PÓLYA concerning the reality of the zeros of the entire function

$$\int_0^\infty \exp(-a\cosh t)\cos zt\,dt, \quad a>0,$$

is a consequence of Ilieff's result because the entire function $a\cosh z$ satisfies the requirements of the above assertion.

Entire functions of the form (6.10) are studied in the paper of N. G. DE BRUIJN, *Duke Math. J.* **17** (1950), 197-226. In fact, ILIEFF'S theorem is a version of THEOREM 1 in DE BRUIJN'S paper. ILIEF'S proof of DE BRUIJN'S theorem is included first in his paper in *Proc. Inst. Math. Acad. Bulgare Sci.* **1** (1954), 147-153 (Bulgarian, German Summary), with the same title as that in *Acta Math. Hung.* It seems that by that time ILIEFF was not familiar with DE BRUIJN'S paper.

The "algebraic" tradition in studying the zero distribution of classes of complex polynomials having suitable integral representation is followed in the paper **On the distribution of the zeros of a class of polynomials and entire functions**, *Annuaire Univ. Sofia, Fac. des Sciences*, **46** (1949/50) Livre 1, 43-72 (Bulgarian, German Summary) of E. BOJOROFF. A typical example is the following statement:

Let the real function $f(t)$ and $\varphi(t)$ satisfy the conditions: 1) $f(t)>0$, $\varphi(t)>0$ *in the interval* $(0,a)$, $a>0$; 2) $f(t)$ *is increasing and* $\varphi(t)$ *is decreasing in* $(0,a)$. *If the zeros of the algebraic polynomial $p(z)$ lie in the strip $\alpha\le\Re z\le\beta$, then the zeros of the polynomial*

$$\int_0^a \{F_\lambda(t)p(z+t)+\Phi_\lambda(t)p(z-t)\}\,dt,$$

where

$$F_\lambda(t)=\int_0^{a-t}\{f(t+u)\varphi(u)+\lambda\varphi(t+u)f(u)\}\,du, \quad |\lambda|=1, \tag{6.11}$$

$$\Phi_\lambda(t)=\int_0^{a-t}\{\varphi(t+u)f(u)+\lambda f(t+u)\varphi(u)\}\,du, \quad |\lambda|=1, \tag{6.12}$$

lie in the same strip.

The proof is based on the following assertion, which is a further extension of ILIEFF'S generalization of I. SCHUR'S theorem, established in [Schur 1917]:

If the zeros of the polynomial $P(z)$ are in the region $|z|\ge 1$ and those of the polynomial $Q(z)$ are in the region $|z|\le 1$, then the zeros of the polynomials

$$P(z)Q^*(z)+\lambda z^s P^*(z)Q(z), \quad |\lambda|=1, \quad s=0,1,2,\dots$$

are on the unit circle.

If $\lambda = 1$, then the zeros of polynomial

$$\int_0^a s(t)\{p(z+t)+p(z-t)\}\,dt,$$

where

$$s(t) = \int_0^{a-t} \{f(t+u)\varphi(u) + f(u)\varphi(t+u)\}\,du,$$

are in the strip $\alpha \le \Re z \le \beta$. It is clear that in general the positive function $s(t)$ is not monotonically increasing in the interval $(0, a)$.

Let $\varphi(t) = 1$ for each $t \in (0, a)$ and suppose that $f(t)$ is continuous. Then, $s(t)$ is differentiable for $t \in [0, a]$ and $s'(t) = -\{f(t) + f(a-t)\}$ there. Hence, in this case $s(t)$ is an even decreasing function in $(0, a)$.

Further, suppose that $\alpha = \beta = 0$ and choose $p(z) = z^n$, $n = 1, 2, 3, \ldots$. Then the zeros of polynomials

$$P_n(z) = \int_0^a s(t)\,\{(1+izt)^n + (1-izt)^n\}\,dt, \quad n = 1, 2, 3, \ldots$$

are real. Therefore, the entire function

$$\int_0^a s(t)\cos zt\,dt = \frac{1}{2}\lim_{n\to\infty} P_n\left(\frac{z}{n}\right)$$

has only real zeros.

If $\lambda = -1$, then the polynomial

$$\int_0^a \sigma(t)\{p(z+t) - p(z-t)\}\,dt,$$

where

$$\sigma(t) = \int_0^a \{f(t+u)\varphi(u) - f(u)\varphi(t+u)\}\,du,$$

has its zeros in the strip $\alpha \le \Re z \le \beta$. Choosing again $p(z) = z^n$, one concludes that the polynomials

$$Q_n(z) = \int_0^a \sigma(t)\{(1+izt)^n - (1-izt)^n\}\,dt, \quad n = 1, 2, 3, \ldots,$$

have only real zeros and, hence, the entire function

$$\int_0^a \sigma(t)\sin zt\,dt = \frac{1}{2i}\lim_{n\to\infty} Q_n\left(\frac{z}{n}\right)$$

has also only real zeros too.

Statements similar to BOJOROFF'S one, were obtained in D. G. DIMITROV'S paper **On the distribution of zeros of certain polynomials and entire**

functions representable in an integral form *Proc. Vish Lesotehnicheski Institute* **8** (1960) 311-326. All these may be considered as extensions of Obreshkoff's earlier results. Here are some of them:

If $p(z)$ is a polynomial with zeros in the half-plane $\Im z \le 0$ and F_λ and Φ_λ are the functions defined in (6.11) and (6.12), with $|\lambda| \le 1$, then the zeros of the polynomial

$$\int_0^a \{F_\lambda(t)p(z+it) + \Phi_\lambda(t)p(z-it)\}\, dt,$$

are in the same half-plane.

Suppose that $P(z), Q(z)$ and $R(z)$ are polynomials, where $P(z)$ is real, $Q(z)$ with real negative zeros, and $R(z)$ with real zeros, and F_λ and Φ_λ are the functions (6.11) *and* (6.12) *with $|\lambda| = 1$. Then, the polynomial*

$$\int_0^a \{F_\lambda(t)P(z+t) + \Phi_\lambda(t)P(z-it)\}\, dt$$

has at least so many real zeros as $P(z)$, the polynomial

$$\int_0^a \{F_\lambda(t)Q(-izt) + \Phi_\lambda(t)Q(izt)\}\, dt$$

and the entire function

$$\int_0^a \{F_\lambda(t)R(-it)\exp(-izt) + \Phi_\lambda(t)R(it)\exp(izt)\}\, dt$$

have only real zeros.

A consequence of the first statement is that the zeros of the polynomials

$$\int_0^a \left\{F_\lambda(t)\left(1+\frac{izt}{n}\right)^n + \Phi_\lambda(t)\left(1-\frac{izt}{n}\right)^n\right\} dt \quad n = 1, 2, 3, \ldots$$

are in the half-plane $\Im z \ge 0$. Hence, the zeros of the entire function

$$\int_0^a \{F_\lambda \exp(izt) + \Phi_\lambda \exp(-izt)\}\; dt$$

are also in the half-plane $\Im z \ge 0$. Moreover, if λ is real and $|\lambda| \le 1$, then the entire functions

$$\int_0^a \{F_\lambda(t)\cos zt + \Phi_\lambda(t)\sin zt\}\; dt$$

and

$$\int_0^a \{F_\lambda(t)\cos zt - \Phi_\lambda(t)\sin zt\}\; dt$$

have only real zeros.

An entire function is said to be in the class **A**, if the series

$$\sum_{n=1}^{\omega} \left| \Im\left(\frac{1}{z_n}\right) \right|,$$

where $\{z_n\}_{n=1}^{\omega}$, $0 \le \omega \le \infty$, are its zeros which are not at the origin, is convergent (B. JA. LEVIN, **Distribution of zeros of entire functions**, Chapter V). In particular, an entire function without zeros ($\omega = 0$) or with finitely many zeros ($1 \le \omega < \infty$) is in the class **A**.

The entire functions of the form (4.7) are also in the class **A** and this follows from the absolute convergence of the series (4.8). It was already observed that a consequence of the last property is that, if $\delta \in (0, \pi/2)$, then in the "angular neighbourhood" $\mathcal{A}_\delta = \{|\arg z| < \delta\} \bigcup \{|\arg(-z)| < \delta\}$ of the real axis contains infinitely many zeros of the function (4.8). In fact, this is a common property of all the functions in the class **A**. A function of this class may have infinitely many zeros outside any $\mathcal{A}_\delta$. However, if $\{\tilde{z}_n\}_{n=1}^{\omega}$ are its zeros outside $\mathcal{A}_\delta$, then the series

$$\sum_{n=1}^{\omega} \frac{1}{|\tilde{z}_n|}$$

is convergent, whatever $\delta \in (0, \pi/2)$ is. This means that, for every such δ, "almost" all the zeros of a function from the class **A** are in $\mathcal{A}_\delta$.

