Special Repo Rates and the Cross-Section of Bond Prices: the Role of the Special Collateral Risk Premium*

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This version: November 19, 2020

Abstract

We estimate the joint term-structure of U.S. Treasury cash and repo rates using daily prices of all outstanding Treasury securities and corresponding special collateral (SC) repo rates. This allows us to derive a risk premium associated to the SC value of Treasuries and to quantitatively link this premium to various price anomalies, such as the on-the-run premium. We show that a time-varying SC risk premium explains 73–90% of the on-the-run premium and a substantial share of other Treasury market anomalies, suggesting a commonality across these anomalies explicitly linked to the SC value of recently-issued nominal Treasury securities—essentially near-money assets.

Keywords: special repo rates, term structure of interest rates, dynamic no-arbitrage models

JEL Classifications: G12, G23, C33

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*We thank Gadi Barlevy, Luca Benzoni, Darrell Duffie, Jean-Sébastien Fontaine, John Griffin, Thomas King, Arvind Krishnamurthy, Gregor Matvos, Daniel Neuhann, Nivine Richie (discussant), Cisil Sarisoy (discussant), Sam Schulhofer-Wohl, Krista Schwarz (discussant), Or Shachar (discussant), Andrea Vedolin, Jan Wrampelmeyer (discussant), Paul Zimmermann (discussant), and seminar participants at the McCombs Finance Brownbag, the 2017 European Winter Meeting of the Econometric Society, the fifth International Conference on Sovereign Bonds at the Bank of Canada, the Federal Reserve Bank of Chicago, the 2018 Annual Meeting of the Central Bank Research Association, the University of North Carolina at Chapel Hill Kenan-Flagler Finance Brownbag, the 2018 Southern Finance Association Annual Meeting, the 2018 Term Structure Workshop at the Deutsche Bundesbank, the 2018 Paris Financial Management Conference, the 2018 Paris December Finance Meeting, the 2019 MFA annual meeting, the SFS Cavalcade North America 2019 meeting, and the 2019 Southern Economic Association annual meeting for helpful comments and conversations. All remaining errors are our own. Iman Dolatabadi and Tim Seida provided excellent research assistance. The views expressed here do not reflect official positions of the Federal Reserve.

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1 Introduction

Previous studies have documented various price anomalies in very large and liquid U.S. fixed-income markets. For example, assets with almost identical payoffs can carry significantly different prices: just-issued (on-the-run) Treasury securities trade at higher prices than previously issued (off-the-run) securities with the same remaining maturity (Krishnamurthy, 2002); prices of nominal Treasury securities can consistently exceed those of inflation-swapped TIPS with exactly matching cash flows (Fleckenstein, Longstaff and Lustig, 2014); and off-the-run Treasury notes can trade at higher prices than off-the-run Treasury bonds with similar duration (Musto, Nini and Schwarz, 2017 and Pancost, 2017). We conjecture that there could be a commonality across these anomalies: a missing risk premium on Treasuries with special collateral (SC) value in the repurchase agreement (repo) market due to their money-like quality.

Holders of nominal Treasury securities that go "on special" in repo markets can effectively receive a dividend by borrowing at below-market rates, using those Treasuries as collateral (Duffie, 1996). That is, this dividend (or convenience yield) is equivalent to the repo special spread. A rational investor, in pricing a Treasury security, would account for the current and expected future special spreads specific to that security. But, this stream of dividends is uncertain, as the Treasury collateral value fluctuates unexpectedly depending on demand and supply dynamics in the repo market, which can cause special spreads to deviate significantly from their predictable auction-cycle dynamics.[1] It is the risk associated with those unexpected deviations from the cyclical component that investors ought to be pricing, commanding a risk premium. This SC risk, being positively (negatively) related to well-known proxies of demand (supply) for near-money assets (e.g., those in Sunderam 2015 or Nagel 2016), appears to be systematic: it increases when investors especially value the safety and liquidity of money-like claims.

In this paper, we focus on pricing SC risk and provide three main contributions. First, we develop a methodology to account for special spreads at the security level within a dynamic term structure model (DTSM), which allows us to decompose the convenience yield on recently-issued Treasuries into expected specialness and SC risk premia. Second, we show that the SC risk premium, and not expected specialness, explains the on-the-run premium. Third, we show that the SC risk premium also explains a substantial share of other Treasury price anomalies, confirming that unexpected fluctuations in the SC value of recently-issued securities are a common risk factor. This, in turn, implies that models that ignore the SC

[1] Keane (1995) and Cherian, Jacquier and Jarrow (2004) show that the cyclicality of specialness is pervasive across auctions: it is higher for the most recently-issued Treasuries and dies out as their time since issuance increases.

risk premium might interpret a spread between actual and model-implied Treasury prices as anomalous, though most of this spread may be justified as compensation for exposure to SC risk.

Our premise is that Treasury market participants price not only expected repo "dividends," but also their associated risk premia. Thus, we develop a DTSM that, by explicitly accounting for observed SC repo risk factors, can identify and estimate SC risk premia separately from other classical risk premia, (i.e., those related to the level, slope, and curvature risk factors). Previous work (Duffie, 1996; Buraschi and Menini, 2002; Cherian, Jacquier and Jarrow, 2004; Fontaine and Garcia, 2012) has used a no-arbitrage framework to study the relation between the Treasury cash premium and expected specialness; however, our DTSM is the first to price the risky component of special spreads, identified by their deviations from the expected auction-cycle dynamics. Subsequent to fitting the model, we study the properties of SC risk premia with the goal of understanding their economic importance, their ability to explain relevant price anomalies, and to serve as an indicator of financial fragility connected to the scarcity of near-money assets.

In our DTSM, individual Treasury securities with the highest potential to go on special are priced by discounting their cash flows with a specialness-risk-adjusted short-rate process. This approach implies a security-specific discounting that effectively links the pricing in the Treasury cash and repo markets. Seminal work by Duffie (1996) and Vayanos and Weill (2008) as well as related empirical evidence (e.g., Jordan and Jordan, 1997; D'Amico, Fan and Kitsul, 2017) have suggested that pricing in these two markets are tightly linked.[2] So far, DTSMs in the literature have been modeling and estimating U.S. Treasury cash and repo rates separately, as the common approach implicitly ignores the possibility that investors might be discounting the stream of future cash-flows of certain Treasury securities at a rate lower than the generic short rate.[3]

Most likely, this omission in the term-structure literature is due to both the lack of data on SC repo rates and the complexity of pricing each Treasury security individually within a DTSM. Using daily prices of all outstanding Treasury securities and BrokerTec data on individual Treasury SC repo rates from 2009 to 2017, we estimate the joint term-structure of U.S. Treasury cash and repo rates. In an economy in which investors often are willing to *pay* up to 3% to *lend* cash overnight to access certain securities, and fails to deliver even

[2]This is nicely illustrated by the formulas in Buraschi and Menini (2002). See also Cherian, Jacquier and Jarrow (2004) and Graveline and McBrady (2011).

[3]The rare exceptions are in the swap spread literature: Grinblatt (2002) and Feldhütter and Lando (2008) approximate the Treasury convenience yield with the Libor-Tbill spread or the Libor-GC spread, respectively, which however is the same across all Treasury securities. Other relevant papers in this literature are Duffie and Singleton (1997) and Liu, Longstaff and Mandell (2006).

seasoned Treasuries are significant, the differences across SC repo rates could have important consequences for cash prices.

An additional feature of our approach is that, by using a large cross-section of bond prices, and because the repo risk factor is observable, we can estimate the SC risk premium without estimating the physical dynamics of the model's latent factors. This is because, with so many more bonds than factors, we can estimate latent factors consistently cross-section by cross-section without worrying about the time-series parameters of the model, as in the sequential regression approach of Andreasen and Christensen (2015). This represents a substantial benefit of using the entire cross-section of actual Treasury bond data rather than a few zero-coupon yields, as our sample length is limited by the SC repo data.

We find that the SC risk premium is large in magnitude at the beginning of the sample, with a peak of 100 basis points in 2009 at the 10-year maturity, as those issues are the "most special" in the repo markets. It fluctuates significantly over time, exhibiting sharp variations following supply shocks, approximated by the FOMC announcements of the Federal Reserve's asset purchase programs, which reduce the net supply of safe and liquid securities.[4] Importantly, at the 10-year maturity, the SC risk premium accounts for 73–90% of the on-the-run premium, consistent with a prediction of Vayanos and Weill (2008). In particular, we show that our model matches the size, persistence, and variability of the on-the-run premium by allowing for a time-varying SC risk premium that also correlates with the state of the economy. This, in turn, indicates that, differently from previous studies (Krishnamurthy, 2002; Cherian, Jacquier and Jarrow, 2004), the expected component of specialness is worth little.

Our SC risk premium does more than explain the on-the-run premium. At the 10-year maturity, it also explains 62% of the off-the-run note-bond spread, 68% of the TIPS-Treasury bond puzzle, and about 60% of measures of TIPS illiquidity relative to their nominal counterparts. This suggests a common underlying economic mechanism across these anomalies, linked to the SC value of recently-issued nominal U.S. Treasuries. These near-money assets command an extra premium due to their exposure to SC risk, which drives a positive wedge between their prices and those of relatively less liquid and safe substitutes.

This paper is related to various studies in the literature on top of those that have motivated our search for a possible common driver of the already-mentioned price anomalies. Duffie (1996) shows in a simple static setting that a security's special repo rate below the generic short rate (i.e., the general collateral repo rate) implies a higher cash price for that security. However, he does not examine whether observed on-the-run premia are consistent with special repo rates within a dynamic setting and in the data. Krishnamurthy (2002)

[4]See Cahill et al. (2013) for an in-depth discussion of supply shocks in the U.S. Treasury market.

finds that high special spreads on the on-the-run 30-year Treasury bond are consistent with a price premium. We go one step further by showing the substantial joint time variation in the special spread and the on-the-run premium across various maturities, and by quantifying the SC risk premium that makes the two consistent with each other.

Importantly, relative to those two seminal papers, our main contribution is to show that the SC risk premium is a common driver of various price anomalies in fixed-income markets and thus is not exclusively related to the on-the-run premium. This aspect of our study is similar in spirit to Fontaine and Garcia (2012). But, while their liquidity factor is a latent factor identified by relative differences in bond ages, our new SC risk factor is an observed factor identified by fluctuations in SC repo rates. And, since only a small part of their liquidity factor is linked ex-post to variations in repo specialness spreads, we do not expect their factor to be strongly related to ours or to the SC risk premium. As shown by Bartolini et al. (2011), the specialness of a Treasury security is not necessarily related to funding liquidity. Further, as in Fontaine and Garcia (2012), Pancost (2017), and Andreasen, Christensen and Rudebusch (2018), our estimation does not rely on zero-coupon yields, as smoothing the data wipes out information that is relevant to our question centered on security-specific characteristics.[5]

Finally, our quest for a risk premium that can make prices in the repo and cash markets consistent with one another is distinct from the question of why an on-the-run premium exists in the first place. For example, Vayanos and Weill (2008) explicitly link the existence of the on-the-run premium to differentials in repo rates and liquidity in a model with search frictions.[6] Pasquariello and Vega (2009) show that the liquidity differential between on- and off-the-run securities varies with the Treasury auction cycle, even after controlling for repo specialness. We document how repo specialness also varies with the auction cycle, and show that the inclusion of a SC risk premium, above and beyond the auction-induced specialness, helps align on-the-run yield differentials with repo special spreads.

The rest of the paper is organized as follows. Section 2 describes the data and motivates our focus on SC risk. Section 3 sets up the model and describes identification and estimation. Section 4 presents our empirical results and shows that the SC risk premium explains

[5]For research questions more focused on macroeconomic dynamics, such as those in Ang and Piazzesi (2003), Bauer and Rudebusch (2014), or Duffee (2018), or econometric properties of DTSMs, such as those in Joslin, Singleton and Zhu (2011) or Creal and Wu (2016), using smoothed or constant-maturity zero-coupon yields may be more appropriate.

[6]Although the model of Vayanos and Weill (2008) is dynamic, they do not allow the on-the-run premium or the repo special spread to vary dynamically. Duffie, Gârleanu and Pedersen (2002) derive a dynamic model with search frictions and heterogenous beliefs that generates an endogenous on-the-run premium that declines over time as pessimists borrow the bond and find progressively less-optimistic traders to whom they short-sell it.

a substantial share of various Treasury price anomalies. Section 5 presents alternative specifications of the model to verify the robustness of our results, and Section 6 offers concluding remarks.

2 Data and Motivation

We use daily prices of Treasury securities from CRSP covering the period from January 2, 2009 to December 29, 2017.[7] We are limited to this sample period by the availability of our SC repo rate data. However, considering that we are interested in understanding how the collateral value of Treasury securities affect their prices in the cash market, this is a very interesting period. In particular, in those years, reduced issuance by the Treasury, the sharp increase in Treasury holdings by the Federal Reserve (Fed), and new financial regulation have reportedly shrunk the availability of Treasury securities, making this high-quality collateral quite scarce in the repo market. This, in turn, might have caused special spreads to be positive also for off-the-run securities and increased off-the-run fails to deliver to unusual levels (see for example Fleming and Keane, 2016*a*).

Our data includes 496,420 price observations across 2,252 trading days and 628 unique CUSIPs; we drop all bonds with remaining maturity of less than one year. We supplement this data with CUSIP-level SC repo rates from BrokerTec, the largest electronic broker-dealer trading platform for Treasury securities. BrokerTec accounts for about 60% of electronic interdealer trading in the on-the-run 2-, 5-, and 10-year notes and about 50% for the 30-year bond (for more details on BrokerTec see Fleming, Mizrach and Nguyen 2017). Virtually all large broker-dealers execute trades on this platform, and since 2004 non-dealer participants, such as hedge funds and HFT firms, have had access. We compute daily averages of SC repo rates quoted between 7:30 and 10 a.m. (Eastern time). This time window is chosen because trading in the repo market begins at about 7 a.m., remains active until about 10 a.m., and then becomes light until the market closes at 3 p.m. Repo transactions with SC are executed on a delivery versus payment basis (i.e., same-day settlement). In these transactions, a collateral security is delivered into a cash lender's account in exchange for funds. The exchange occurs via FedWire or a clearing bank. Finally, we use Treasury general collateral (GC) repo rates from the General Collateral Finance (GCF) Repo Index, which is a tri-party repo platform maintained by the Depository Trust & Clearing Corporation (DTCC).[8] This market is characterized as being primarily inter-dealer, although some commercial banks

[7] Source: CRSP®, the Center for Research in Security Prices, Booth School of Business, The University of Chicago. Used with permission. All rights reserved. crsp.uchicago.edu

[8] DTCC GCF rate data are publicly available at http://www.dtcc.com/charts/dtcc-gcf-repo-index.aspx.

and Fannie Mae also participate. It is a fairly active market although its size is still small compared to that of the overall tri-party repo market.[9] Table 1 provides some descriptive statistics of our data.