It is well known that an entire function of exponential type, which is bounded on the real axis, is in the class **A**. This follows immediately from Theorem 2 on p. 225 in B. JA. LEVIN'S book. In particular, every entire function of the form

$$E_a(f; z) = \int_{-a}^{a} f(t) \exp(izt)\, dt, \quad 0 < a < \infty, \quad f \in L(-a, a), \tag{6.13}$$

is in the class **A** and the same holds for the entire functions

$$U_a(f; z) = \int_0^a f(t) \cos zt\, dt \tag{6.14}$$

and

$$V_a(f; z) = \int_0^a f(t) \sin zt\, dt \tag{6.15}$$

where $f \in L(0, a)$.

On p. 291 in his paper [Titchmarsh 1926] E. TITCHMARSH gives an example of an entire function of the form (6.13) with infinitely many zeros on the "positive" imaginary half-axis. As we have just mentioned, there exists a function in the class **A** with infinitely many zeros outside every set $\mathcal{A}_\delta$ with $\delta \in (0, \pi/2)$. On the other hand, the entire function

$$\frac{1}{z^2}\{\exp(-iz) - 1 + iz\} = -\int_0^1 (1-t) \exp(-izt)\, dt,$$

considered by L. Tschakaloff, has finitely many zeros outside every set $\mathcal{A}_\delta$ with $\delta \in (0, \pi/2)$

The asymptotic behaviour of the zeros of functions of the form (6.13) is studied in one of papers of N. Obrechkoff's that we have already surveyed. It is proved that if the function $f(t), -a \leq t \leq a$, $0 < a < \infty$ admits an L-integrable derivative and if $f(-a)f(a) \neq 0$, then there exists a strip $\alpha \leq \Im z \leq \beta$, $\alpha < 0 < \beta$ containing all the zeros of the entire function (6.13). Further, it is proved that if $N(\mu, \lambda)$, $\mu < \lambda$ is the number of the zeros of this function whose real parts are in the interval (μ, λ), then

$$N(\mu, \lambda) = \frac{a}{\pi}(\lambda - \mu) + O(1).$$

The asymptotics of the zeros of the entire functions (6.14) and (6.15) is investigated in P. Rusev's paper **On the asymptotic behaviour of the zeros of a class of entire functions**. *Proc. Inst. Math. Acad. Bulgare Sci.* **4** (1960), 67-73 (Bulgarian, Russian and German Summaries). Without loss of generality, one may assume that $a = 1$. Recall that in this case the function (6.14) and (6.15) are denoted by $U(f; z)$ and $V(f; z)$, respectively. For any $\lambda \in \mathbb{R}$, denote by $G(\lambda)$ the set of the real positive functions $\gamma(t)$, defined for $0 < t < \infty$, such that $\liminf_{t \to \infty} t^\lambda \gamma(t)$ exists and is positive. Let $H(\lambda)$ be the class of entire functions F with the property that, for every $\gamma \in G(\lambda)$, F has finitely many zeros outside the region

$$S(\gamma) = \{z = x + iy \ : \ -\infty < x < \infty, \ |y| < \gamma(|x|)\}.$$

The results in [Rusev 1960] are:

Let f be an Lebesgue-integrable function in the interval $[0, 1]$. If

$$\int_0^1 f(t) \cos(2n+1)\frac{\pi t}{2}\, dt = o\left(\frac{1}{n^{\delta+3}}\right), \quad n \to \infty,$$

for some $\delta > 0$ and

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) \int_0^1 f(t) \cos(2n+1)\frac{\pi t}{2}\, dt \neq 0,$$

then, for any $\varepsilon \in (0, \delta)$, the entire function $U(f; z)$ is in the class $A(\delta - \varepsilon)$.

Let f be an Lebesgue-integrable in $[0, 1]$. If

$$\int_0^1 f(t) \sin n\pi t\, dt = o\left(\frac{1}{n^{\delta+3}}\right), \quad n \to \infty,$$

for some $\delta > 0$ and

$$\sum_{n=1}^{\infty} (-1)^n n \int_0^1 f(t) \sin n\pi t\, dt \neq 0,$$

then, for any $\varepsilon \in (0, \delta)$, *the entire function* $V(f; z)$ *is in the class* $A(\delta - \varepsilon)$.

The proofs are based on a theorem of N. OBRECHKOFF, stated on p. 144 of his paper [Obrechkoff 1936], which concerns the distribution of zeros of meromorphic functions of the form

$$-\gamma + \sum_{n=-\infty}^{\infty} \frac{A_n}{z - a_n},$$

where $\gamma \neq 0$, and $A_n, a_n \in \mathbb{R}$ for $n \in \mathbb{Z}$, as well as on the MITTAG-LEFFLER decompositions of the meromorphic functions $U(f; z)/\cos z$ and $V(f; z)/\sin z$.

Recall that G. PÓLYA proved in [Pólya 1918] that the entire function

$$V(z) = \int_0^1 t \sin zt \, dt = \frac{1}{z} \left(\frac{\sin z}{z} - \cos z \right)$$

has only real zeros and that every interval $((2n-1)\pi/2, (2n+1)\pi/2)$, $n \in \mathbb{N}$, contains only one zero α_n of this function. Moreover,

$$\lim_{n \to \infty} \left((2n+1)\frac{\pi}{2} - \alpha_n \right) = 0$$

and the last limit relation is intuitively clear, if we look at the graph of

$$V(z) = \frac{\cos z}{z^2} (\tan z - z).$$

A similar result, proved by P. RUSEV in the paper **Asymptotic properties of the zeros of a class of meromorphic functions**, *Annuaire Univ. Sofia, Math. Fac.* **58** (1963/64) 241-271 (Bulgarian, English Summary), is the following one:

Let F *be an even entire function of normal exponential type and let* σ *be its type. Let* F *satisfy the following conditions*:

a) $F(x) = O\left(\dfrac{1}{|x|^\mu} \right)$ as $|x| \to \infty$, $\mu > \dfrac{1}{2}$;

b) $(-1)^n F\left((2n+1)\dfrac{\pi}{2\sigma} \right) > 0$, $n = 0, 1, 2, \ldots$;

c) $F\left((2n+1)\dfrac{\pi}{2\sigma} \right) = O\left(\dfrac{1}{n^{3+\delta}} \right)$, $\delta > 0$, $n \to \infty$.

Then F *has only real zeros and every interval* $((2n-1)\pi/(2\sigma), (2n+1)\pi/(2\sigma))$, $n \in \mathbb{N}$ *contains only one zero of* F. *If this zero is* z_n, *then*

$$(2n+1)\frac{\pi}{2\sigma} - z_n = O\left(\frac{1}{n^{\delta - \varepsilon}} \right)$$

for every positive $\varepsilon < \delta$.

The asymptotics of the zeros of the entire functions $U(f; z)$ and $V(f; z)$ is studied also in the paper of I. KASANDROVA **Distribution of the zeros of**

a class of entire functions of exponential type. *Université de Plovdiv "Paissi Hilendarski", Travaux scientifiques, vol.* **13** *(1975) - Mathematiques, 339-345* (Bulgarian, English Summary) under weaker assumptions than those in the papers of P. RUSEV. Here are the corresponding results:

Let f be a real and Lebesgue-integrable function in $[0,1]$. If

$$\int_0^1 f(t)\cos n\pi t\,dt = O\left(\frac{1}{n^{2+\varepsilon}}\right)$$

for some $\varepsilon > 0$ and

$$\sum_{n=1}^{\infty}(-1)^n \int_0^1 f(t)\cos n\pi t\,dt \neq 0,$$

then the zeros of the entire function $U(f;z)$ belong to a strip parallel to the real axis.

Let f be a real and Lebesgue-integrable function in $[0,1]$. If

$$\int_0^1 f(t)\sin(2n+1)\frac{\pi t}{2}\,dt = O\left(\frac{1}{n^{2+\varepsilon}}\right)$$

for some $\varepsilon > 0$ and

$$\sum_{n=0}^{\infty}(-1)^n \int_0^1 f(t)\sin(2n+1)\frac{\pi t}{2}\,dt \neq 0,$$

then the zeros of the entire function $V(f;z)$ are in a strip parallel to the real axis.

It is clear that, since the entire functions $U(f;z)$ and $V(f;z)$ are real when f is real, any strip parallel to the real axis and containing their zeros, is symmetric with respect to it.

We recall once again that G. PÓLYA was the first to observe the relation between the reality of the zeros of FOURIER'S transforms of a real R-integrable function f on the interval $[0,1]$ and the distribution of zeros of the polynomials

$$P_n(f;z) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) z^k, \quad n = 1,2,3,\ldots \tag{6.16}$$

His idea was to apply the principle of argument and the ENESTRÖM-KAKEYA theorem. Thus, he succeeded to prove that the entire functions $U(f;z)$ and $V(f;z)$ have only real and, in general, interlacing zeros when f is a non-negative and monotonically increasing in $[0,1]$. L. ILIEFF'S approach to the same problem is based on his generalization of a theorem of I. SCHUR and on an algebraic theorem of N. OBRECHKOFF. In the paper **Über die Verteilung der Nullstellen einer Klasse ganzer Funktionen**, *C. R. Acad. Bulgare Sci.* **14**

(1961) No 1, 7-9 (Russian Summary), P. RUSEV studies the relation between the asymptotics of the zeros of polynomials (6.17) when $n \to \infty$ and the zero distribution of the entire functions $U(f;z)$ and $V(f;z)$. The following assertion can be considered as a generalization of a particular case of L. ILIEFF'S theorem of SCHUR'S type:

Let $P(z)$ be a polynomial of n-th degree with zeros in the region $|z| \geq r$, $0 < r \leq 1$. Then the zeros of the polynomial $P(z) + \varepsilon z^n P^*(z), |\varepsilon| = 1$, are in the circular ring

$$\frac{1-\sqrt{1-r^2}}{r} \leq |z| \leq \frac{1+\sqrt{1-r^2}}{r}.$$

Note that if $P(z) = z - r$, $0 < r < 1$, then the zeros of the polynomial $P(z) - zP^*(z)$ are at the points $(1-\sqrt{1-r^2})/r$ and $(1+\sqrt{1-r^2})/r$, so that this result is sharp.

Suppose that f is a real R-integrable function in $[0,1]$, such that there exists a nonnegative constant λ_f with the property that, for every positive number δ, all the zeros of the polynomials (6.16) are in the disk $|z| \leq 1 + (\lambda_f + \delta)n^{-2}$, provided n is large enough. Then, under the additional assumption that $f(0)f(1) \neq 0$, it was proved that the zeros of the entire functions $U(f;z)$ and $V(f;z)$ are in the strip $|\Im z| \leq \sqrt{2\lambda_f}$.

These results we sharpened by K. DOČEV'S in his paper **Über die Verteilung der Nullstellen einer Klasse ganzer Funktionen** *C. R. Acad. Bulgare Sci.* **15** (1962), 239-241 (Russian Summary). There he introduces the class $L_\alpha(\lambda)$, $\alpha, \lambda > 0$ of the complex-valued and R-integrable functions f on the interval $[0,1]$ with the property that, for any $\delta > 0$, all the zeros of the polynomial (6.16) lie in the disk $|z| < 1+(\lambda+\delta)n^{-\alpha}$ provided n is large enough. The first results announced in this paper are:

If $f \in L_1(\lambda)$, then the zeros of the entire function

$$E(f;z) = \int_0^1 f(t)\exp(izt)\,dt \tag{6.17}$$

are in the half-plane $\Im z \geq -\lambda$.

If $f \in L_\alpha(\lambda)$ with $\alpha > 1$, then the zeros of the entire function (6.18) are in the half-plane $\Im z \geq 0$.

Further, it is claimed that from the above assertion and HERMITE-BIEHLER'S theorem for entire function of first order, it follows that, if the function f is real, then the entire functions $U(f;z)$ and $V(f;z)$ have only real and interlacing zeros. This is true if the defect of the entire function (6.17) is positive and this condition is satisfied if the function f satisfies additional requirements. The other results in K. DOČEV'S paper are the following:

If the function f satisfies the LIPSCHITZ *condition $|f(t')-f(t'')| \leq M|t'-t''|$ in the interval $[0,1]$, $|f(1)| \geq |f(0)|$ and $f(1) \neq 0$, then $f \in L_1(\lambda)$ with $\lambda = M|f(1)|^{-1}$.*

If f satisfies Lipschitz's *condition in the interval* $[0,1]$ *with constant* M *and* $|f(1)| \leq |f(0)|$, *then all the zeros of entire function* (6.17) *are in the strip* $-M|f(0)|^{-1} \leq \Im z \leq (|f(0)| - |f(1)| + M)|f(1)|^{-1}$.

The fact that the defect of the entire function $E(f;z)$ is, in general, non-negative, plays an essential role in proving that the entire functions $U(f;z)$ and $V(f;z)$ have only real zeros when $f \in L_\alpha(\lambda)$ with $\alpha > 1$. It is interesting what happens when $\alpha = 1$. An answer of this question for a class of entire functions of exponential type is given in I. M. Kasandrova's paper **A theorem of Hermite-Biehler type for a class of entire functions**. *C. R. Acad. Bulgare Sci.* **29** (1976), 1245-1248 (Russian). The corresponding assertion is:

Let $\omega(z) = u(z) + iv(z)$ be an entire function of exponential type satisfying the following conditions:

a) the defect d_ω of the function ω is positive;

b) the zeros of the function $\omega(z)$ are in the half-plane $\Im z \geq -\lambda, 0 \leq \lambda < \infty$;

c) the function $\omega(z)$ is bounded on the real axis;

d) if $\delta > \lambda$, then $M_\omega(\delta,\lambda) = \sup_{t\in\mathbb{R}} |\omega(t+i\delta)/\overline{\omega}(t+i\delta)| < \infty$.

Then the zeros of the real entire functions $u(z)$ and $v(z)$ belong to the strip $|\Im z| \leq \delta + (2d_\omega)^{-1} \log^+ M_\omega(\delta,\lambda)$.

As an application of the above theorem Kasandrova obtains the following result:

Let the zeros of the function (6.17) be in the half-plane $\Im z \geq -\lambda$, $\lambda \geq 0$, the function f be continuous and $f(1) \neq 0$, and let $M_E(\delta,\lambda) < \infty$ for $\delta > \lambda$. Then the zeros of the entire functions $U(f;z)$ and $V(f;z)$ belong to the strip $|\Im z| \leq \delta + M_E(\delta,\lambda)$.

For a complex function F, defined on the interval $[0,1]$, denote by $B_n(F;z)$ its n-th S. N. Bernstein polynomial,

$$B_n(F;z) = \sum_{k=0}^{n} \binom{n}{k} F\left(\frac{k}{n}\right) z^k (1-z)^{n-k}, \quad n = 1,2,3,\ldots$$

Let $\mathcal{B}$ be the set of functions F for which $\lim_{n\to\infty} B_n(F;t) = F(t)$ almost everywhere in $[0,1]$. Further, denote by $\mathcal{F}$ the set of function F, such that the zeros of the polynomials

$$Q_n(F;z) = \sum_{k=0}^{n} \binom{n}{k} F\left(\frac{k}{n}\right) z^k, \quad n = 0,1,2\ldots$$

are in the unit disk when n is large enough. Finally, let $\mathcal{E}$ be the set of the complex functions f defined in $[0,1]$, such that the zeros of the polynomials (6.16) are located in the unit disk for every sufficiently large n.

The following assertion is proved in P. Rusev's paper **On an application of S. N. Bernstein's polynomials**. *C. R. Acad. Bulgare Sci.* **26** (1973), 585-586 (Russian):

If $F \in \mathcal{B} \cap \mathcal{F}$ *with* $F(1) \neq 0$ *is real and bounded and* $f \in \mathcal{E}$ *is real and* R*-integrable, then the entire functions*

$$U(F, f; z) = \int_0^1 F(t)f(t)\cos zt\,dt$$

and

$$V(F, f; z) = \int_0^1 F(t)f(t)\sin zt\,dt$$

have only real zeros.

The proof is based on the equalities

$$B_n(F; z) = (1-z)^n Q_n\left(F; \frac{z}{1-z}\right), \quad n = 1, 2, 3, \ldots,$$

which imply that if n is large enough, then the zeros of the polynomial $B_n(F; z)$ are in the half-plane $\Re z \leq 1/2$. By OBRECHKOFF'S h-theorem the entire functions

$$U_n(F, f; z) = \int_0^1 B_n(F; t)f(t)\cos zt\,dt,$$

$$V_n(F, f; z) = \int_0^1 B_n(F; t)f(t)\sin zt\,dt$$

have only real zeros. The sequences $\{U_n(F, f; z)\}_{n=1}^{\infty}$ and $\{V_n(F, f; z)\}_{n=1}^{\infty}$ are uniformly bounded on every compact set $K \subset \mathbb{C}$ and, by a well-known theorem of LEBESGUE, they converge to $U(f; x)$ and $V(f; x)$, respectively, for every $z \in \mathbb{R}$. Hence, by VITALI'S theorem, these sequences converge locally uniformly to the entire functions $U(F, f; z)$ and $V(F, f; z)$, respectively, and, by HURWITZ'S theorem, the latter functions possess only real zeros.