[Table 1 about here.]

We highlight three features of our data from Table 1. First, on average there are over 200 bonds per cross-section, so that there is plenty of variation for identifying 3 latent pricing factors. Second, although all maturities trade special, the 10-year on-the-run bond is the "most special," in the sense that it has the highest special spread of any maturity. As shown in the fourth column, the average special spread is over 35 bps, which is large given that the average GC repo rate in our sample is 30 bps. The difference implies that on average, holders of the 10-year on-the-run bond are *paid* about 5 bps to *borrow* cash using their special collateral. Other maturities have lower special spreads, about 20 bps on average. The 30-year on-the-run bond trades at a much lower average special spread of about 8 bps.

Third, the higher special spreads on the 10-year bond correspond to higher on-the-run premia in the cash market. The last column of Table 1 reports, for on-the-run bonds, the average price residual (actual minus fitted) implied by the Gürkaynak, Sack and Wright (2007)—henceforth, GSW—model estimates.[10] In our sample, this premium is about 68 bps of par on average for the 10-year on-the-run bond; maturities besides 10 years tend to have much lower cash premia. The on-the-run 30-year bond trades on average only about 22 bps above what it "should" according to the GSW model. This finding is unique to our sample period: before 2009 the 30-year on-the-run bond featured a large and time-varying on-the-run premium, often as large as the 10-year note. However, because we do not have special spreads before 2009, we omit this time period from our analysis.

[Figure 1 about here.]

Because our model is most conveniently formulated in terms of coupon bond prices, rather than yields (see equations 15 and 16 in the next section), in this paper we define the on-the-run premium in prices rather than in yields. This choice is only for convenience and does not materially affect the results: our model produces on-the-run yield spreads that mirror other measures of the on-the-run premium. This is clearly illustrated in the top panel of Figure 1, which plots our 10-year on-the-run premium in yields (red line) together with two other

[9]For more detail about the GCF Repo Index see Fleming and Garbade (2003).

[10]GSW omit on-the-run and first-off-the-run bonds from their estimation because these bonds usually trade at a premium.

measures of the same premium commonly used in the literature. Both the blue and red on-the-run premia are obtained by subtracting the actual yield on the 10-year on-the-run note from the yield on a synthetic bond with the exact same maturity and coupon structure. The red line estimates the synthetic bond's yield with our 3-factor DTSM ignoring repo spreads, while the blue line uses the Svensson yield curve estimated by GSW. Both synthetic yields are constructed using parameters estimated only on off-the-run bonds. The GSW on-the-run premium has a correlation coefficient of 0.98 with the one implied by our 3-factor DTSM.

While the two on-the-run measures based on synthetic bonds are very similar, a simpler "2-bond" measure looks completely different. The black line in the top panel of Figure 1 reports the on-the-run premium calculated as the difference between the yield on the first off-the-run 10-year note and the yield on the on-the-run 10-year note (Krishnamurthy, 2002). This 2-bond measure is model-free, but as noted by GSW, it will tend to understate the on-the-run premium when the yield curve is upward sloping, because the first off-the-run bond usually has a lower duration than the on-the-run bond. This bias can be so severe that the 2-bond measure even returns an on-the-run *discount*, albeit a small one, as in the top panel of Figure 1. In our sample, this premium is generally an order of magnitude smaller than the premium implied by the other two measures. The reason is that, as shown in Table 1, not only on-the-run securities but also off-the-run securities can have special collateral value in the repo market, and thus, at times, the first off-the-run, being the closest substitute for the on-the-run, can be almost as valuable as collateral. In addition, security-level data from the DTCC show that fails to deliver seasoned Treasury securities—defined as securities issued more than 180 days prior—were increasing in early 2011 through late 2012 from previously-negligible levels, the same period in which we observe the 2-bond premium fall into negative territory.[11]

While the top panel of Figure 1 shows that the 10-year on-the-run premium is large and time-varying; the bottom panel of Figure 1 shows that the 10-year note also has a large and time-varying special spread. The figure plots the SC repo rates on 10-year on-the-run notes over time (blue line), as well as the overnight GC repo rate (red line). Consistent with a large and positive cash premium, the 10-year SC repo rate is often substantially lower than the GC rate, often dropping as low as -3% (annualized). Therefore, in what follows we explore in a reduced-form fashion whether cash price residuals are related to special spreads, which would indicate an omitted factor.

Duffie (1996) shows in a static setting that a security on special should have a higher cash price than an equivalent security that is not on special, and the price differential should be increasing in the specialness spread. In Table 2, we regress each CUSIP's price residuals,

[11] See Fleming et al. (2014) and Fleming and Keane (2016*a*,*b*).

$\eta_{i,t}$ (from the GSW model) on its special spreads y_t^i and its lagged price residuals:

$$\eta_{i,t} = \alpha_i + \beta_1 y_t^i + \beta_2 \eta_{i,t-1} + \xi_{i,t}. \tag{1}$$

The first column of Table 2 confirms Duffie's prediction by showing a positive and significant relation between price residuals and special spreads. In other words, the higher the special spread, the larger the amount of underpricing implied by the GSW model, which does not account for that risk factor.[12]

[Table 2 about here.]

The second column of Table 2 shows that higher price residuals continue to correlate significantly with higher special spreads even after including the lagged price residual, which reduces the magnitude of the coefficient on special spreads. The third and fourth columns show that these effects remain statistically and economically significant when we include bond fixed effects.

The reduced-form analysis in Table 2 shows that standard bond-pricing models that ignore special spreads underprice bonds that are on special. In this paper, we show that incorporating special spreads into a DTSM can explain a substantial fraction of special-bond prices, but only after allowing for time-varying risk premia linked to special repo spreads. To better illustrate why these spreads can be a source of risk, we analyze the nature of their fluctuations, in particular the way they vary around the Treasury auction cycle.

2.1 Specialness: Expected and Unexpected

Often, the SC repo rates in the bottom panel of Figure 1 tend to spike downwards at regular intervals. This "seasonality" is a feature of the Treasury auction cycle, which has three important periodic dates: the auction announcement date, the auction date, and the issuance date. There is usually about one week from the announcement to the auction. During a typical auction cycle, the supply of Treasury collateral available to the repo market is at its highest level when the security is issued, therefore the repo specialness spread should be close to zero. As time passes, more and more of the security may be purchased by holders who are not very active in the repo market, consequently the security's availability may decline over time and the repo specialness spread may increase. When forward trading in the next security begins on the auction announcement date, holders of short positions will

[12] Results are similar when we use price residuals generated by our 3-factor model instead of those from the GSW model.

usually roll out of the outstanding issue, implying that demand for that specific collateral should decrease and its repo specialness spread will rapidly decline (Fisher 2002).

[Figure 2 about here.]

The top panel of Figure 2 illustrates the auction-cycle dynamics of repo special spreads by showing their averages as function of days since issuance for bonds maturing in 2, 3, 5, 7, 10, and 30 years. As in Keane (1995), specialness spreads on the securities that are auctioned monthly (2-, 3-, 5-, and 7-year notes) climb with the time since the last auction until around the announcement of the next auction, after which they decline sharply. In contrast, the quarterly auction cycle of the 10-year note looks quite different, mainly because since 2009 the Treasury has introduced two regular re-openings following each 10-year note auction. Therefore, it is possible to observe three separate auction sub-cycles: the most dramatic run-up in specialness spread takes place before the first reopening, approximately 30 days after the auction; a second run-up, similar in shape but smaller in magnitude, immediately follows and peaks just before the second reopening, approximately 60 days after the auction; and finally, during the third sub-cycle the specialness spread is practically flat. This would suggest that the increased availability of the on-the-run security after each reopening strongly diminishes the impact of the seasonal demand for short positions around these dates (Sundaresan 1994).

The evidence of a recurrent auction cycle suggests that some time variation in special spreads is predictable and therefore is not itself a source of risk. Thus, when modeling repo special spreads we should carefully account for their deterministic component. However, there is a good deal of unpredictable variation around the averages plotted in the top panel of Figure 2. The bottom panel of the same figure shows a scatterplot of the square root of the observed special spreads around the auction-cycle component, estimated using a smoothing spline.[13] It is precisely the risk of the special spread varying significantly around the deterministic auction-cycle component that market participants ought to be pricing and that should generate a SC risk premium. This implies that the risky component of repo special spreads should capture those substantial deviations from the deterministic component and should be associated with a market price of risk.

To incorporate these features of the data in our DTSM (detailed in the next section), we specify the individual repo special spread y_t^i as the sum of three observable factors:

$$y_t^i = \left[y_{(t,i)}^D + y_t^S + x_t^i \right]^2 \tag{2}$$

[13]We report the square root because we will model the square root of the special spread over the auction cycle, rather than the level, to ensure that the special spread is weakly positive.

where $y^D_{(t,i)}$ is the deterministic auction-cycle process that depends on calendar time t only through the number of days since bond i was issued, y^S_t is a stochastic process meant to capture the risky component, and x^i_t is a bond-specific residual. The square in equation (2) ensures that special spreads are always weakly positive. We estimate $y^D_{(t,i)}$ on each maturity separately using smoothing splines, as in the bottom panel of Figure 2. We then identify the repo risk factor y^S_t as the average deviation of SC repo rates from their deterministic component across all n_t special securities (on- the-runs and first-off-the-runs) of a certain maturity at time t:

$$y^S_t = \frac{1}{n_t}\sum_{i=1}^{n_t}\left[\sqrt{y^i_t} - y^D_{(t,i)}\right]. \tag{3}$$

y^S_t is our measure of aggregate SC risk. The top panel of Figure 3 plots the time series of the SC repo risk factors for three maturity groups: the 2- and 3-year special notes together (red line), the 10-year special notes (blue line), and the 30-year special bonds (black line). These repo risk factors should capture SC risk from the short to the very long end of the maturity spectrum. It is easy to note that all three series exhibit the same upward time trend, indicating that SC repo risk has increased over our sample period. Further, the three time-series behave quite differently in the first half of the sample, while towards the end of the sample they tend to move together, though the 10-year SC risk factor displays the sharpest fluctuations. Next, we explore the systematic nature of our SC risk factors, that is, whether times of high SC risk are also times of high demand/low supply of near-money assets, likely times of high marginal utility of those assets.

[Figure 3 about here.]

To show that those unexpected fluctuations in SC repo rates expose investors (and in particular short sellers) to systematic risk and therefore should command a risk premium, we relate the SC risk factors to well-known proxies of demand and supply of near-money assets. The rationale being the following. Recently-issued Treasury notes and bonds are close substitutes of other near-money assets, such as Treasury Bills, hence their convenience yields should be interlinked (Sunderam, 2015; Nagel, 2016). As a result, increased demand for near-money assets due to stronger investors' preferences for the safety and liquidity of money-like claims will be associated with higher prices of recently-issued Treasuries and higher SC value of those securities in the repo market. In response, special spreads will spike and deviate from their auction cycle. This exposes short sellers to high interest rate risk as those speculators tend to short sell recently-issued Treasuries, hoping to buy them back at a lower price. And, if they cannot purchase them at a lower price, they will have to keep borrowing them in the SC repo market, paying a larger special spread. Similarly, reduced supply of near-money assets will also be associated with large deviations of SC repo rates from their auction cycle, and therefore with higher interest rate risk for short sellers. For example, this should happen when broker-dealers, the largest lenders of recently-issued Treasuries in the repo market, obtain a smaller share of those securities at auctions or become less willing to engage in repo transactions. In what follows we will investigate these possibilities.

[Table 3 about here.]

The first three variables reported in the first column of Table 3 are proxies of demand for money-like claims: the spread between 3-month GC repo rate and 3-month Treasury Bill rate used in Nagel (2016); the one- and 3-month OIS-Treasury Bill spreads used in Sunderam (2015). The remaining variables are proxies of supply of near-money assets in repo markets (Adrian et al., 2014), including auction allotments (as percentage of total issue) of Treasury securities to dealers and brokers, as well as the amount outstanding of overnight (and continuing) US Treasury repos and reverse repos in primary dealers' balance sheets (from the FR 2004 report). The last two measures should approximate primary dealers' willingness to engage in repo transactions. "Dealers and brokers," a category that includes primary dealers, other commercial bank dealer departments, and other non-bank dealers and brokers, are the most active lenders of recently-issued Treasuries in the SC repo market.

As can be noted in Table 3, all three SC risk factors are positively and significantly related to all three measures of convenience yield on Treasury Bills, with the coefficients and R^2 larger for the short-term (2-to-3-year) SC risk factor (third and fourth columns). This is not surprising as that factor captures the risky component of the convenience yields on recently-issued short-term Treasuries, which are very close substitutes of Treasury Bills. And, coefficients and R^2 are also quite large for the 10- and 30-year special bonds (last four columns). This suggests that SC repo risk increases when investors highly value money-like assets. Further, the SC risk factors are negatively and significantly related to the proxies of supply of near-money assets (last three rows), indicating that the SC repo risk tends to increases when the broker-dealers' ability and willingness to lend recently-issued Treasuries decreases, exposing short sellers to higher interest rate risk.

The goal of our model is to price SC risk within a DTSM, and quantitatively link the premium associated to this observed risk factor to the cash premium plotted in the top panel of Figure 1. Because the 10-year on-the-run note displays the largest premium in the cash market, and also has the largest special spread on average, we begin with a model specification that considers only the 10-year on-the-run and first off-the-run bonds as "special," that is, exposed to repo risk factors. Then, we extend the baseline model to incorporate special spreads and cash price premia for other maturities as well.

3 Model

We assume that the prices of Treasury bonds depend on a $k \times 1$ vector X_t that consists of both observable and unobservable (latent) factors, which evolve according to

$$X_{t+1} = \mu + \Phi X_t + \Sigma \varepsilon_{t+1} \tag{4}$$

where the vector of shocks ε_{t+1} are independent of X_t and of each other across time, and are normally distributed:

$$\varepsilon_{t+1} \sim \mathbb{N}\left(\vec{0}, I\right).$$

The generic short rate of interest is assumed to be an affine function of the factors:

$$\log\left(1 + R_t\right) = \delta_0 + \delta_1' X_t. \tag{5}$$

The stochastic discount factor (SDF) is given by

$$\log \frac{M_{t+1}}{M_t} = -\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1}, \tag{6}$$

where

$$\lambda_t \equiv \lambda + \Lambda X_t.$$

So far, the specification of our DTSM is quite standard, but next we introduce into the model special repo spreads at the security level, which requires a more extensive explanation because of its novelty.

3.1 Special Repo Spreads

A repo contract can be thought of as a collateralized loan, where the repo seller borrows at the repo rate in exchange for a Treasury bond, and regains the bond when she repays the loan plus interest at maturity. Duffie (1996) shows that SC repo rates can be below the GC repo rate, without creating an arbitrage opportunity, because the supply of special collateral is fixed.

A SC repo rate that is below the GC rate is a dividend that is proportional to the collateral's current price. Let $1 + R_t$ denote the current (gross) GC rate, and $1 + r_t^i$ the SC rate on bond i with current price P_t. Assume no haircut for simplicity. At time t, the owner of the special collateral borrows P_t against the collateral, at r_t^i, and simultaneously lends an amount Δ at the GC rate R_t. At time $t+1$, she earns

$$(1 + R_t)\,\Delta - \left(1 + r_t^i\right) P_t$$

so that if $\Delta = \frac{1+r_t^i}{1+R_t} P_t$, she has no gain or loss at $t+1$, and earns

$$P_t - \Delta P_t = \left(1 - \frac{1 + r_t^i}{1 + R_t}\right) P_t$$

at time t.

In what follows, it will be convenient to parameterize the log gross special spread on bond i as

$$y_t^i \equiv \log \frac{1 + R_t}{1 + r_t^i} \geq 0, \tag{7}$$

which implies that the price of a zero-coupon bond that is on special, with special spread equal to y_t^i, must have a price given by

$$\begin{aligned} P_t &= \left(1 - e^{-y_t^i}\right) P_t + E_t^* P_{t+1} \\ &= e^{y_t^i} E_t^* P_{t+1}, \end{aligned}$$

where E_t^* is the risk-neutral expectation. Because SC rates are always weakly smaller than the GC rate, the "special dividend" $e^{y_t^i} \geq 1$ or, equivalently, $y_t^i \geq 0$.

In order to ensure that y_t^i is nonnegative for all values of the state vector X_t, we parameterize it as a quadratic form in the factors:

$$y_t^i = X_t' \Gamma_i X_t, \tag{8}$$

where Γ_i is a symmetric positive semidefinite matrix that picks out some i-specific factors from the state vector X_t. This modeling device is commonly used in DTSMs accounting for the zero lower bound on the short rate (e.g., Ahn, Dittmar and Gallant, 2002; Kim and Singleton, 2012), and leads to the following proposition for pricing zero-coupon bonds that are on special for their entire life. We show later how we allow bonds to be on special for only a limited (deterministic) time, for example while they are on-the-run and first off-the-run.

Proposition 1. *Consider a zero-coupon bond on special with τ periods to maturity, where* *the repo spread is given by equation (8). Then the zero-coupon bond price satisfies*

$$\log P^Z_{t,\tau} = A_\tau + B'_\tau X_t + X'_t C_\tau X_t \tag{9}$$

where the A_τ, B_τ, and C_τ loadings are given by

$$\begin{aligned} C_\tau &= \Gamma + \Phi^{*\prime} C_{\tau-1} D_{\tau-1} \Phi^* \\ B'_\tau &= -\delta'_1 + \left(2\mu^{*\prime} C_{\tau-1} + B'_{\tau-1}\right) D_{\tau-1}\Phi^* \\ A_\tau &= -\delta_0 + A_{\tau-1} + \frac{1}{2} B'_{\tau-1}\Sigma G_{\tau-1}\Sigma' B_{\tau-1} \\ &\quad + \frac{1}{2}\log|G_{\tau-1}| + \left(\mu^{*\prime} C_{\tau-1} + B'_{\tau-1}\right) D_{\tau-1}\mu^* \end{aligned} \tag{10}$$

and

$$\begin{aligned} G_{\tau-1} &= \left[I - 2\Sigma' C_{\tau-1}\Sigma\right]^{-1} \\ D_{\tau-1} &= \Sigma G_{\tau-1}\Sigma^{-1}, \end{aligned}$$

$C_0 = \vec{0}_{k\times k}$, $B_0 = \vec{0}_{k\times 1}$, *and* $A_0 = 0$, *and the risk-neutral parameters* μ^* *and* Φ^* *are given by*

$$\begin{aligned} \mu^* &\equiv \mu - \Sigma\lambda \\ \Phi^* &\equiv \Phi - \Sigma\Lambda. \end{aligned} \tag{11}$$

Proof. See Appendix A. ∎

The loadings in equation (10) include the loadings of an affine term structure model as a special case when $\Gamma = \vec{0}$, since in this case $C_\tau = \vec{0}$ for all τ and therefore $G_\tau = D_\tau = I$ for all τ. Further, these loadings are usually obtained in quadratic-Gaussian term-structure models (e.g., Kim 2004; Breach, D'Amico and Orphanides forthcoming).

Recall that in Section 2.1, we specify y^i_t as the squared sum of three components: a deterministic auction-cycle component, $y^D_{(t,i)}$, the SC risk factor y^S_t, and a security-specific residual x^i_t. The latter can be obtained as follows:

$$x^i_t = \sqrt{y^i_t} - y^S_t - y^D_{(t,i)}. \tag{12}$$

Importantly, all the elements of equation (12) can be estimated directly on the special-spread data, independently of the DTSM. In this sense they are observable pricing factors, in contrast to the first three latent factors of the model.[14]

[14] Further, including the deterministic component $y^D_{(t,i)}$ in equation (2) also allows us to include more bonds,

After estimating the repo factors $y^D_{(t,i)}$, y^S_t, and x^i_t, we add them to the state vector X_t and assume, along with the three latent factors, that the full vector X_t follows a VAR process as in equation (4). Then, in the basic case of a bond that is on special for its entire life, the matrix Γ from equation (10) is pinned down by

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{13}$$

where the three repo factors enter as the fourth, fifth, and sixth components of the state. Notably, we allow the SC risk factor y^S_t to have unrestricted VAR dynamics and prices of risk. We assume that the idiosyncratic special spread residuals x^i_t follow the process

$$x^i_{t+1} = \rho x^i_t + \sigma_x \varepsilon^i_{t+1}, \tag{14}$$

where ε^i_{t+1} is a standard normal random variable that is iid over time and independent of the aggregate VAR shocks ε_{t+1}. We also assume that x^i_t is unconditionally mean-zero in order to allow the average repo spreads to be governed by the auction-cycle component and the SC risk factor. Because the x^i_t are idiosyncratic, we assume that their dynamics are identical under the physical and risk-neutral measures, so that the shocks ε^i_{t+1} do not enter the SDF or carry any price of risk.

Because the auction-cycle component y^D_τ does not depend directly on calendar time and is not random, we model its effect on bond prices by inserting it directly into the appropriate element of μ, and setting the corresponding rows of Φ and Σ to zero.

Overall, this setup implies that for "non-special" bonds (i.e., second off-the-runs and older) the price loadings are given by equation (10) with $\Gamma = \vec{0}$ and initial conditions $A_0 = B_0 = C_0 = 0$. In other words, we price these bonds with a standard 3-factor Gaussian DTSM. In contrast, a zero-coupon bond that is on special until maturity would have price loadings again given by equation (10), with the same initial condition, but with Γ given by equation (13). Thus the price of a special bond is driven by six factors rather than three.

at different points in the auction cycle, into the aggregate repo factor y^S_t, as it strips out the auction-cycle component so we can compare apples to apples.

3.2 Coupon-bearing bonds

So far, we have derived the price of zero-coupon bonds that can be on special for their entire life. However, special spreads typically accrue to coupon-bearing bonds that only trade special for a limited time. Coupon-bearing bonds are linear combinations of the zero-coupon bonds priced in Proposition 1, weighted by their coupon payments: the price of bond i at time t is given by

$$\begin{aligned} P_t^i &= \sum_j c_j \exp\left\{A_{\tau_j} + B'_{\tau_j} X_t + X'_t C_{\tau_j} X_t\right\} \\ &\equiv \sum_j c_j P^Z\left(\tau_j, X_t\right) \\ &\equiv \vec{P}^Z \left\{\vec{c}\,(i)\,, \vec{\tau}\,(i)\,, X_t\right\}, \end{aligned} \tag{15}$$

where c_j denotes the jth coupon payment, τ_j its time to maturity, and the last two lines define notation. Equation (15) includes the repo special spread at time t through the C_{τ_j} loadings (which all contain a Γ term). Details on how we identify the term structure of special spread loadings on actual coupon bonds are given in Section 3.3.

Treasury bonds in our sample pay the same coupon amount every six months; accounting for these coupons, and pricing accrued interest, implies that the price of bond i is given by

$$P_t^i = P^Z\left(\tau_i, X_t\right) + \frac{c_i}{2 \times 100}\left[\sum_{j=1}^{N_{it}} P^Z\left(\tau_{ij}, X_t\right) - \frac{\tau_{i1}}{\xi_{it}} P^Z\left(\tau_{i1}, X_t\right)\right], \tag{16}$$

where N_{it} is the number of remaining coupon payments for bond i at time t, and ξ_{it} is the time between the next and previous coupon payment for bond i at time t.[15] The last term in equation (16) accounts for the accrued interest on bond i at time t, which is shared pro rata between the buyer and seller depending on the time remaining to the next coupon payment. Equation (16) describes how the coupon rate c and time to maturity τ of a given bond i translate into the cash-flows $\vec{c}\,(i)$ and their maturities $\vec{\tau}\,(i)$ in equation (15).

[15]For the bonds that mature on February 29th, we assign their coupon payments in non-leap-years to February 28th.

Stacking all n_t bonds at time t yields the measurement equation

$$\vec{P}_t = \begin{bmatrix} \vec{P}^Z \left\{ \vec{c}\,(1)\,, \vec{\tau}\,(1)\,, X_t \right\} \\ \vec{P}^Z \left\{ \vec{c}\,(2)\,, \vec{\tau}\,(2)\,, X_t \right\} \\ \ldots \\ \vec{P}^Z \left\{ \vec{c}\,(n_t)\,, \vec{\tau}\,(n_t)\,, X_t \right\} \end{bmatrix} + \vec{\eta}_t \tag{17}$$

which, along with equation (4), constitute the state-space system to be estimated. The number of bonds n_t in each cross-section is so large relative to the number of factors X_t that the latter can be estimated on each cross-section individually without regard to the state equation (4) (see Andreasen and Christensen 2015 for a proof); that is, our method is similar to a formal non-linear Kalman filter. However, unlike an estimation with a non-linear Kalman filter, our method does not require estimating the physical measure parameters. This is important because our sample of repo special spreads is only about nine years long, and does not include enough business cycles to reliably estimate physical parameters.

Coupon payments complicate the computation of prices for special securities somewhat, in particular because, as shown in Figure 2 and the fourth and fifth columns of Table 1, bonds do not remain on special for their entire life. Bonds are "more special," in the sense of a higher special spread, when they are on-the-run. However, we are helped by the fact that securities go off-the-run at a fixed time since issuance, at which point their special spreads are close to zero (see Figure 2). Thus, it is reasonable to make two assumptions. First, we assume that second off-the-run and older securities have zero special spreads. Second, since the Treasury auction calendar is very regular, we assume that each bond's remaining maturity—as well as each of its coupons' remaining maturity—when they go "off special" are known with certainty at issuance.

We model these complications as follows. Let $m\,(j)$ denote the remaining maturity of coupon payment j when the bond goes off special. Then, for each bond i in our sample, we compute the price loadings of each coupon j and principal payment using the off-the-run loadings with $\Gamma = \vec{0}$ if the remaining maturity is between 0 and $m\,(j)$. For maturities bigger than $m\,(j)$, we use Γ as in equation (13), as over this time interval the bond is considered special. Thus not only the principal, but also every coupon payment of a special bond is discounted using the special spread and the SC risk factor; and the sensitivity to that risk factor varies across coupons according to their remaining maturity to ensure absence of arbitrage.

This way of pricing "special" coupons can be formalized in the following way. Recall that A_τ, B_τ, and C_τ denote the loadings from equation (10) with the "usual" initial conditions

$A_0 = 0$, $B_0 = \vec{0}$, and $C_0 = \vec{0}$. Because everything that follows applies equally to the A, B, and C loadings, for brevity we detail only the A loadings. The price loading $A_\tau^{m(j)}$ of a special coupon is given by

$$A_\tau^{m(j)} = \begin{cases} A_\tau & \tau \leq m(j) \\ \text{equation (10) and (13), with initial condition } A_\tau & \tau > m(j) \end{cases}.$$

Figure 4 illustrates how we use equation (10) to discount each coupon individually depending on its time to maturity. Each bar along the y-axis represents a different coupon payment, while the x-axis (and thus the length of the bars) represents time to maturity. The vertical dashed line is the time at which this bond and all its remaining coupons become non-special. Because the first coupon pays out before the bond goes off-special, its entire payment is discounted using the SC risk factor (in addition to the usual latent factors) for its whole life. The second coupon payment, by contrast, occurs after the bond has gone off-special; to price this coupon, we use the $A_\tau^{m(2)}$ price loadings. Other coupons are priced analogously, but each coupon has a different $m(j)$ and thus a different initial condition for its $A_\tau^{m(j)}$ loadings. All of this bond's coupons are priced as special coupons for the same amount of time, marked in blue in Figure 4.

[Figure 4 about here.]

This modeling approach implies that we must compute the recursion (10) once for each coupon maturity for each bond in the data, varying the initial condition according to the coupon's remaining maturity when the bond in question goes off-special (i.e., becomes a second off-the-run). Notice that this is the only way to ensure that there is no arbitrage from, for example, short-selling an on-the-run bond to buy an off-the-run bond of the same maturity the day before the on-the-run bond goes off-special. On that day, this method ensures that the special bond's price incorporates its exposure to the SC risk factor for one more day; on the next day, it will be priced according to its exposure to only the latent factors. Two days before the bond becomes non-special, this method ensures that all coupons are priced to incorporate two more days of exposure to the repo factor, and so on.