The class $\mathcal{E}$, under another notations, is used by different authors. It seems that the class $\mathcal{F}$ was introduced in E. BOJOROFF'S paper **On some questions related to the theory of integral polynomials**. *Annuaire de l'Institut Chimico-Technologique*, **2** *(1955), Livre* 2, 151-161 (Bulgarian, Russian and German Summaries). There, on the basis of G. SZEGÖ'S version of the classical theorem of GRACE, it is proved that if $F \in \mathcal{F}, \Phi \in \mathcal{F}$ and $f \in \mathcal{E}$, then $F\Phi \in \mathcal{F}$ and $Ff \in \mathcal{E}$. In addition, it is shown that an algebraic polynomial $p(z)$ having its zeros in the half-plane $\Re z \leq 1/2$ is in the set $\mathcal{F}$. This is a corollary of the equality

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - \zeta\right) z^k = (1+z)^{n-1}(z(1-\zeta) - \zeta), \quad n = 1, 2, 3, \ldots$$

and the fact that $\Re\zeta \leq 1/2$ implies $|\zeta(1-\zeta)^{-1}| \leq 1$.

Let $h(z)$ be a real polynomials having its zeros in the half-plene $\Re z \leq 1/2$ and let the function $f \in \mathcal{E}$. Then, it follows from the above statement that

$hf \in \mathcal{E}$. Hence, the entire functions $U(hf;z)$ and $V(hf;z)$ have only real zeros. Thus, as BOJOROFF mentions, one obtains another proof of OBRECHKOFF'S h-theorem.

G. PÓLYA pointed out that there are kernels f, for which the entire functions $U(f;z)$ and $V(f;z)$ have only real zeros and infinitely many common zeros. This happens, if, for example, f is an exceptional function, i.e. f is non-negative, increasing and has finitely many points of growth, all of which are rational numbers. Indeed, here is a simple example:

$$U(1;z) = \int_0^1 \cos zt\, dt = \frac{\sin z}{z} = \frac{2}{z}\sin\frac{z}{2}\cos\frac{z}{2},$$

$$V(1;z) = \int_0^1 \sin zt\, dt = \frac{1-\cos z}{z} = \frac{2}{z}\sin\frac{z}{2}\sin\frac{z}{2}.$$

Let $\eta(t)$ be the function defined in $[0,1]$ by

$$\eta(t) = \begin{cases} 1, & 0 \le t < 1/2, \\ 0, & t = 1/2, \\ -1, & 1/2 < t \le 1. \end{cases}$$

Then a simple calculation shows that

$$U(\eta;z) = \frac{2}{z}\left(1-\cos\frac{z}{2}\right)\sin\frac{z}{2}, \quad V(\eta;z) = -\frac{2}{z}\left(1-\cos\frac{z}{2}\right)\cos\frac{z}{2}.$$

A wide classes of functions $U(f;z)$ and $V(f;z)$ with only real zeros and having infinitely many common zeros are given in P. RUSEV'S paper **Some results about the distribution of zeros of entire functions of the form** $\int_0^1 f(t)\cos zt\,dt$ **and** $\int_0^1 f(t)\sin zt\,dt$. *Proc. Inst. Math. Acad. Bulgare Sci.* **15** (1974) 33-62 (Bulgarian, Russian and English Summaries). Theorem 3 on p. 38 states:

Let τ be a real number, such that $|\tau| \ge 1$, and let $\mathcal{E}_\tau$ be the set of the real R-integrable function f in $[0,1]$, satisfying the following conditions:

1) $f(t) = \tau f(1-t)$ *if* $\frac{1}{2} < t \le 1$;

2) *all the zeros of the polynomial*

$$\sum_{k=0}^{n-1} f\left(\frac{k}{2n-1}\right) z^k$$

are in the region $|z| \ge 1$ when n is large enough.

Then

a) *for each $f \in \mathcal{E}_\tau$ the entire functions $U(f;z)$ and $V(f;z)$ have only real zeros*;

b) *if $f \in \mathcal{E}_1$, then*

$$U(f;z) = R(z)\cos\frac{z}{2}, \quad V(f;z) = R(z)\sin\frac{z}{2},$$

where $R(z)$ is an entire function with only real zeros. Moreover, $R(z) = (2/z)\sin(z/2)$ *if and only if* $f(t) = 1$ *almost everywhere in the interval* $[0, 1]$.

c) *if* $f \in \mathcal{E}_{-1}$, *then*

$$U(f;z) = S(z)\sin\frac{z}{2}, \quad V(f;z) = -S(z)\cos\frac{z}{2},$$

where $S(z)$ is an entire function with only real zeros. Moreover, $S(z) = (2/z)(1-\cos(z/2))$ *if and only if* $f(t) = \eta(t)$ *almost everywhere in* $[0, 1]$.

It is clear that if a function $f \in \mathcal{E}_\tau$ is positive and monotonically decreasing in $[0, 1/2)$, then it satisfies condition 2).

It was already mentioned that the possibility to apply a theorem of HERMITE-BIEHLER'S type to the entire function $E(f;z)$ leads to the conclusion that the entire functions $U(f;z)$ and $V(f;z)$ have only real and, in general, interlacing zeros. More detailed information about the distribution of their zeros can be obtained by A. HURWITZ'S approach to this problem. We emphasize that HURWITZ uses MITTAG-LEFFLER'S decompositions of the meromorphic functions $U(f;z)/\cos z$, $V(f;z)/\sin z$, $U(f;z)/\sin z$ and $V(f;z)/\cos z$. Then, as it is well-known, the number of the non-real zeros of $U(f;z)$ and $V(f;z)$ depends on the number of variations in the sequences of FOURIER'S coefficients of functions, which are related to the function f. Moreover, the real zeros of $U(f;z)$ and $V(f;z)$ may interlace either with the zeros of $\cos z$ or with those of $\sin z$.

In I. M. KASANDROVA'S paper **Some results about a class of entire functions with only real zeros**, *C. R. Acad. Bulgare Sci.* **30** (1977), 965-968 (Russian), sufficient conditions ensuring this interlacing are established in the case when $U(f;z)$, and $V(f;z)$ have only real zeros and the distance between every two consecutive zeros of each function is not greater that π. Under these assumptions, the corresponding results can be formulated as follows:

If $U(f;0) \neq 0$ *and if* $\{\mu_n,\ n \in \mathbb{Z}\}$ *is a sequence of positive numbers with* $\sum_{n=-\infty}^{\infty} \mu_n = 1$ *and* $|U(f;(2n+1)\pi/2)| < \mu_n |U(f;0)|$ *for* $n \in \mathbb{Z}$, *then every interval* $[(2n-1)\pi/2, (2n+1)\pi/2]$, $n \in \mathbb{Z}$, *contains only one zero of the function* $U(f;z)$.

If $U(f;0) \neq 0$ *and if* $\{\mu_n, n \in \mathbb{Z}^*\}$ *is a sequence of positive numbers such that* $\sum_{n=1}^{\infty}(\mu_n + \mu_{-n}) = 1$ *and* $n|U(f;n\pi)| < \mu_n |U(f;0)|, n \in \mathbb{Z}^*$, *then the zeros of* $U(f;z)$ *are separated by that of the function* $\sin z$.

If $V'(f;0) \neq 0$ *and* $\{\mu_n, n \in \mathbb{Z}^*\}$ *is a sequence of positive numbers such that* $\sum_{n=1}^{\infty}(\mu_n + \mu_{-n}) = 1$ *and* $|V(f;n\pi)| < \mu_n |V'(f;0)|$, $n \in \mathbb{Z}^*$, *then every interval* $[n\pi, (n+1)\pi], n \in \mathbb{Z}^*$ *contains only one zero of the function* $V(f;z)$.

If $V(f;\pi/2) \neq 0$ *and if* $\{\mu_n, n \in \mathbb{Z}^*\}$ *is a sequence of positive numbers such that* $\sum_{n=1}^{\infty}(\mu_n + \mu_{-n}) = 1$ *and* $n|V(f;(2n+1)\pi/2)| < \mu_n |V(f;\pi/2)|$, $n \in \mathbb{Z}^*$, *then the zeros of* $V(f;z)$ *are separated by that of the function* $\cos z$.

If $\delta > 0$, then the entire function

$$V_\delta(z) = \int_0^1 (1+\delta t)\sin zt\,dt$$

has only real and simple zeros. Evidently,

$$\lim_{\delta\to 0} V_\delta(z) = V_0(z) = \int_0^1 \sin zt\, dt$$

uniformly on every compact subset of the complex plane. The zeros of the function $V_0(z)$ are at the points $0, \pm 2\pi, \pm 4\pi, \ldots$. Hence, by the theorem of Hurwitz, if δ is small enough, then there are consecutive zeros of the function $V_0(z)$ such that the distance between them is greater than π. This example shows that the requirement for reality of the zeros, e.g for functions of the form $V(f;z)$ does not imply that the distances between their consecutive zeros are not greater that π.

A similar problem is discussed in another paper of I.M. Kasandrova entitled **Distribution of zeros of a class of entire functions**. *Complex Analysis and Applications*, Sofia, 1984, 272-275. Under the assumption that f is a real function, defined in $[0,1]$, with an integrable second derivative there, the following statements are proved:

If $f(1) \neq 0$ and if the function $U(f;z)$ has only real zeros, then every interval $[(2n-1)\pi/2, (2n+1)\pi/2]$, $n \in \mathbb{Z}$, contains only one zero u_n of the function $U(f;z)$ provided $|n|$ is sufficiently large, and this zero is simple.