Appendix C also illustrates this issue algebraically with a simple two-coupon example.

3.3 Identification

By Proposition 1, in order to price the cross-section of bonds we need only the "risk-neutral" parameters of the model: μ^*, Φ^*, Σ, δ_0, δ_1, and Γ. The remaining time-series parameters μ and Φ (or, equivalently, the price of risk parameters λ and Λ) can be identified by the dynamics of the factors X_t.[16] However, because some of the elements of X_t are latent, they must be invariant to translation and rotation; this means that to identify all elements of μ^*, Φ^*, δ_0, δ_1, and Σ, additional restrictions need to be imposed.

Our benchmark model without repo specials has three latent factors; such a model has only 10 free risk-neutral parameters (Joslin, Singleton and Zhu, 2011). We achieve identification by assuming that $\Sigma = I/365$ (so that it is equal to the identity matrix at an annual frequency), $\mu^* = \vec{0}$, δ_0 and δ_1 are free parameters, and the top-left corner of Φ^* has the form

$$\Phi^*_{\text{TL}} = \begin{bmatrix} \phi^*_1 & \phi^*_2 & 0 \\ \phi^*_3 & \phi^*_4 & 0 \\ 0 & \phi^*_5 & \phi^*_6 \end{bmatrix}. \tag{18}$$

Given these identifying assumptions, the only risk-neutral parameters to be estimated are δ_0 (one parameter), δ_1 (three parameters), and Φ^*_{TL} (six parameters). In addition, some parameters, including ρ and σ_x, can be estimated directly on the repo special spread data without needing to estimate a DTSM. In particular, we estimate x^i_t for each CUSIP i using equation (12). We then estimate ρ and σ_x from equation (14) via OLS, pooling across CUSIPs.

To estimate the remaining parameters we proceed in steps. First, we minimize the sum of squared price residuals using only second off-the-run and older bonds and ignoring repo special spreads, as follows. Given values for the 10 parameters in δ_0, δ_1, and Φ^*_{TL}, each day we estimate the latent factors X_t that minimize the pricing residuals. These estimates of X_t are consistent as the number of bonds in each cross-section goes to infinity. We search over parameters values to maximize the combined log-likelihood from the residuals in equation (17) pooled across all dates t.[17]

Second, to estimate the repo parameters of the model, we expand the vector X_t to include the repo risk factor y^S_t in addition to the estimated latent factors from the previous step. Holding the latent factors (and y^S_t) fixed, we search over the remaining parameters in μ^*, Φ^*_{BL}, and Φ^*_{BR} to minimize the sum of squared price residuals of on-the-run and first off-the-run bonds. Third, we estimate the bottom row of Σ (corresponding to the SC risk factor) directly from a VAR including the three latent factors and y^S_t.

[16] Although for the analysis in this paper, we need not do so, as described earlier.

[17] We ignore the residuals that come from equation (4) in this exercise. In practice, the more than 200 bonds in each cross-section contain much more information than the three factos X_{t-1} from the previous cross-section, so that our method is in practice very similar to a formal non-linear Kalman filter. In addition, nothing in our main results requires estimates of the physical measure parameters μ and Φ.

4 Results

In this section, we estimate models of increasing complexity to illustrate the importance of including a time-varying risk premium on SC risk factors for pricing Treasury securities with special value in the repo market; we label this premium the SC risk premium.

In light of the evidence showing that the 10-year bond has the largest GSW price residuals and the largest special spreads (Table 1, Figure 2), in our benchmark specification we include only the 10-year on-the-run and first off-the-run notes in equation (3). We then proceed in stages.

First, we estimate a standard 3-factor model that ignores special spreads completely and thus includes only non-special bonds in the estimation (i.e., all securities with maturity at issuance other than 10 years, and 10-year bonds that are second off-the-run or older); and, we use the estimated parameters to price the 10-year on-the-run and first off-the-run bonds (the "special" bonds), continuing to ignore the special repo factors. We report these parameters in Panel A of Table 4.

Second, we use the 3-factor model parameters, the estimated latent factors, and the estimated dynamics of the special spreads to price the SC risk factors in a risk-neutral fashion. That is, we force the parameters pertaining to y_t^S to be the same under the physical and risk-neutral measures.[18] To do so, we set μ^* and Φ^* such that the fourth element of λ is zero, and the fourth row and column of Λ are all zeroes.

[Table 4 about here.]

Third, we incorporate a constant risk premium on special bonds linked to the repo factor y_t^S by allowing the corresponding elements of μ^* to differ from μ. Finally, we allow for a time-varying risk premium associated to the repo factor y_t^S by estimating new values for the elements of μ^*, as well as for the bottom rows of Φ^* (that is, Φ^*_{BL} and Φ^*_{BR}). Panel B of Table 4 reports the SC risk factor parameters for the 10-year only model. It is worth noticing that Φ^*_{BR} switches sign when going from the risk-neutral to the time-varying risk evaluation, which, using equation (11), implies that the price of risk is inversely related to y_t^S. In other words, on-the-run bonds are more valuable to their holders when there are large and positive deviations of special spreads from the expected auction-cycle component.[19] In this sense it might be more precise to call our SC risk premium, a risk *discount*.

[18] Notice that although we need to estimate the physical dynamics of X_t (in particular the repo factor elements of X_t) in order to label these estimates "risk-neutral," we do not need to know the physical dynamics to estimate the full model with time-varying risk premia, since we estimate μ^* and Φ^* from equation (11) directly.

[19] The magnitude of Φ^*_{BR} (0.713) suggests a low persistence of the repo factor. It is possible that this estimate is driven by our short sample period, and that with a higher persistence, the risk-neutral valuation plotted in red in Figure 5 would explain more of the on-the-run premium. However, this is not the case. If

In addition, for robustness, in Section 5 we also explore two alternative specifications that differ from the benchmark model in the bonds that they include as "special," and how they price specialness. We estimate a model that includes on-the-runs and first-off-the-runs of four different maturities (not just the 10-year) in equation (3); we label this estimation the "pooled" model. Then, we move away from a single SC risk factor and extend y_t^S to be a 3×1 vector that averages over 30-year special spreads, 10-year special spreads, and 2- and 3-year special spreads, separately; we label this estimation the "split" model. This model, differently from the benchmark and pooled models, is a 6-factor model that includes 3 latent factors and 3 observed SC risk factors. Similarly to the benchmark model, also for the pooled and split models we will estimate the risk-neutral, constant risk premium, and time-varying risk premium versions.

4.1 The SC risk premium

Although the results in Table 2 suggest that standard DTSMs are missing something by ignoring special spreads, we can do much more to quantify their importance. The top panel of Figure 5 plots the price residuals on the 10-year on-the-run bond implied by our benchmark model, in which we treat 10-year on-the-run and first off-the-run bonds as "special." The black line (barely visible beneath the red line) is the price residual ignoring special spreads completely, which we report for comparison.

The red line is the price residual assuming that the risky special spreads are priced as risk-neutral dividends. In theory, if the profits from short-selling the on-the-run bond and going long a bond with similar cash-flows, for example the first off-the-run bond, were about zero on average, then the red line should hover near zero. That is, the profits from selling the more-expensive bond would be roughly offset by the cost of borrowing that bond (earning a negative interest rate) in the repo market. This is what Krishnamurthy (2002) finds for the 30-year on-the-run bond over his 1995–1999 sample period: the special spread and cash price premium are roughly consistent with risk-neutral repo special spreads. In contrast, in our sample period, the 10-year on-the-run premium is much larger than the 30-year premium (in prices, almost 1% of par on average according to Table 1; and in yields as high as 60 bps at times as shown in the top panel of Figure 1); and average special spreads, which accrue to the bond only for the first six months, are not high enough to wipe out the short-selling profits (i.e., the on-the-run premium).[20] This is not consistent with a risk-neutral valuation

we replace the 0.713 with a 1, and re-estimate the other VAR parameters accordingly, we get almost identical results.

[20] Allowing bonds to be on special for their entire life and re-estimating the model does not solve the problem, because special spreads on far off-the-run bonds are so low on average (fourth column of Table 1).

of special spreads, and therefore with previous studies such as Krishnamurthy (2002).

There are at least four explanations for the difference between our results and those in Krishnamurthy (2002). First, we focus on 10-year notes rather than 30-year bonds; as shown in Table 1, 10-year notes have a much higher on-the-run premium, as well as a somewhat higher average special spread, than 30-year bonds. Second, special spreads of 10-year notes exhibit larger deviations from their auction cycle than those of 30-year bonds: the deviations plotted in the bottom panel of Figure 2 are 40% more volatile for the 10-year note than they are for the 30-year bond. Third, the arbitrage trade analyzed in Krishnamurthy (2002) shorts the on-the-run bond and longs the first off-the-run bond; as we show in the top panel of Figure 1, in our sample this understates the on-the-run premium because the first off-the-run bond often trades at a premium. Fourth, the on-the-run premium in our sample is on average much larger than in the 1995–1999 period that Krishnamurthy (2002) studies (compare the top panel of Figure 1 with Figure 2 of Krishnamurthy 2002).

[Figure 5 about here.]

The green and blue lines in the top panel of Figure 5 plot price residuals for the models with constant and time-varying risk premia on the repo factor y_t^S, respectively. The model with constant risk premia shifts residuals down, which improves the fit somewhat until about 2012, after which the new residuals hover near zero (or go slightly negative). Any further increase in the risk premium would fit the financial crisis period even better, but at the cost of over-pricing after 2012, pushing residuals into negative territory. The model with time-varying risk premia, on the other hand, is able to make the pricing errors almost mean-zero during the crisis without sacrificing much of the fit in the later part of the sample; although, in the second half of 2013, it overestimates the price by as much as 50–75 bps of par. Interestingly, this time period coincides with the immediate aftermath of the "Taper Tantrum" episode that we will discuss shortly.

The bottom panel of Figure 5 plots our yield measure of the SC risk premium: the yield difference between two 10-year *zero-coupon* bonds, both receiving the repo dividend y_t^S for 180 days, but one estimated using a risk-neutral valuation of this stream of repo dividends, and the other using our estimated time-varying prices of SC risk. This measure of the SC risk premium, relative to the difference between the red and blue lines plotted in the top panel of Figure 1, eliminates some of the seasonality by pricing a "constant maturity" synthetic 10-year bond that is consistently exposed to the repo factor for the next 180 days (90 days as an on-the-run and 90 days as a first off-the-run).

The 10-year SC risk premium is large in 2009–2011, with a peak of 100 basis points, and then hovers around 25 basis points. Thus, its magnitude is similar to other classical risk premia, such as the real and inflation risk premium, estimated in the literature over this sample period (e.g., Ajello, Benzoni and Chyruk, 2014 or Breach, D'Amico and Orphanides, forthcoming). To try to understand what drives our SC risk premium, the vertical lines in the bottom panel of Figure 5 mark the times of major macroeconomic events particularly relevant for U.S. Treasury bonds. In particular, they indicate supply shocks approximated by the FOMC announcements of the Federal Reserve's asset purchase programs (Cahill et al. 2013). By far, the largest drop downward in the SC risk premium occurred near the beginning of the sample, on March 18, 2009 when the Federal Reserve announced the first round of quantitative easing (QE1). It is possible that the prospect of a large new investor (the Fed) willing to buy seasoned Treasuries, at that time regarded as considerably less liquid than recently-issued securities, increased their expected trading opportunities, making the 10-year on- and first off-the-run relatively less special. Similarly, the FOMC announcement on August 10, 2010, in which the Fed promised to reinvest agency-MBS principal payments into Treasury securities, also seems to have induced a decline in the SC risk premium.

In contrast, the second round of quantitative easing (QE2), which was announced on November 3, 2010, and the September 21, 2011 announcement of the Maturity Extension Program (MEP), in which the Fed promised to sell shorter-term Treasuries in order to buy longer-term securities to extend the duration of its portfolio, produce a spike upwards in the SC risk premium. One possibility is that those additional rounds of Fed purchases, mostly focused on longer-term Treasuries, were perceived as potentially increasing the relative scarcity of 10-year bonds, making them more special. However, the QE3 announcement on December 12, 2012 had essentially no effect on the repo risk premium.

Another event that seems to have impacted the SC risk premium is Chairman Bernanke's Congressional testimony on May 22, 2013, in which his remarks about the possibility of moderating the pace of asset purchases later that year led to the subsequent "Taper Tantrum." That is, a sharp and large increase in longer-term yields that lasted for about 3 months, also accompanied by increased volatility. This, is turn, induced a sudden reversal in hedging and speculative positions involving short-selling of the 10-year bond, potentially pushing the SC premium up. Overall, major policy announcements changing the availability of safe and liquid securities have significantly affected the SC risk premium of recently-issued Treasuries, often viewed by investors as near-money assets. This is probably not surprising given the evidence on the systematic nature of the SC risk, provided in Section 2.

4.2 A commonality across price anomalies

To analyze the ability of the SC risk premium to explain relative price anomalies in large and liquid fixed-income markets, Table 5 presents correlations and regression R^2 between our measures of the 10-year SC risk premium and various measures of price anomalies documented in the literature. The first row shows that the SC risk premium has a 91% correlation with, and explains 83% of the GSW on-the-run premium, plotted over our sample period in the top panel of Figure 1. This series is a standard measure of the on-the-run premium (see for example Adrian, Fleming and Vogt 2017) that is independent of our model. This finding is in line with the calibration exercise of Vayanos and Weill (2008) showing that the majority of the on-the-run premium is due to specialness rather than liquidity.

[Table 5 about here.]

The 10-year SC risk premium is also highly correlated with four other Treasury price anomalies documented in the literature: the 10-year TIPS-Treasury bond puzzle of Fleckenstein, Longstaff and Lustig (2014), which we proxy with the wedge between the 10-year inflation swap rate and the 10-year TIPS break-even rate;[21] the off-the-run note-bond spread of Musto, Nini and Schwarz (2017), which we derive as the difference between the average yields of off-the-run Treasury notes and bonds with duration between 7 and 9 years to make their maturity as close as possible to 10 years;[22] the 10-year TIPS liquidity premium of D'Amico, Kim and Wei (2018); and the average absolute nominal yield curve fitting errors, which can be interpreted as a measure of limits to arbitrage (see for example Hu, Pan and Wang 2013). Except for the latter, which is derived from all outstanding Treasuries, the other measures are all roughly *10-year* spreads to either a comparable *nominal* Treasury security or to a *more recently-issued* Treasury security. As such, those spreads may appear anomalous when the relatively higher-quality security is particularly valuable as collateral and that value is not priced in as a risk factor.