If $f(0) = 0, f(1) \neq 0$ and if the function $V(f;z)$ has only real zeros, then every interval $[n\pi, (n+1)\pi]$, $n \in \mathbb{Z}$, contains only one zero v_n of the function $V(f;z)$ provided $|n|$ is sufficiently large, and this zero is simple.

If $f(0) = 0, f(1) \neq 0$ and if the functions $U(f;z)$ and $V(f;z)$ have only real zeros, then their zeros u_n and v_n interlace provided $|n|$ is large enough.

An algorithm allowing to answer the question whether the zeros of a given algebraic polynomial are in the unit disk is due to I. Schur **Über algebraische Gleichungen die nur Wurzeln mit negativen Realteilen besitzen**. *Z. angew. Math. Mech.* **1** (1921), 75-88. It is applied in the papers of M. Kostova. **On the functions of the class $\mathcal{E}$** II, *Université de Plovdiv "Paissi Hilendarski", Travaux scientifiques,* **11** *(1973) 3-Mathematiques*, 33-36 (Bulgarian, Russian and German Summaries) and **Einige Anwendungen eines Schurs Theorem**. *Université de Plovdiv "Paissi Hilendarski", Bulgarie, Nature*, **6** 1973, 1, 43-47 (Russian Summary). The main result in the first of them is the following one:

Let the function $f(t)$ be real, nonnegative and monotonically increasing in the interval $[0,1]$. Then, for any positive integer s, the function $F(t) = t^s(f(1-t) - f(t))$ is in the class $\mathcal{E}$ (see Theorem 1 on p. 33).

We point out that this assertion is true under the only assumption that the function f is in the class $\mathcal{E}$. An immediate corollary is the following one (see Theorem 3 on p. 35):

If $f \in \mathcal{E}$, then the entire functions $U(F;z)$ and $V(F;z)$ have only real zeros.

In the second paper Kostova provides a procedure which allows, given a function of the class $\mathcal{E}$, to generate a sequence of functions of the same class.

Let f be a real function in $[0,1]$ and let us define it to be zero outside this interval. If $s \in \mathbb{N}$ and $\alpha \in [0,1/2]$, then $f_0(t;\alpha) := f(t)$ and $f_s(t;\alpha) := f_{s-1}(t;1)f_{s-1}(t;\alpha) - f_{s-1}(\alpha;\alpha)f_{s-1}(1-t+\alpha;\alpha)$. Suppose that $f \in \mathcal{E}$ and $p < q$ are positive integers, such that $(p,q) = 1$ and $2p < q$. Let, for every $s = 1,2,3,\ldots,p+1$, $F_s(t)$ be defined by $F_s(t) = f_s(t;(s-1)/q)$ for $t \in ((s-1)/q, 1]$ and $F_s(t) = 0$ for $0 \le t \le (s-1)/q$. Then $F_s \in \mathcal{E}$ for each $s = 1,2,3,\ldots,p+1$.

Let $p(x)$ be a nonconstant real algebraic polynomial and let $m(p;a)$ be the multiplicity of a as a zero of $p(z)$, that is, $m(p;a) = \ell$ if $p(a) = p'(a) = p''(a) = \cdots = p^{(\ell-1)}(a) = 0$, $p^{(\ell)}(a) \neq 0$. In the paper of S. TODORINOV **On the distribution of zeros of a class of entire functions**. *Annuaire Univ. Sofia, Phys.-Math. Fac.* **52** *(1957/58) Livre 1-Mathematiques*, 145-147 (Bulgarian, French Summary), the following assertion is proved:

Let $p(x)$ be a nonconstant real algebraic polynomial with $m(p;0) = k$ and $m(p;1) = q$. If $k \le q$, then there exists a positive number a_0 such that, for every $a > a_0$, the entire functions

$$\int_0^1 \exp(at)p(t)\cos zt\,dt, \qquad \int_0^1 \exp(at)p(t)\sin zt\,dt \tag{6.18}$$

have only real and interlacing zeros. If $k > q$, then, for every positive a, the above functions cannot have only real and interlacing zeros.

The proof is based on BIEHLER-HERMITE'S theorem for entire functions of exponential type. In fact, it is proved that if $k \le q$, then there exists a positive a_0, such that the zeros of the entire function

$$\int_0^1 p(t)\ \exp zt\,dt$$

are in the half-plane $\Re z \le a_0$. Hence, the zeros of the entire function

$$\int_0^1 p(t)\ \exp\{(a+iz)t\}\,dt$$

are in the half-plane $\Im z > 0$ provided $a > a_0$.

If $k > q$, then the assumption that the entire functions (6.18) have only real and interlacing zeros for some positive a contradicts a result about the zero distribution of quasi polynomials without main term due to L. S. PONTRIAGIN **On the zeros of certain elementary transcendental functions**. *Izv. Akad. Nauk SSSR, Ser. Math.* **6** (1942) 115-131 (Russian). Pontriagin's result states that the real parts of the zeros of such quasi polynomials are not bounded.

Comments and references

1. If $P(z)$ is a polynomial having its zeros in the unit disk, then the zeros of the polynomial $P(z) + \gamma P^*(z)$, $|\gamma| = 1$, are on the unit circle. This assertion is due to I. Schur and appears on p. 230 in his paper **Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind**. *J. r. angew. Math.* **147** (1917) 205-232, p. 230, XII. Ilieff's key lemma may be considered as its extension. Indeed, suppose that the origin is a zero of multiplicity m of the polynomial $P(z)$, i.e. $P(z) = z^m Q(z)$, $Q(0) \neq 0$, and define $P_\delta(z) = (z+\delta)^m Q(z)$, $0 < \delta < 1$. Then the zeros of the polynomial $P_\delta^*(z)$ are outside the unit disk and then the key-lemma with $k = 0$ gives that the zeros of the polynomial $\gamma\{P_\delta^*(z) + \gamma^{-1} P_\delta(z)\} = (z+\delta)^m Q(z) + \gamma(1+\delta z)^m Q^*(z)$ are on the unit circle. Letting $\delta \to 0$, we obtain that the zeros of the polynomial $z^m Q(z) + \gamma Q^*(z) = P(z) + \gamma P^*(z)$ are on the unit circle too. It turns out that Schur's theorem is a corollary of Ilieff's key lemma, but the converse is not true.

2. Ilieff's results about the zeros of the entire function (6.10) are generalized and extended by Alfred Rényi in his paper **Remarks concerning the zeros of certain integral functions**. *C. R. Acad. Bulgare Sci.* **3** (1950) no 2-3, 9-10. Rényi proves the following:

Theorem A. *Let n and m denote non-negative integers, and let us suppose that $n+m$ is odd. Let $f(t)$ denote a real function, with $f \in C^n(0,1)$, which satisfies the following conditions*:

a) $f^{(k)}(1) = 0$ *for* $k = 1, 2, 3, \ldots, n-1$;

b) $f^{(2k+1)}(0) = 0$ *for* $1 \leq 2k+1 < n$;

c) $g(t) = t^{-m} f^{(n)}(t)$ *is integrable, non-negative and non-decreasing in* $(0,1)$.

It follows that the functions

$$F(z) = \int_0^1 f(t) \cos zt \, dt \quad \text{and} \quad \Phi(z) = \int_0^1 f(t) \sin zt \, dt$$

have only real roots.

Theorem B. *Let n and m be non-negative integers with $n+m$ is even. Let the real function $f \in C^n(0,1)$ satisfies conditions* a) *and* c) *of Theorem A, but instead of* b) *obeys the following property*:

b′) $f^{(2k)}(0) = 0$ *for* $2 \leq 2k < n$.

It follows that the functions $F(z)$ and $\Phi(z)$ have only real roots.

The proofs are based on the general property that the derivatives of real entire functions of order not greater than one with only real zeros have only real zeros too. This property is applied to the entire functions $\int_0^1 g(t) \sin zt \, dt$ and $\int_0^1 g(t) \cos zt \, dt$.

3. P. Rusev, **Distribution of the zeros of a class of entire functions**, *Phys.-Math. J.* **4(37)** (1961), 130-135 (Bulgarian) makes an attempt to extend Pólya's classical results concerning the zero-distribution of the entire functions (5.1) and (5.2) to entire functions defined by means of Riemann-Stieltjes' integrals. He proves the following:

Suppose that $f(t)$ and $\psi(t)$, $0 \leq t \leq 1$ are real functions such that the function $F(u) = \int_0^t f(u) \, d\psi(u)$, $0 \leq t \leq 1$ is increasing and convex. Then *the entire functions* $\int_0^1 f(t)\cos zt\,d\psi(t)$ *and* $\int_0^1 f(t)\sin zt\,d\psi(t)$ *have only real zeros.*

7. The Hawaii school and the Hungarian connection

There are some remarkable examples of scientific collaboration of outstanding mathematicians during the twentieth century. Among the most convincing are G. PÓLYA and G. SZEGÖ, the authors of **Aufgaben und Lehrsätze aus der Analysis** I,II, Berlin, 1925, G. HARDY and J. LITLEWOOD for their contributions to the Analytic Number Theory as well as N. WIENER and R. PALEY whose joint work made the FOURIER transform in the complex domain a powerful tool in Classical Analysis.

If we think about a collaboration of mathematicians who most contributed to the topic of the present survey, probably the first names are those of GEORGE CSORDAS and THOMAS CRAVEN. As a result of their very long activity at the University of Hawaii, a considerable number of interesting joint papers appeared. Among them are **Zero-diminishing linear transformations**, *Proc. Amer. Math. Soc.* **80** (1980), 544-546, **An inequality for the distribution of zeros of polynomials and entire functions**, *Pacific J. Math.*, **95** (1981), 263-280, **On the number of real roots of polynomials**, *Pacific J. Math.* **102** (1982), 15-28, **On the Gauss-Lucas theorem and Jensen polynomials**, *Trans. Amer. Math. Soc.* **278**, (1983), 415-429, the already mentioned **Jensen polynomials and the Turán and Laguerre inequalities**, *Pacific J. Math.* **136** (1989), 241-260, **Differential operators of infinite order and the distribution of zeros of entire functions**, *J. Math. Anal. Appl.* **186** (1994), 799-820, **On a converse of Laguerre's theorem**, *Electron. Trans. Numer. Anal.*, **5** (1997), 7-17, **Hermite expansions and the distribution of zeros of entire functions**, *Acta Sci. Math. (Szeged)*, **67** (2001) 177-196, **The Fox-Wright functions and Laguerre multiplier sequence**, *J. Math. Anal. Appl.* **314** (2006), 109-125.

It has to be noted that G. CSORDAS is a co-author of several other papers treating problems concerning entire Fourier transforms. In his joint paper with TIMOTHY S. NORFOLK and RICHARD S. VARGA **The Riemann hypothesis and the Turán inequalities**, *Trans. Amer. Math. Soc.* **296** (1986), 521-541, a fifty-eight year-old problem of Pólya [Pólya 1927b] related to the Riemann Hypothesis is solved. More precisely, Pólya asked the natural question if the Turán inequalities (3.44) hold for the Maclaurin coefficients of RIEMANN'S ξ-function. It is a matter of straightforward calculation to transform them into the following inequalities for the moments of the kernel $\Phi(t)$:

$$\hat{b}_m^2 > \frac{2m-1}{2m+1}\hat{b}_{m-1}\hat{b}_{m+1}, \quad m = 1, 2, 3, \ldots \tag{7.1}$$

where

$$\hat{b}_m := \int_0^\infty t^{2m} \hat{\Phi}(t)\, dt, \quad m = 0, 1, 2, \ldots, \tag{7.2}$$

and

$$\hat{\Phi}(t) = \sum_{n=1}^{\infty} (2\pi^2 n^4 \exp(9t) - 3\pi n^2 \exp(5t)) \exp(-\pi n^2 \exp(4t)). \tag{7.3}$$

The delicate nature of the inequalities (7.3) is revealed by comparing them with the inequalities

$$\hat{b}_m^2 \le \hat{b}_{m-1}\hat{b}_{m+1}, \quad m = 1, 2, 3, \ldots$$

which follow immediately from the CAUCHY-SCHWARZ inequality applied to the numbers $\{\hat{b}_m\}_{m=0}^{\infty}$ after representing them in the form

$$\hat{b}_m = \int_0^\infty t^{(2m-2)/2} \sqrt{\Phi(t)} t^{(2m+2)/2} \sqrt{\Phi(t)}\, dt, \quad m = 1, 2, 3, \ldots.$$

Thus, if $K(t)$ is an even kernel, whose cosine transform is in $\mathcal{LP}$, then its moments $b_m = \int_0^\infty t^m K(t) dt$ must satisfy

$$1 \le \frac{b_{m-1} b_{m+1}}{b_m^2} \le \frac{2m+1}{2m-1} \quad \text{for} \quad m \in \mathbb{N}.$$

The clue idea of the proof of (7.1) is to establish first the following sufficient conditions which guarantee that the TURÁN inequalities hold. These read as follows:

Set $b_m = \int_0^\infty t^{2m} K(t)\, dt$ *and* $\gamma_m = m!/(2m)!, \quad m = 0, 1, 2, \ldots$. *If* $\log K(\sqrt{t})$ *is strictly concave for* $0 < t < \infty$, *then the Turán inequalities*

$$\gamma_m^2 - \gamma_{m-1}\gamma_{m+1} > 0, \quad m = 1, 2, 3, \ldots,$$

or equivalently

$$(2m+1) b_m^2 - (2m-1) b_{m-1} b_{m+1} > 0, \quad m = 1, 2, 3, \ldots,$$

hold.

This principal idea for establishing TURÁN'S inequalities for entire function that are represented by cosine transforms of even positive kernels was explored further.

It is worth mentioning that $\log K(\sqrt{t})$ is strictly concave for $0 < t < \infty$ if and only if

$$t\{K^2(t) - K(t)K''(t)\} + K(t)K'(t) > 0 \quad \text{for} \quad 0 < t < \infty.$$

Then the authors prove that

$$t\{(\Phi(t))^2 - \Phi(t)\Phi''(t)\} + \Phi(t)\Phi'(t) > 0 \tag{7.4}$$

holds for $t > 0$. It is worth mentioning that the proof of (7.4), which takes up nearly 13 pages, is very technical and by no means it could be qualified as elementary one.

GEORGE CSORDAS and RICHARD VARGA, **Moments inequalities and the Riemann Hypothesis**, *Constr. Approx.* **4** (1988), 175-198, prove an extension of (7.1) for much more general kernels. The authors consider function in $\mathcal{LP}$ of the form

$$f(z) = Cz^n \prod_{k=1}^{\omega} \left(1 - \frac{z^2}{z_k^2}\right), \quad 0 \le \omega \le \infty, \tag{7.5}$$

where C is a real nonzero constant, n is a nonnegative integer, z_k, $1 \le k \le \omega$ are real and positive and $\sum_{k=1}^{\omega} z_k^{-2} < \infty$. It is supposed also that $i^n C$ is a positive number. Then $f(iz)$ is also a real entire function having its zeros on the imaginary axis. Then, for any $\lambda \ge 0$, the function $\exp(\lambda t^2) f(it)$ is an universal factor in the sense of PÓLYA. The authors prove that the MACLAURIN coefficients of the entire function

$$G(z; f, \lambda) = \int_0^\infty \exp(\lambda t^2) f(it) \Phi(t) \cos zt \, dt,$$

satisfy TURÁN'S inequalities. Let

$$\hat{b}_m(f, \lambda) = \int_0^\infty t^{2m} \exp(\lambda t^2) f(it) \Phi(t) \, dt, \quad m = 0, 1, 2, \dots.$$

The main result in [Csordas Varga 1988] is Theorem 2.4 which says:

For any $f(z)$ of the form (7.4) *the following Turán inequalities*

$$(\hat{b}_m(f; \lambda))^2 > \left(\frac{2m-1}{2m+1}\right) \hat{b}_{m-1}(f, \lambda) \hat{b}_{m+1}(f, \lambda) \tag{7.6}$$

hold for all $m = 1, 2, 3, \dots$ and all real λ.

It is clear that the inequalities (7.3) are particular case of (7.5) when $f \equiv 1$ and $\lambda = 1$.

In **Fourier transforms and the Hermite-Biehler theorem**, *Proc. Amer. Math. Soc.* **107** (1989), 645-652, G. CSORDAS and R.S. VARGA establish necessary and sufficient conditions for a real entire functions, represented by Fourier transforms, to have only real zeros and apply their result to the RIEMANN ξ-function. They study entire functions of the form

$$F(z; K) = \int_{-\infty}^{\infty} K(t) \exp(izt) \, dt. \tag{7.7}$$

The authors call the function $K : \mathbb{R} \to \mathbb{R}$ an *admissible kernel* if it satisfies the following requirements:

(i) $K(t) > 0$ for each $t \in \mathbb{R}$,

(ii) $K(-t) = K(t)$ for $t \in \mathbb{R}$,

(iii) $K(t) = (\exp(-|t|^{2+\varepsilon}))$ for some $\varepsilon > 0$ as $t \to \infty$.