The first two columns of Table 5 show that the 10-year SC risk premium can explain a substantial fraction of these price anomalies. In particular, it accounts for 68% of the variation in the TIPS-Treasury bond puzzle, 62% of the variation in the off-the-run note-bond spread, and 58% of the variation in the TIPS liquidity premium. It is not surprising

[21] Fleckenstein, Longstaff and Lustig (2014) exactly match all cash-flows of pairs of TIPS and nominal Treasuries by converting the inflation-indexed cash flow of a TIPS into fixed payouts using inflation swaps. The strategy consists of three parts: the purchase of the bond, the sales of several zero-coupon CPI swaps, and the purchase or sale of several Treasury STRIPS. Since the arbitrage profits of this strategy arise in large part because inflation swap break-evens are always higher than TIPS nominal break-evens, our proxy, although less precise, captures the core of the TIPS-Treasury bond puzzle.

[22] The average duration of the on-the-run 10-year note is usually about eight years.

that, relative to the 10-year on-the-run premium, a smaller share of those anomalies is explained by the 10-year SC risk premium. This is because, as the lower-quality asset used in the spread becomes increasingly dissimilar from the recently-issued off-the-run nominal security (either because it is more seasoned or because it is issued as a TIPS), factors other than collateral value should start playing a bigger role. For example, trading liquidity might explain part of the remaining variation.

In addition, as shown in the last row, the 10-year SC risk premium also accounts for 36% of the variation in nominal yield curve fitting errors, indicating that relative differences in the observed SC value of Treasury securities play a significant role in explaining their daily deviations from the yield curve. In other words, it seems reasonable that the cost of short-selling Treasury securities might affect investors' ability to take advantage of arbitrage opportunities, proxied by yield curve fitting errors.

5 Robustness

Up to this point we have defined only the 10-year on-the-run and first off-the-run notes as "special" for calculating the value of y_t^S in equation (3). To examine the robustness of our results to alternative definitions of the SC risk factor, we also estimate a "pooled" model incorporating special spreads on short-maturity (2- and 3-years) and long-maturity (30-year) bonds in addition to 10 years.[23] This "pooled" SC risk factor is plotted in red in the bottom panel of Figure 3, and the model's estimated parameters are reported in Table 6. We then estimate a "split" model in which we allow for three separate y_t^S factors: one for 30-year bonds, one for 10-year bonds, and one for 2- and 3-year bonds together, shown in the top panel of Figure 3 and discussed in Section 2.1. The parameters of this third model are reported in Table 7.

[Table 6 about here.]

[Table 7 about here.]

For each model we estimate risk-neutral, constant risk premium, and time-varying risk premium versions and use them to price the 10-year on-the-run note, as in the top panel of Figure 5; the results are plotted in Figure 6. They are qualitatively similar to the 10-year only model, although the pooled model, being forced to explain more variation with the same number of parameters, tends to capture substantially less of the 10-year price premium

[23]We exclude the 5- and 7-year bonds from this exercise for clarity; results are similar if we include them.

and generates a smaller SC risk premium than the other two models (the difference between the red and blue lines).

[Figure 6 about here.]

To more easily compare results across models, Table 8 quantifies the amount of variation in on-the-run price residuals explained by our estimated models and summarized by the sum of squared residuals for each model and its alternative versions: no special spreads, risk-neutral evaluation, a constant risk premium, and a time-varying risk premium.

[Table 8 about here.]

In all three model specifications, a time-varying SC risk premium is necessary to price 10-year on-the-run and first off-the-run bonds, as shown by the drastic reduction in the sum of squared residuals in Table 8. The same is true for the 30-year bond. In contrast, this is not the case for short-term securities, as evidenced by the lack of improvement across the first 4 columns.

The last three columns report the share of variation explained, for each model and maturity, in the form of an R^2, defined as follows:

$$R_i^2 \equiv 1 - \frac{\eta_i' \eta_i}{\eta_0' \eta_0}, \tag{19}$$

where η_0 is the vector of price residuals under the risk-neutral evaluation, and η_i is the vector of price residuals from the ith alternative model. Thus, for example, in the first row of Table 8, the 10-year only model delivers, for 10-year special bonds, a sum of squared residuals of 0.379; with a time-varying SC risk premium this value drops to 0.0423, which is an R^2 of 88.9%. The model achieves this 89% fit on 2,252 price observations with only 5 free parameters (as the other repo parameters reported in Panel B of Table 4, ρ, σ_x, Σ_{21}, and Σ_{22}, are the same across the risk-neutral, constant, and time-varying risk premium models).

While the explanatory power of the pooled and split models is roughly similar to the benchmark model at the the 10-year maturity, they perform differently at other maturities. In particular, the pooled model actually has a negative R^2 for maturities other than 10 years. This is because the price residuals on the special 10-year note are so much larger than those for the other maturities that, to minimize the total sum of squared residuals, the pooled model sacrifices the fit of other maturities to better fit the 10-year. However, with a single y_t^S that averages across special spreads of 4 maturities and only five free parameters, the pooled model is still not able to fit the 10-year prices as well as the 10-year only model.

Nevertheless, the SC risk premium that comes out of this model has a correlation coefficient of 0.98 with the one implied by our benchmark model.

In contrast, the split model is able to explain the 10-year prices as well as (actually slightly better than) the 10-year only model. It also treats the three other maturities very differently: for the 30-year maturity, a constant risk premium explains only 5% of the price premium, while the time-varying risk premium explain almost half of the variation in price premia. At the short end of the yield curve (2 and 3 years), a constant risk premium actually does a bit better at explaining price residuals than a time-varying one. A worse fit at the short end is "optimal" in the sense that those on-the-run price residuals are already quite small, as can be seen in the first column of Table 8, relative to the 10- and 30-year sum of squared residuals.[24] Thus, the split model makes the fit at these maturities just a bit worse in order to better explain the larger deviations at the 10- and 30-year maturities.

[Figure 7 about here.]

As shown in the top panel of Figure 7, the 2-year SC risk premium never exceeds 25 basis points even during the crisis, and it is in general much smaller and noisier than the 10-year SC risk premium (Figure 5). This also illustrates that the SC risk premium is different from the liquidity premium of near-money assets (Nagel, 2016) as it is much larger for 10-year Treasury securities than shorter-term ones. However, the bottom panel of Figure 7 shows that the 30-year SC risk premium is essentially constant and below 10 bps, apart from the first few months of 2009. This suggests that the SC risk premium is not necessarily monotonically increasing with maturity.

Finally, to verify the robustness of the relation between the SC risk premium and the various price anomalies, the last four columns of Table 5 show that whether we derive the 10-year SC risk premium including various maturities or just the 10-year, and whether we "split" y_t^S into three factors or pool maturities together, the SC risk premium can always explain a substantial fraction of these price anomalies. In particular, the second two columns of Table 5 report correlations and R^2 for the "pooled" model, while the remaining columns report the same quantities for the 10-year SC risk premium from the "split" model. In all cases, the R^2 varies between 27% and 81%, suggesting that these price anomalies are largely justified by time variation in SC risk.

[24] In fact, the on-the-run price residuals on the 2- and 3-year bonds are generally of the same size as for non-special bonds. In this sense, there is not much variation to be explained, relative to the 10- and 30-year bonds.

6 Conclusion

This paper shows that in traditional DTSMs of U.S. nominal Treasury securities there seems to be a missing special collateral risk factor, which we obtain from aggregate deviations of special repo spreads from expected auction-cycle dynamics. This, in turn, implies that there is a missing risk premium: the special-collateral repo risk premium. This omission has led some to interpret price differences relative to the highest quality collateral—recently-issued nominal Treasury securities—entirely as price anomalies. In contrast, if observed special-collateral repo risk factors are explicitly priced in, these anomalies are mostly justified by compensation for exposure to special-collateral risk, and thus are not that anomalous. That is, rational investors seem to account for the uncertain stream of expected repo dividends (i.e., the convenience yield), in addition to the classical risk factors, when pricing nominal Treasuries.

This paper has only begun to scratch the surface of what is possible with the special repo rate data, and how this data can improve our understanding of convenience yields on near-money assets. For example, it would be great if, on top of pricing jointly Treasury cash and repo rates, we could also bring into the model Treasury future rates. This would complicate further the security-level pricing within a DTSM and we leave these challenges to future research.

A Proofs

A.1 Proof of Proposition 1

The result follows by induction. First note that a zero-coupon bond pays \$1 at maturity, so that $A_0 = 0$, $B_0' = \vec{0}$, and $C_0 = \vec{0}$ as in equation (10) prices bonds at maturity. Next, fix τ and assume that at any time t, the price of an $\tau - 1$ period bond satisfies

$$\log P_t^{(\tau-1)} = A_{\tau-1} + B_{\tau-1}' X_t + X_t' C_{\tau-1} X_t. \tag{20}$$

It then suffices to show that equation (20) implies equation (10) for bonds with maturity τ.

The log price of an τ-period zero-coupon bond at time t with special spread y_t is given by

$$\begin{aligned}
\log P_t^{(\tau)} &= y_t + \log E_t \left\{ \frac{M_{t+1}}{M_t} P_{t+1}^{(\tau-1)} \right\} \\
&= X_t' \Gamma X_t + \log E_t \left\{ \frac{M_{t+1}}{M_t} \exp \left\{ A_{\tau-1} + B_{\tau-1}' X_{t+1} + X_{t+1}' C_{\tau-1} X_{t+1} \right\} \right\} \qquad (21) \\
&= X_t' \Gamma X_t + \log E_t \exp \Bigg\{ -\delta_0 - \delta_1' X_t - \frac{1}{2} \lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1} \\
&\qquad + A_{\tau-1} + B_{\tau-1}' \left(\mu + \Phi X_t + \Sigma \varepsilon_{t+1} \right) \\
&\qquad + \left(\mu + \Phi X_t + \Sigma \varepsilon_{t+1} \right)' C_{\tau-1} \left(\mu + \Phi X_t + \Sigma \varepsilon_{t+1} \right) \Bigg\} \\
&= X_t' \Gamma X_t - \delta_0 - \delta_1' X_t - \frac{1}{2} \lambda_t' \lambda_t + A_{\tau-1} + B_{\tau-1}' \left(\mu + \Phi X_t \right) + \left(\mu + \Phi X_t \right)' C_{\tau-1} \left(\mu + \Phi X_t \right) \\
&\qquad + \log E_t \exp \left\{ m' \varepsilon_{t+1} + \varepsilon_{t+1}' \Sigma' C_{\tau-1} \Sigma \varepsilon_{t+1} \right\}
\end{aligned}$$

where the second line uses equations (8) and (20), the next line plugs in equations (4) and (6), and m in the last line is given by

$$\begin{aligned}
m &\equiv -\lambda_t + \Sigma' B_{\tau-1} + 2 \Sigma' C_{\tau-1} \left(\mu + \Phi X_t \right) \\
&= -\lambda + \Sigma' B_{\tau-1} + 2 \Sigma' C_{\tau-1} \mu + \left(2 \Sigma' C_{\tau-1} \Phi - \Lambda \right) X_t.
\end{aligned}$$

Because ε_{t+1} is a standard multivariate normal random variable, we have that

$$\log E_t \exp \left\{ m' \varepsilon_{t+1} + \varepsilon_{t+1}' \Sigma' C_{\tau-1} \Sigma \varepsilon_{t+1} \right\} = \frac{1}{2} m' G_{\tau-1} m + \frac{1}{2} \log \left| G_{\tau-1} \right|, \tag{22}$$

where $G_{\tau-1} = \left[I - 2 \Sigma' C_{\tau-1} \Sigma \right]^{-1}$ and $|G_{\tau-1}|$ denotes the determinant of $G_{\tau-1}$. Equation (22) holds provided $G_{\tau-1}$ is positive semi-definite, and can be derived by completing the square.

Plugging equation (22) into equation (21) and combining quadratic, linear, and scalar terms yields

$$
\begin{aligned}
\log P_t^{(n)} &= \left(\begin{array}{cc} 1 & X_t' \end{array} \right) M_{\tau-1} \left(\begin{array}{c} 1 \\ X_t \end{array} \right) \\
M_{\tau-1} &\equiv -\widetilde{\Delta} - \frac{1}{2}\widetilde{\Lambda}'\widetilde{\Lambda} + \frac{1}{2}\widetilde{\Lambda}'G_{\tau-1}\widetilde{\Lambda} + \widetilde{\Phi}'C_{\tau-1}\widetilde{\Phi} + 2\widetilde{\Phi}'C_{\tau-1}\Sigma G_{\tau-1}\Sigma' C_{\tau-1}\widetilde{\Phi} \\
&\quad - \left(\widetilde{\Lambda}'G_{\tau-1}\Sigma' C_{\tau-1}\widetilde{\Phi} + \widetilde{\Phi}'C_{\tau-1}\Sigma G_{\tau-1}\widetilde{\Lambda}\right) \\
&\quad + \widetilde{B}_{\tau-1}'\left(\widetilde{\Phi} - \Sigma G_{\tau-1}\widetilde{\Lambda} + 2\Sigma G_{\tau-1}\Sigma' C_{\tau-1}\widetilde{\Phi}\right) + \widetilde{A}_{\tau-1}
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
\widetilde{\Delta} &\equiv \left[\begin{array}{cc} \delta_0 & \frac{1}{2}\delta_1' \\ \frac{1}{2}\delta_1 & -\Gamma \end{array} \right] \\
\widetilde{\Lambda} &\equiv \left[\begin{array}{cc} \lambda & \Lambda \end{array} \right] \\
\widetilde{\Phi} &\equiv \left[\begin{array}{cc} \mu & \Phi \end{array} \right] \\
\widetilde{B}_{\tau-1} &\equiv \left[\begin{array}{cc} B_{\tau-1} & \vec{0}_{n\times n} \end{array} \right] \\
\widetilde{A}_{\tau-1} &\equiv \left[\begin{array}{cc} A_{\tau-1} + \log|G_{\tau-1}| + \frac{1}{2}B_{\tau-1}'\Sigma G_{\tau-1}\Sigma' B_{\tau-1} & \vec{0}_{1\times n} \\ \vec{0}_{n\times 1} & \vec{0}_{n\times n} \end{array} \right].
\end{aligned}
$$

The remainder of the proof consists of showing that equation (23) is equivalent to equation (10). To do so, we use the fact (proven below in Lemma 1) that the matrix $C_{\tau-1}D_{\tau-1} = C_{\tau-1}\Sigma G_{\tau-1}\Sigma^{-1}$ is symmetric. For notational simplicity, in what follows we drop all the $\tau - 1$ subscripts.