Applying Hermite-Biehler's theorem to the entire function $F(z;K)$, where $K(t)$ is an admissible kernel, they obtain the following:

Suppose that all the zeros of the real entire function $F(z;K)$ lie in the strip

$$S(\tau) := \{z \in \mathbb{C} : |\Im z| < \tau\} \quad \text{for some} \quad \tau > 0. \tag{7.8}$$

Then, for each fixed $\mu \geq \tau$,

$$P_\mu(z;K) = \int_0^\infty K(t)\cosh(\mu t)\cos(zt)\,dt \ \in \ \mathcal{LP}$$

and

$$Q_\mu(z;K) = \int_0^\infty K(t)\sinh(\mu t)\sin(zt)\,dt \ \in \ \mathcal{LP}.$$

Define

$$\Delta(x,y;K) = \int_{-\infty}^\infty \int_{-\infty}^\infty K(t)K(s)\exp(i(t+s)x)\exp((t-s)y)(t-s)^2\,dt\,ds.$$

Further, on the basis of Proposition 3.1, a criterion of Jensen's type is established:

Suppose that the zeros of $F(z;K)$ lie in the strip (7.8) *for some fixed τ. Then, $F(z;K) \in \mathcal{LP}$ if and only if $\Delta(x,y;K) \geq 0$ for $0 < x < \infty$ and $0 \leq y < \tau$.*

At the end of the paper the above criterion is applied to the entire function

$$\xi\left(\frac{z}{2}\right) = F(z;\Phi) = \int_{-\infty}^\infty \Phi(t)\exp(izt)\,dt.$$

Since $\Phi(t)$ is an admissible kernel and the zeros of $\xi(z/2)$ are in the strip $S(1)$, the authors formulate the following criterion:

$F(z;\Phi) \in \mathcal{LP}$ if and only if

$$\Delta(x,y;\Phi) \geq 0 \tag{7.9}$$

for $0 < x < \infty$ and $0 \leq y < 1$.

In other words, Riemann's Hypothesis is true if and only if the inequality (7.9) holds.

In **Integral transform and the Laguerre-Pólya class**, *Complex Variables*, **12** (1989), 211-230, George Csordas and Richard S. Varga investigate again entire Fourier transforms of the form (7.7) under the assumption that the admissible kernel $K(t)$ satisfies the the additional conditions

(iv) $K'(t) < 0$ for $t > 0$,

(v) There exist a positive $\tau = \tau(K)$, such that $K(t)$ has a holomorphic extension in the strip (7.8).

The following results are proved under the assumptions (i)-(v):

Suppose that $\log K(\sqrt{t})$ *is strictly concave for* $0 < t < \infty$. *Let* $f \in \mathcal{LP}$, $f \not\equiv 0$ *be even and normalized so that its first non-zero Taylor coefficient is positive. Set*

$$c_m(K; f) = \int_0^\infty t^{2m} f(it) K(t)\, dt \quad m = 0, 1, 2, \ldots.$$

Then

$$(c_m(f; K))^2 > \left(\frac{2m-1}{2m+1}\right) c_{m-1}(f; K)\, c_{m+1}(f; K), \quad m \in \mathbb{N}.$$

A brief outline of the paper of GEORGE CSORDAS, RICHARD S. VARGA and ISTVÁN VINCZE, **Jensen Polynomials with Applications to the Riemann ξ-Function**, *J. Math. Anal. Appl.* **153** (1990), 112-135, is given in its abstract, where the authors claim that they establish generalizations of some known results for JENSEN polynomials, pertaining to (i) convexity, (ii) the TURÁN inequalities, and (iii) the LAGUERRE inequalities, and that these results are then applied in general to real entire function, which are represented by Fourier transform, and in particular to the RIEMANN ξ-function.

In Section 2 of the same paper real entire functions of the form (1.9) are considered, i.e.

$$f(z) = \sum_{k=0}^\infty \frac{\gamma_k}{k!} z^k,$$

where $\gamma_k > 0$, $k = 0, 1, 2, \ldots$ and $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \ge 0$ for $k \in \mathbb{N}$. Under these weak assumptions it is shown that not only the LAGUERRE inequalities (1.12) hold, but also that $L_p^{(\nu)}(t) \ge 0$ for $t \ge 0$, $\nu, p = 0, 1, 2, \ldots$ (see Theorem 2.5). The main result in this section, which is Theorem 2.7, states that under the same conditions on the numbers γ_k, $k = 0, 1, 2, \ldots$, the inequality $\Delta_{n,p}^{(\nu)}(t) \ge 0$ holds for $t \ge 0$ and for $\nu, p = 0, 1, 2, \ldots$. This extends inequality (1.11).

Entire functions of the form

$$F_c(z) := F_c(z; K, f) = \int_{-\infty}^\infty f(it) K(t) \cosh(t\sqrt{z})\, dt,$$

where $f \in \mathcal{LP}$ and K is an admissible kernel, are studied in Section 3. Let

$$L_p(t; F_c) = (F_c^{(p+1)}(t))^2 - F_c^{(p)}(t) F_c^{(p+2)}(t), \quad p = 0, 1, 2, \ldots.$$

Then, under the additional assumptions that f is even, $f(0) = 1$ and the function $\log K(\sqrt{t})$ is strictly concave for $0 < t < \infty$, it is proved in Theorem

3.3 that that $L_p^{(\nu)}(t;F_c) \geq 0$ for $t \geq 0$ and $\nu, p = 0,1,2,\ldots$. This result is applied to the entire functions

$$F_c(z;\Phi,f) = \int_{-\infty}^{\infty} f(it)\Phi(t)\cosh(t\sqrt{z})\,dt,$$

and this leads to the inequality $L_p^{(\nu)}(t;F_c) \geq 0$ for $t \geq 0$ and $\nu, p = 0,1,2,\ldots$.

Further, the authors focus their attention on a relation between the LAGUERRE inequalities and the RIEMANN hypothesis stating that the latter is true if and only if

$$L_0^{(0)}(t;P_\mu) + L_0^{(0)}(t;Q_\mu) \geq 0, \quad t \geq 0, \quad 0 \leq \mu < 1,$$

where

$$P_\mu(z) = 2\int_0^{\infty} \cosh(\mu t)\Phi(t)\cos(zt)\,dt$$

and

$$Q_\mu(z) = 2\int_0^{\infty} \sinh(\mu t)\Phi(t)\sin(zt)\,dt.$$

The main result in the paper of GEORGE CSORDAS, **Convexity and the Riemann ξ-function**, *Glasnik Matematički*, **33** (1998) 37-50, stated as Theorem 2.12, is that the function $\Phi(\sqrt{t})$ is strictly convex for $t > 0$, that is, $(\Phi(\sqrt{t}))'' > 0$ for $t > 0$ (). As an application of this fact, it is shown in Corollary 2.13 that the inequality

$$\int_0^{\infty} \Phi(\sqrt{t})\cos(xt)\,dt > 0$$

holds for every $x \in \mathbb{R}$. This means that the cosine transform of the function $\Phi(\sqrt{t})$ cannot have any real zeros. Let us point out that it is shown in Section 6 of PÓLYA'S paper in *Math. Z.*, **2** (1918) 352-383, that the sine transform of a function which is positive and decreasing in $(0,\infty)$, has no positive zeros. A corollary of this fact is that if, in addition, the function is convex, then the same holds for its cosine transforms.

Comments and references

Extensions of TURÁN'S inequalities

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \quad k \in \mathbb{N}, \tag{7.10}$$

were obtained by D. K. DIMITROV in [Dimitrov 1998], where the inequalities

$$H_k := 4(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \geq 0, \quad k \in \mathbb{N}, \tag{7.11}$$

were shown to be another necessary conditions that the entire function $\psi(x)$, defined by

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!},$$

is in $\mathcal{LP}$. In [Dimitrov 1998] (7.11) were called the *higher order Turán inequalities*. The recent paper [Dimitrov Lucas 2011] of D. K. DIMITROV and F. R. LUCAS shows that, while the TURÁN inequalities provide necessary and sufficient conditions for the second degree JENSEN polynomials to be hyperbolic, (7.11) hold if and only if the generalized JENSEN polynomials of degree three $g_{3,k-1}(x)$ possess only real zeros. The result reads as follows:

Let $k \in \mathbb{N}$. Then the real polynomial

$$g_{3,k-1}(x) = \gamma_{k-1} + 3\gamma_k x + 3\gamma_{k+1} x^2 + \gamma_{k+2} x^3$$

with nonzero leading coefficient γ_{k+2} is hyperbolic if and only if the inequalities

$$\gamma_{k+1}^2 - \gamma_k \gamma_{k+2} \geq 0,$$

and

$$4(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \geq 0$$

hold simultaneously.

If $k(t)$ is an even admissible kernel and it Fourier transform is

$$F(x) = \frac{1}{2}\int_{-\infty}^{\infty} K(t) e^{ixt} dt = \int_0^{\infty} K(t)\cos(xt) dt,$$

then its moments are

$$b_m := \int_0^{\infty} t^{2m} K(t) dt, \quad m = 0, 1, 2, \ldots.$$

The change of variable, $z^2 = -x$ shows that

$$F_1(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} b_k \frac{x^k}{(2k)!}, \qquad \gamma_k := \frac{k!}{(2k)!} b_k$$

belongs to $\mathcal{LPI}$ if and only if $F \in \mathcal{LP}$. Moreover, straightforward calculations yield $H_k = d_k \widetilde{H_k}$, where

$$d_k = \frac{[k!]^2[(k+1)!]^2}{(2k)!(2k+1)![(2k+3)!]^2}$$

and

$$\begin{aligned} \widetilde{H_k} \;\; = \;\; & 4(2k+3)\left[(2k+1)b_k^2 - (2k-1)b_{k-1}b_{k+1}\right] \\ & \quad \times\left[(2k+3)b_{k+1}^2 - (2k+1)b_k b_{k+2}\right] \\ & -(2k+1)\;\left[(2k+3)b_k b_{k+1} - (2k-1)b_{k-1}b_{k+2}\right]^2. \end{aligned}$$

The following result was proved in [Dimitrov Lucas 2011]:

If $K(t)$ is an admissible kernel with moments $b_k = \int_{-\infty}^{\infty} t^{2k} K(t)dt$ and

$$(\log K(\sqrt{t}))'' < 0 \quad \text{for} \quad t > 0,$$

then $\widetilde{H_k} \geq 0$ for every $k \in \mathbb{N}$.

Its rather curious and surprising that the logarithmic concavity of $K(\sqrt{t})$ guarantees not only that the TURÁN inequalities hold but also that the higher order ones are true. Then an immediate consequence of the fact the $\Phi(\sqrt{t})$ is logarithmically concave, established in [Csordas Norfolk Varga 1986], is that $\widetilde{H_k} \geq 0$ for the moments of the kernel $\Phi(t)$. Thus, all JENSEN polynomials of degree three $g_{3,k-1}(x)$, associated with the RIEMANN function $\xi_1(z)$ are hyperbolic.

Another interesting extension of TURÁN inequalities is due to CRAVEN and CSORDAS. In [Craven Csordas 1989] they considered entire functions whose MACLAURIN coefficients γ_k, $k = 0, 1, \ldots$, obey the so-called double TURÁN inequalities

$$E_k = E_k(\gamma) := T_k^2 - T_{k-1}T_{k+1} \geq 0, \qquad k = 2, 3, 4, \ldots.$$

and proved the following:

If

$$\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \in \mathcal{LPI}, \quad \text{where} \quad \gamma_k \geq 0 \quad \text{for} \quad k = 0, 1, 2, \ldots,$$

then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies the double Turán inequalities

$$E_k = T_k^2 - T_{k-1}T_{k+1} \geq 0, \qquad k = 2, 3, 4, \ldots.$$

The problem of finding sufficient conditions which guarantee that the MACLAURIN coefficients of an entire function, defined as a Fourier transform of an even admissible kernel, satisfy the double TURÁN inequalities was studied by G. CSORDAS and D. K. DIMITROV. The relevant result is Theorem 2.4 b), established in [Csordas Dimitrov 2000] which reads as follows:

Let $K(t)$ be an even admissible kernel and let b_k denote its moments. Let $s(t) := K(\sqrt{t})$ and $f(t) := s'(t)^2 - s(t)s''(t)$. If both $\log K(\sqrt{t})$ and $\log f(t)$ are concave for $t > 0$, that is, if $(\log K(\sqrt{t}))'' < 0$ and

$$(\log f(t)))'' < 0 \qquad \text{for} \quad t > 0, \tag{7.12}$$

hold, then the double Turán inequalities hold.

The natural question if inequalities (7.12) are true when $K(t)$ is the kernel $\Phi(t)$, associated with the RIEMANN ξ-function, remains open.

8. Variations on classical themes

In the paper of D. K. DIMITROV, **Fourier transforms in the Laguerre-Pólya class and Wronskians of orthogonal polynomials**, (submitted), a criterion the Fourier transform of a real, even and positive function $K(t)$, $-a < t < a$, $0 < a \le \infty$ to be in the $\mathcal{LP}$ class is established. It is assumed that all the moments

$$\mu_k = \int_{-a}^{a} t^k K(t)\,dt, \quad k = 0, 1, 2, \ldots$$

exist and $\lim_{k\to\infty}(|\mu_k|/(k!))^{1/k} = 0$. Denoting by $\{p_n(z)\}_{n=0}^{\infty}$ the system of polynomials which are orthogonal on the interval $(-a, a)$ with respect to the kernel $K(t)$, the author proves that:

The entire function

$$F(z; K) = \int_{-a}^{a} K(t)\exp(izt)\,dt = 2\int_{0}^{a} K(t)\cos(zt)\,dt$$

is in the $\mathcal{LP}$ class if and only if for any even integer n all the zeros of the Wronskian $W(p_1(z), p_2(z), \ldots, p_n(z))$ are purely imaginary.

It seems for the first time Wronskians of systems of orthogonal polynomials were investigated systematically in the long paper of S. KARLIN and G. SZEGÖ, **On certain determinants whose elements are orthogonal polynomials**, *J. Analyse Math.* **8** (1961), 1-157. There the authors, inspired by TURÁN'S inequality $P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \ge 0$, $x \in [-1, 1]$, which hold for the LEGENDRE polynomials, study certain determinants whose elements are orthogonal polynomials.

An interesting extension of a result of G. PÓLYA is given in the paper of DAVID A. CARDON, **Convolution operators and zeros of entire functions**, *Proc. Amer. Math. Soc.* **130** (2000) No 6, 1725-1734. The author denotes by $\{X_i\}_{i=1}^{\infty}$ a sequence of independent random variables such that X_i takes values ± 1 with equal probability and by F_n the distribution function of the normalized sum $(a_1X_1 + \cdots + a_nX_n)/s_n$, $s_n = a_1^2 + \cdots + a_n^2$, $n \in \mathbb{N}$, where $\{a_i\}_{i=1}^{\infty}$ is a nonincreasing sequence of positive numbers.

It is proved in Lemma 1 that, under the above assumptions, the sequence $\{F_n\}_{n=1}^{\infty}$ converges pointwise to a continuous distribution F which is either normal, or not depending on whether $\lim_{n\to\infty} s_n$ is equal or less than infinity.

Let $G(z)$ be a real entire function of order less than 2 with only real zeros. In other words, it is in the class $\mathcal{LP}^*$ which consists of $\mathcal{LP}$ functions of order less than 2. The main result in the paper is Theorem 1:

Define H by the integral

$$H(z) = (G * dF)(z) := \int_{-\infty}^{\infty} G(z - is)dF(s).$$

Then, H is a real entire function of order less than 2. If H is not identically zero, then it has only real zeros.

The author points out that his main result generalizes a "very interesting observation of Pólya about the zeros of certain entire functions", having in mind **Hilfssatz II** in Pólya's paper in *Acta Math.* **48** (1926) 305-317 concerning the zeros of entire function of the form (3.31). The main result is applied then to the case when the function G can be represented as Fourier transform, i.e.

$$G(z) = \int_{-\infty}^{\infty} K(t) \exp(izt)\, dt,$$

where the real function K is (locally) absolutely integrable and satisfies $K(t) = O(\exp(-|t|^{2+\varepsilon}))$ for some $\varepsilon > 0$ when $|t| \to \infty$. Let

$$L(t) = \int_{-\infty}^{\infty} \cosh(ts) dF(s)$$

with F as above. Then, Theorem 3 states:

If the entire function

$$H(z) = \int_{-\infty}^{\infty} K(t)L(t) \exp(izt)\, dt$$

is not identically zero, then it has only real zeros.

It is observed at the end of the paper that the realization of Riemann ξ-function as a convolution $G * dF$ with $G \in \mathcal{LP}^*$ "would prove the Riemann Hypothesis. However, it seems unlikely that this approach would be fruitful because the formula

$$\xi(t) = 2 \int_{0}^{\infty} \Phi(u) \cos(tu)\, du$$

with

$$\Phi(u) = 2 \sum_{N=1}^{\infty} (2\pi^2 n^4 \exp(9u/2) - 3\pi n^2 \exp(5u/2)) \exp(-\pi n^2 \exp(2u))$$

suggests that it may be more natural to consider $\xi(t)$ as a Fourier integral than as a convolution $G * dF$".

Pólya's characterization of the universal factors in [Pólya 1927a] inspired the main result in Cardon, **Fourier transforms having only real zeros**, *Proc. Amer. Math. Soc.* **133** (2005), 1349-1356, which states:

Let H be the Fourier transform of $G(it)$ with respect to the measure dF, that is,

$$H(z) = \int_{-\infty}^{\infty} G(it) \exp(izt) dF(t).$$

Then H is an entire function of order ≤ 2 that is real on the real axis. If H is not identically zero, then it has only real zeros.

The author compares his main result with Pólya's one and claims that "while there is some overlap between Proposition 1 (Pólya's result) and Theorem 1 (the latter statement), neither implies the other".

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Received February 11, 2011

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