Equation (23) can be written as

$$
\begin{aligned}
M = &-\widetilde{\Delta} - \frac{1}{2}\widetilde{\Lambda}'G\left(G^{-1} - I\right)\widetilde{\Lambda} + \widetilde{\Phi}'C\Sigma G\Sigma'\left(2C + J^{-1}\right)\widetilde{\Phi} \\
&- \left(\widetilde{\Lambda}'G\Sigma' C\widetilde{\Phi} + \widetilde{\Phi}'C\Sigma G\widetilde{\Lambda}\right) \\
&+ \widetilde{B}'\Sigma G\left(G^{-1}\Sigma^{-1}\widetilde{\Phi} - \widetilde{\Lambda} + 2\Sigma' C\widetilde{\Phi}\right) + \widetilde{A},
\end{aligned}
$$

where $J \equiv \Sigma G \Sigma'$ so that

$$\begin{aligned} J^{-1} &= \Sigma'^{-1} G^{-1} \Sigma^{-1} \\ &= \Sigma'^{-1} \left(I - 2\Sigma' C \Sigma \right) \Sigma^{-1} \\ &= \Sigma'^{-1} \Sigma^{-1} - 2C. \end{aligned} \tag{24}$$

Plugging in equation (24) and the fact that $G^{-1} = I - 2\Sigma' C \Sigma$ and rearranging yields

$$\begin{aligned} M &= -\widetilde{\Delta} + \widetilde{\Lambda}' G \Sigma' C \Sigma \widetilde{\Lambda} + \widetilde{\Phi}' C \Sigma G \Sigma^{-1} \widetilde{\Phi} \\ &\quad - \left(\widetilde{\Lambda}' G \Sigma' C \widetilde{\Phi} + \widetilde{\Phi}' C \Sigma G \widetilde{\Lambda} \right) \\ &\quad + \widetilde{B}' \Sigma G \left(\Sigma^{-1} \widetilde{\Phi} - \widetilde{\Lambda} - 2\Sigma' C \widetilde{\Phi} + 2\Sigma' C \widetilde{\Phi} \right) + \widetilde{A} \\ &= -\widetilde{\Delta} + \widetilde{\Lambda}' G \Sigma' C \left(\Sigma \widetilde{\Lambda} - \widetilde{\Phi} \right) + \widetilde{\Phi}' C \Sigma G \Sigma^{-1} \left(\widetilde{\Phi} - \Sigma \widetilde{\Lambda} \right) \\ &\quad + \widetilde{B}' \Sigma G \Sigma^{-1} \left(\widetilde{\Phi} - \Sigma \widetilde{\Lambda} \right) + \widetilde{A} \\ &= -\widetilde{\Delta} + \left(\widetilde{\Phi}' C \Sigma G \Sigma^{-1} - \widetilde{\Lambda}' \Sigma' \Sigma^{-1\prime} G \Sigma' C \right) \left(\widetilde{\Phi} - \Sigma \widetilde{\Lambda} \right) \\ &\quad + \widetilde{B}' \Sigma G \Sigma^{-1} \left(\widetilde{\Phi} - \Sigma \widetilde{\Lambda} \right) + \widetilde{A} \\ &= -\widetilde{\Delta} + \left(\widetilde{\Phi} - \Sigma \widetilde{\Lambda} \right)' C \Sigma G \Sigma^{-1} \left(\widetilde{\Phi} - \Sigma \widetilde{\Lambda} \right) + \widetilde{B}' \Sigma G \Sigma^{-1} \left(\widetilde{\Phi} - \Sigma \widetilde{\Lambda} \right) + \widetilde{A}, \end{aligned}$$

where the last line applies Lemma 1. Separating M into quadratic, linear, and scalar terms gives the loadings in equation (10).

Lemma 1. *The matrix $C_{\tau-1} D_{\tau-1} = C_{\tau-1} \Sigma G_{\tau-1} \Sigma^{-1}$ is symmetric for all τ.*

Proof. Using equation (23), so long as Γ is symmetric, then $C_{\tau-1}$ and $G_{\tau-1}$ are both symmetric for all τ. In what follows we drop the $\tau - 1$ subscripts. Let $H \equiv C \Sigma G \Sigma^{-1}$, so that we need to show that $H = H' = \Sigma'^{-1} G \Sigma' C$.

By definition, $G^{-1} = 1 - 2\Sigma' C \Sigma$, so that

$$\begin{aligned} \Sigma G^{-1} \Sigma^{-1} &= \Sigma \left(1 - 2\Sigma' C \Sigma \right) \Sigma^{-1} \\ &= I - 2\Sigma \Sigma' C \end{aligned} \tag{25}$$

$$\begin{aligned} \Sigma'^{-1} G^{-1} \Sigma' &= \Sigma'^{-1} \left(I - 2\Sigma' C \Sigma \right) \Sigma' \\ &= I - 2C \Sigma \Sigma'. \end{aligned} \tag{26}$$

Then we have that

$$\begin{aligned}
H &= C\Sigma G\Sigma^{-1} \\
&= \underbrace{\left(\Sigma'^{-1}G\Sigma'\right)\left(\Sigma'^{-1}G^{-1}\Sigma'\right)}_{=I} C\Sigma G\Sigma^{-1} \\
&= \left(\Sigma'^{-1}G\Sigma'\right)\underbrace{\left(I - 2C\Sigma\Sigma'\right)}_{\text{by equation (26)}} C\Sigma G\Sigma^{-1} \\
&= \left(\Sigma'^{-1}G\Sigma'\right)\left(C - 2C\Sigma\Sigma' C\right)\Sigma G\Sigma^{-1} \\
&= \left(\Sigma'^{-1}G\Sigma' C\right)\left(I - 2\Sigma\Sigma' C\right)\Sigma G\Sigma^{-1} \\
&= H' \underbrace{\left(\Sigma G^{-1}\Sigma^{-1}\right)}_{\text{by equation (25)}} \Sigma G\Sigma^{-1} \\
&= H'.
\end{aligned}$$

∎

B Standard Errors

We compute the asymptotic variance of our model estimates by treating the estimation of the risk-neutral parameters as a single non-linear least-squares problem. Stacking equation (17) over all cross-sections, the model can be written as

$$P_i = Z(z_i, \theta^*) + \eta_i \qquad\qquad i = 1, 2, ..., 496420$$

where z_i is a vector of bond-time characteristics—including time to maturity, coupon rate, and the factors X_t—and the function $Z(\cdot)$ captures both the measurement equation (17) as well as the minimization of squared residuals over the unobserved factors X_t.[25] Notice that each security-time pair has a separate entry i so that each bond occurs multiple times, once for each observed price.

We then calculate asymptotic standard errors using the standard nonlinear least-squares formula:

$$\text{asymVar}(\theta^*) = \left(\mathbb{X}'\mathbb{X}\right)^{-1} \mathbb{X}'\Sigma_M\mathbb{X} \left(\mathbb{X}'\mathbb{X}\right)^{-1} \tag{27}$$

where

$$\Sigma_M = \text{diag}\left\{\Sigma_t^M\right\}$$
$$\Sigma_t^M \equiv \frac{1}{n_t}\sum_{i=1}^{n_t} \eta_{i,t}^2$$

is the maximum-likelihood estimator of the specification-error variance, diagonalized to have the proper dimensionality, and the matrix $\mathbb{X}$ is the $N \times M$ matrix of derivatives of the measurement equation with respect to the parameters for each bond-time observation, where M is the number of parameters under consideration (11 for the 3-factor model, 5 for the 4-factor models, and 21 for the 6-factor model). We compute these derivatives analytically, by differentiating equation (10) with respect to each risk-neutral parameter individually. Notice that, as noted in Appendix C below, these "derivative loadings" must be computed for each daily coupon maturity in the data, i.e. 853 times, for each parameter.

Equation (27) corrects for the heteroskedasticity over time in $\eta_{i,t}$. Standard errors for ρ, σ_x, and the non-normalized elements of Σ are the usual OLS and VAR standard errors.

[25]The latent factors in X_t are nuisance parameters whose asymptotic variance is of little interest.

C Computing Special-Factor Loadings for Coupon Bonds

Consider a 10-year bond that trades special while it is on-the-run and first off-the-run. This bond has 3,650 days to maturity and is special for 180 days; assume, as is true in the data, that its first coupon payment is made just before it goes off special, at (say) 175 days after issuance.[26] For simplicity, assume that this bond makes only three payments: the first coupon payment at 175 days, another coupon payment at 365 days, and a principal plus coupon payment at maturity. Denote the price loadings of each coupon payment j of this bond by $A_{\tau_j}^{m(j)}$, $B_{\tau_j}^{m(j)}$, and $C_{\tau_j}^{m(j)}$, where $m(j)$ is the maturity of the jth coupon when it goes off special, while A_τ, B_τ, and C_τ denote price loadings of non-special bonds. The latter triplet of loadings follow equation (10) but with $\Gamma = \vec{0}$. If the bond is issued at time t, its price is given by:

$$P_t = ce^{A_{175}^{0}+B_{175}^{0\prime}X_t+X_t'C_{175}^{0}X_t} + ce^{A_{365}^{185}+B_{365}^{185\prime}X_t+X_t'C_{365}^{185}X_t} + (1+c)\,e^{A_{3650}^{3470}+B_{3650}^{3470\prime}X_t+X_t'C_{3650}^{3470}X_t}.$$

Notice that there is no e^{y_t} term, since this is implicit in equations (8) and (10).

The idea is that although all three sets of loadings satisfy equation (10), they each have a different dependence on the matrix Γ coming through their initial conditions, which depend on the maturity of each payment at the time the bond goes off special. No matter when the bond goes off special, the current level of the special spread needs to be taken into account, and crucially, this special spread is a dividend proportional to the value of the entire bond (all coupon payments). This is because, for the first 180 days, the principal and all coupon payments are exposed to the SC risk factor shocks.

The complication arises because each coupon has a different remaining time to maturity when the bond goes off special. Consider the first coupon payment, which occurs in 175 days. On this date, the bond is still special (and will be for an additional 5 days), so that the A^0, B^0, and C^0 loadings follow equation (10) with Γ from equation (13) and initial condition $A^0 = B^0 = C^0 = 0$. These loadings are valid for dates after t (issuance) up to the maturity of that coupon payment at $t+175$. After $t+175$ this coupon no longer appears in the pricing equation for this bond, because it has already been paid.

The second coupon payment in 365 days uses a different set of loadings, because when the bond goes off special in 180 days, that coupon still has $365-180 = 185$ days to maturity. Thus the A^{185}, B^{185}, and C^{185} loadings follow equation (10) with Γ from equation (13) but with initial condition $A_{185}^{185} = A_{185}$, $B_{185}^{185} = B_{185}$, and $C_{185}^{185} = C_{185}$, because at $t+180$ this

[26] Because we include both on-the-runs and first off-the-runs as "special" in our benchmark model, bonds are special for roughly six months. In practice the first semi-annual coupon payment of each 10-year note arrives a few days before the next note is issued and the bond becomes a second off-the-run. Thus the first coupon payment of each on-the-run bond is indeed "special" its entire life.

bond is no longer exposed to the SC risk factor. Likewise, the final coupon and principal payment follows yet another set of loadings, with Γ from equation (13) but with initial condition $A_{3470}^{3470} = A_{3470}$, $B_{3470}^{3470} = B_{3470}$, and $C_{3470}^{3470} = C_{3470}$.

In all cases, price loadings follow equation (10); the issue is which intial condition to use. In order to price this bond for the 3650 days that it is observed in the data, we must compute loadings using the recursion in equation (10) four times: once for each coupon payment, and once for non-special bonds.[27] Given these loadings, the price of the bond depends only on calendar time through the factors X_t. Thus, for example, the price 10 days after issuance would be

$$\begin{aligned} P_{t+10} &= ce^{A_{165}^{0} + B_{165}^{0\prime} X_{t+10} + X_{t+10}^{\prime} C_{165}^{0} X_{t+10}} + ce^{A_{355}^{185} + B_{355}^{185\prime} X_{t+10} + X_{t+10}^{\prime} C_{355}^{185} X_{t+10}} \\ &\quad + (1+c)\, e^{A_{3640}^{3470} + B_{3640}^{3470\prime} X_{t+10} + X_{t+10}^{\prime} C_{3640}^{3470} X_{t+10}} \end{aligned}$$

while the prices at $t+177$ (when the first coupon payment has been paid, but the bond is still special) and $t+200$ (when the bond is off special) are given by

$$\begin{aligned} P_{t+177} &= ce^{A_{188}^{185} + B_{188}^{185\prime} X_{t+177} + X_{t+177}^{\prime} C_{188}^{185} X_{t+177}} + (1+c)\, e^{A_{3473}^{3470} + B_{3473}^{3470\prime} X_{t+177} + X_{t+177}^{\prime} C_{3473}^{3470} X_{t+177}} \\ P_{t+200} &= ce^{A_{165}^{185} + B_{165}^{185\prime} X_{t+200} + X_{t+200}^{\prime} C_{165}^{185} X_{t+200}} + (1+c)\, e^{A_{3450}^{3470} + B_{3450}^{3470\prime} X_{t+200} + X_{t+177}^{\prime} C_{3450}^{3470} X_{t+200}} \\ &= ce^{A_{165} + B_{165}^{\prime} X_{t+200} + X_{t+200}^{\prime} C_{165} X_{t+200}} + (1+c)\, e^{A_{3450} + B_{3450}^{\prime} X_{t+200} + X_{t+177}^{\prime} C_{3450} X_{t+200}} \end{aligned}$$

where the last line follows from the chosen initial conditions of the "special" loadings, which now coincide with the loadings of non-special bonds.

Notice that there is nothing crucial about the assumption that the bond is no longer special at the switching date $t+180$. For example, we could extend the model to allow for a second, "off-the-run" repo factor $\widetilde{y}_t^S$ that applies to bonds that are second off-the-run or older. The only way this would change the computations would be that now the off-special loadings A_τ, B_τ, and C_τ would have a Γ different from zero, loading on that new factor, and equation (13) would have to be modified accordingly.

[27] To price the actual Treasury bonds in the data, we must compute the recursion (10) a total of 853 times: because we price bonds with time to maturity as long as 30 years, and Treasury bonds pay semi-annual coupons, in the data we have as many as sixty different maturities at off-special, per bond, for which to compute loadings. However, there are many "shared" coupon maturities, so that the calculation does not need to be redone for every single coupon of every single bond. For example, the 38 different 10-year notes in our sample have 8 maturities when they go off special (between 3,465 and 3,472 days), but there are only 5 different maturities at off-special for their penultimate coupons (between 3,284 and 3,288 days). In all 38 cases the first coupon payment is made just before the bond goes off special (i.e., becomes a second off-the-run bond).

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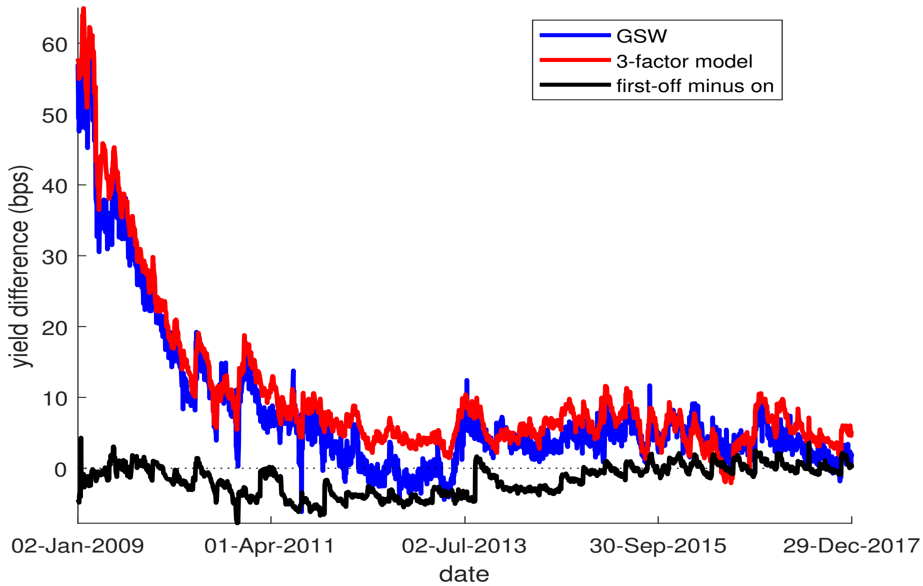


(a) On-the-Run Yield Premia, 10-year

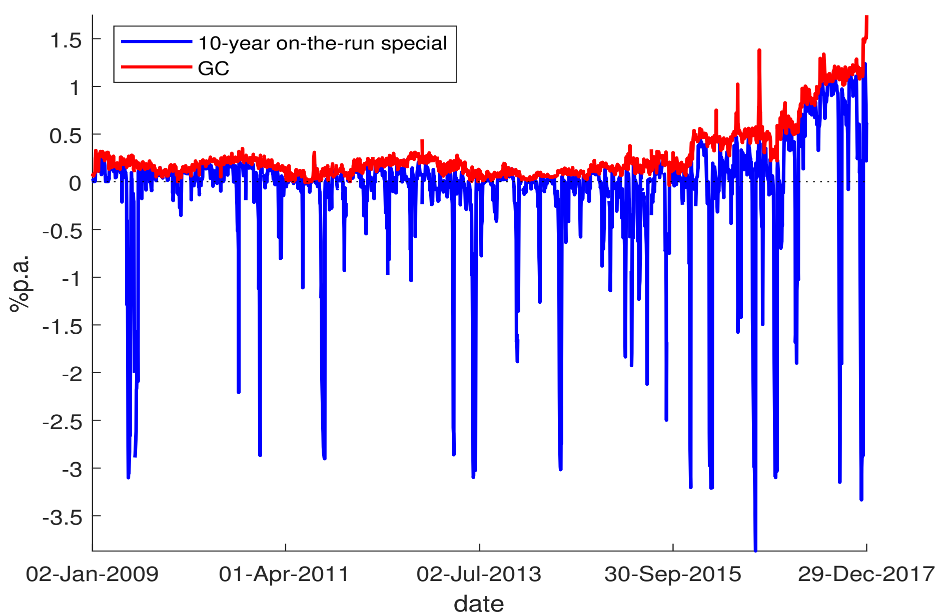
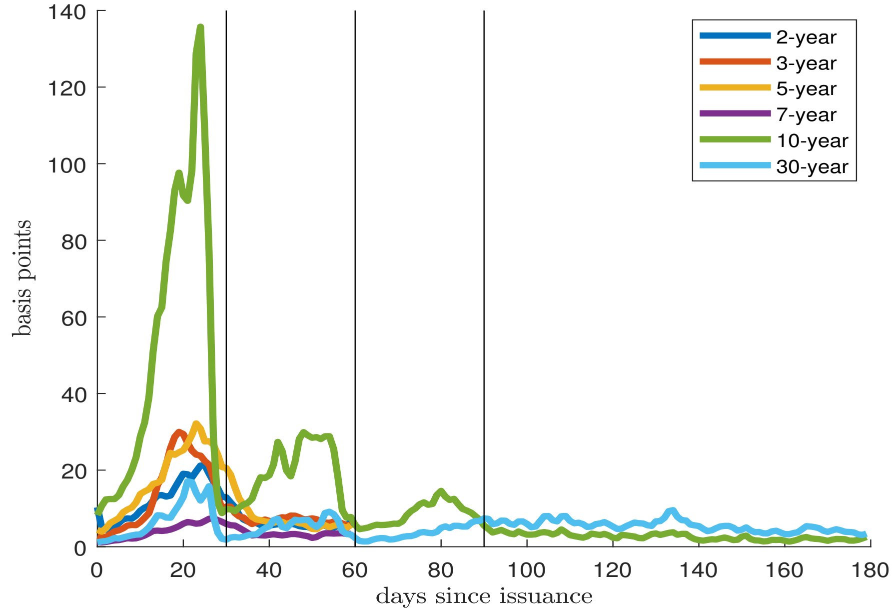


Figure 1. On-the-Run Premia and Special Rates
The top panel plots the difference between the 10-year on-the-run yield and the 10-year off-the-run yield from either the first off-the-run bond (black line), or from the yield curve estimated by GSW (blue line), or from our estimated 3-factor DTSM (red line), whose parameters are reported in Panel A of Table 4. Each line subtracts the on-the-run yield from the alternative off-the-run yield. The bottom panel plots the GC repo rate (red line) along with the SC repo rate for the 10-year on-the-run bond (blue line) over time, in annualized percent.



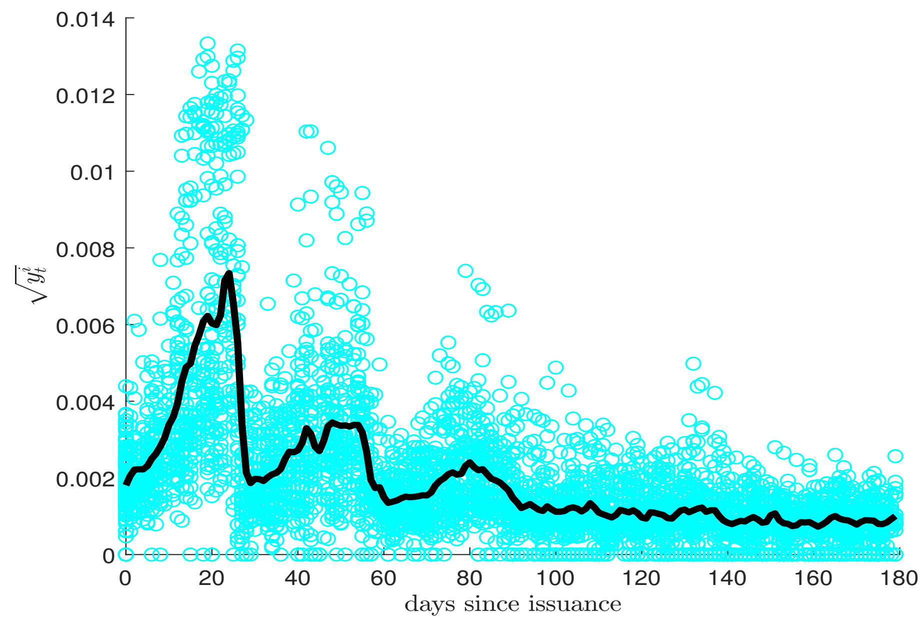


Figure 2. Special Spreads over the Auction Cycle
The top panel plots average special spreads, in annualized basis points, as a function of days since issuance for Treasury notes and bonds of all maturities. The vertical lines at 30-, 60-, and 90-days indicate the timings of the next auction for 2-, 3-, 5-, and 7-year notes, and of the first and second re-openings and the next auction for 10-year notes and 30-year bonds. The bottom panel plots the observed square root of the special spread (not annualized), $\sqrt{y_t^i}$, for the 10-year note (blue circles) against its estimated deterministic auction-cycle component, y_τ^D, as a function of time since issuance, τ (black line).

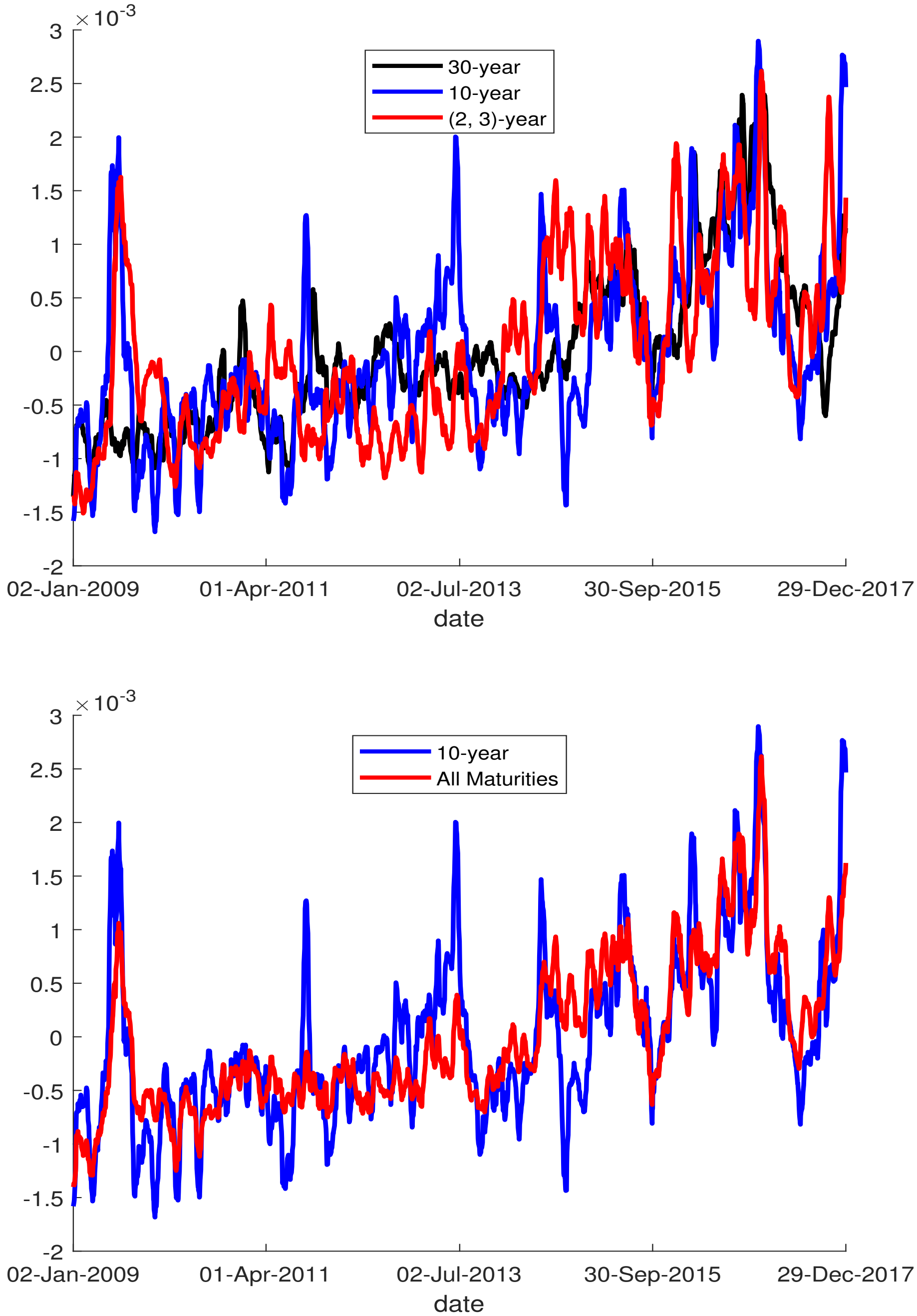


Figure 3. Both panels plot the estimated SC risk factors y_t^S, defined in equation (3), over time. To better distinguish the lines, each figure plots 15-day moving averages of y_t^S. The top panel plots the three separate y_t^S series; the first averages over 30-year bonds (black line), the second averages over 10-year notes (blue line), and the third averages over 2- and 3-year notes (red line). The bottom panel plots a "pooled" y_t^S including all maturities as a red line, with the 10-year y_t^S in blue for comparison.

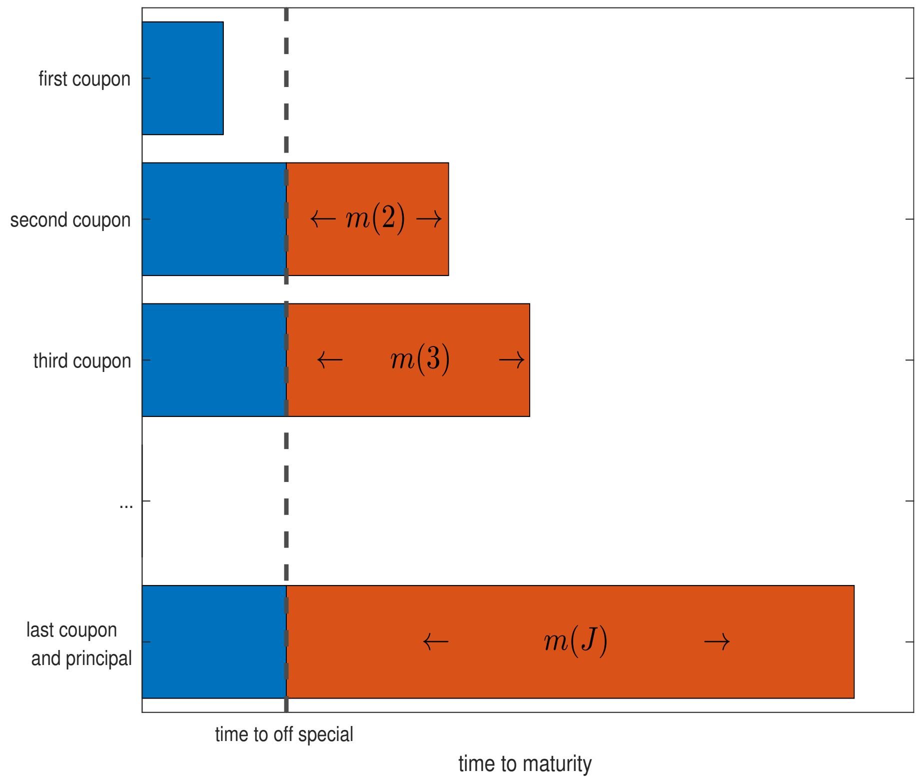
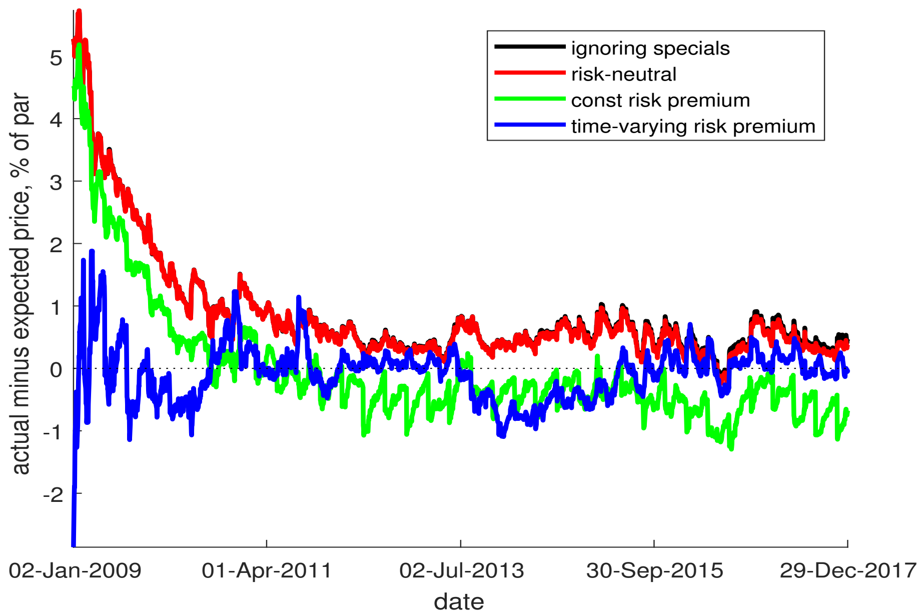


Figure 4. Discounting Special Coupons
The figure illustrates the way we discount individual coupons of bonds that are on special. The x-axis denotes time to maturity, while each bar on the y-axis is one of the bond's coupons. The dashed vertical line denotes the time at which the bond and all its coupons will cease to be "special," i.e. the date at which it becomes a second off-the-run bond. The blue portion of each bar represents the remaining time that each coupon will be exposed to the SC risk factors; the orange bars represent time when the bond is exposed only to the three latent factors.



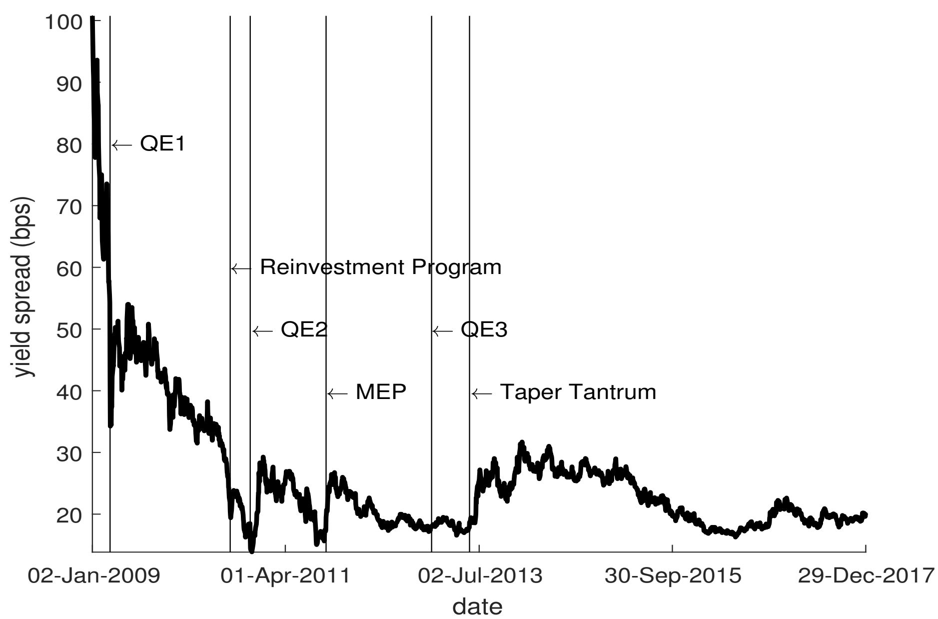
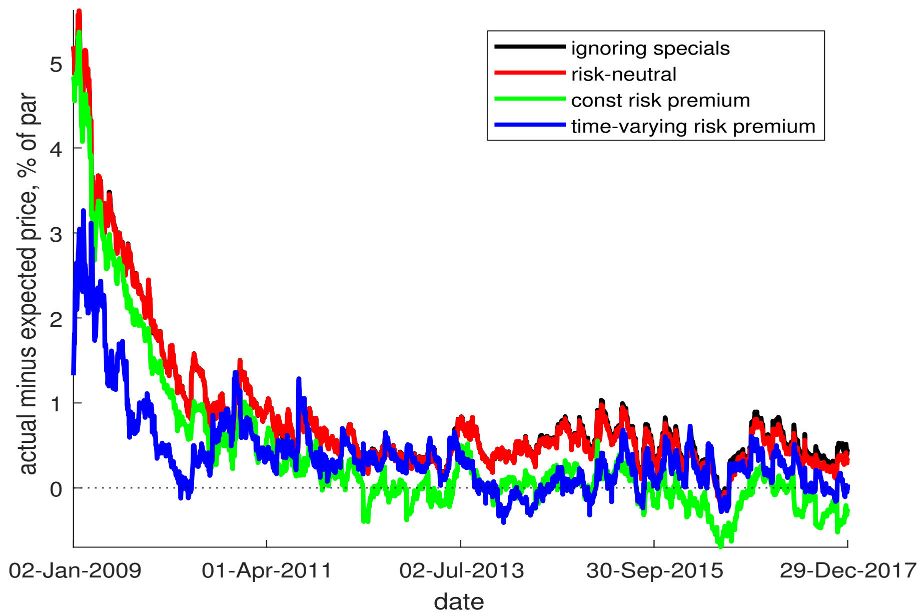


Figure 5. The top panel plots the price residuals from equation (17) on the 10-year on-the-run bond for four different specifications of the 10-year only model: a model ignoring special spreads (black), a model with risk-neutral special spreads (red); a model with a constant risk premium on the repo risk factor (green); and a model with a time-varying risk premium tied to the repo risk factor (blue). The bottom panel plots the SC repo risk premium, i.e. the estimated yield spread between two 10-year special zero-coupon bonds that are exposed to the aggregate repo risk factor for 180 days, but in one case the special spread is evaluated in a risk-neutral way and in the other case using our estimated prices of risk. The vertical lines mark events related to the Fed asset purchase programs.



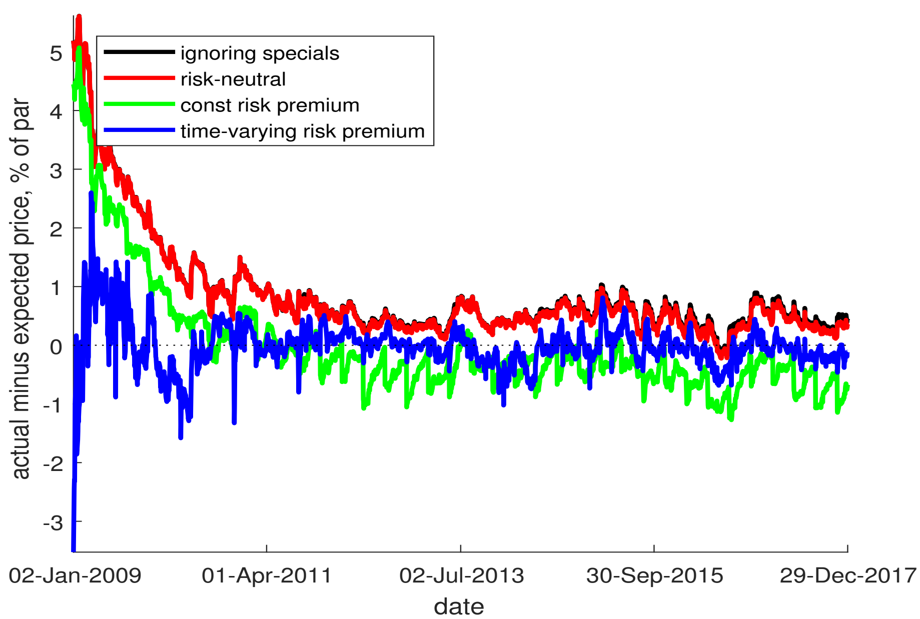
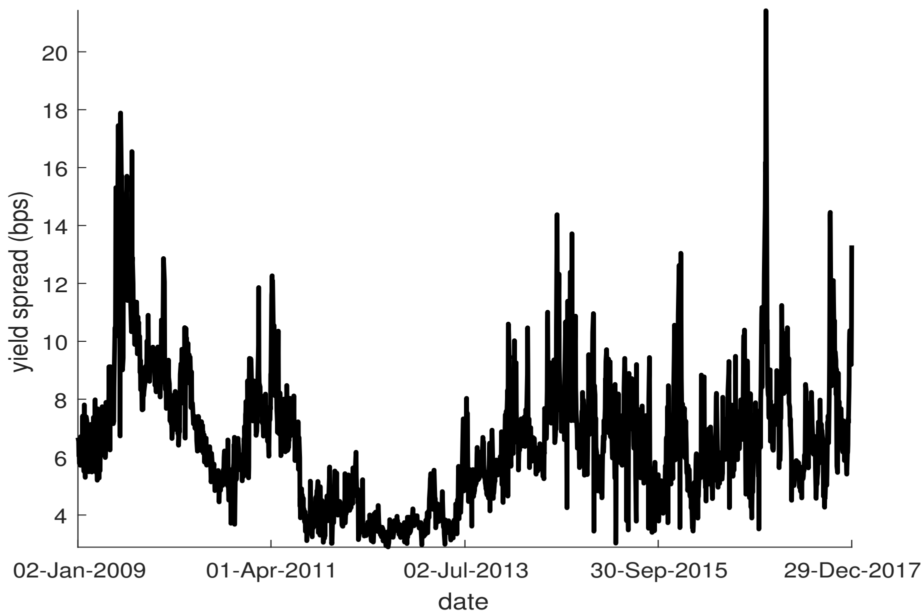


Figure 6. Both panels plot the price residuals from equation (17) on the 10-year on-the-run bond for four different specifications: a model ignoring special spreads (black), a model with risk-neutral special spreads (red); a model with a constant risk premium on the repo risk factor (green); and a model with time-varying risk premia tied to the repo risk factor (blue). The top panel uses the "pooled" model, whose parameters are reported in the last three columns of Panel B of Table 4, while the bottom panel uses the "split" model whose parameters are reported in Table 7.



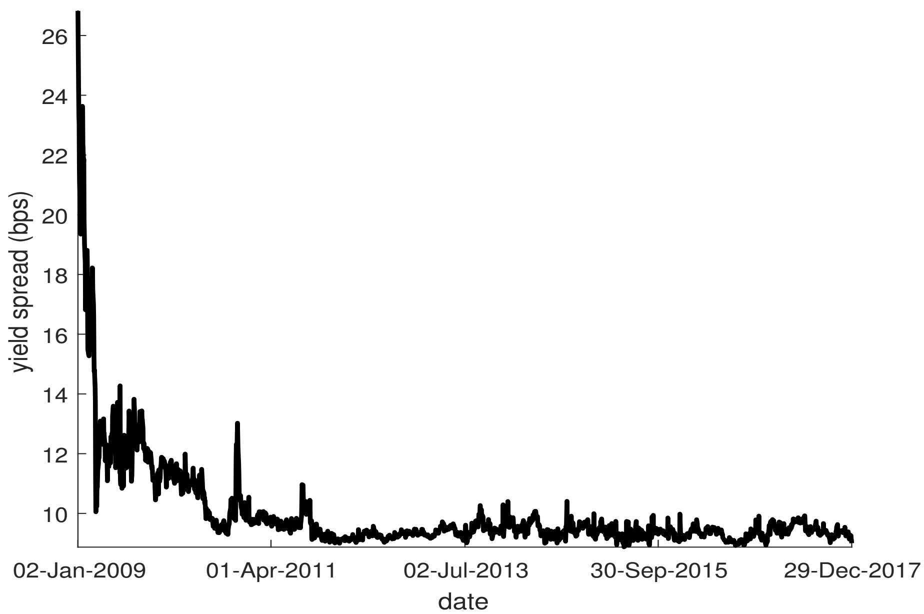


Figure 7. Both panels plot the SC repo risk premium, i.e. the estimated yield spread between two special zero-coupon bonds that are exposed to the aggregate SC risk factor, but in one case the special spread is evaluated in a risk-neutral way and in the other case using our estimated prices of risk. Both panels use the split model estimates given in Table 7. The top panel plots the 2-year SC repo risk premium, which uses the 2- and 3-year y_t^S factor plotted in red in the bottom panel of Figure 3 and assumes the bond is on special for 60 days and matures in 2 years. The bottom panel plots the 30-year SC repo risk premium, which uses the 30-year y_t^S factor plotted in black in the bottom panel of Figure 3 and assumes the bond is on special for 180 days and matures in 30 years.

Table 1. Summary Statistics
The table reports summary statistics for our sample from January 2, 2009 to December 29, 2017, by maturity at issuance in years (first column). The second column reports the average number of bonds in each cross-section with the indicated maturity at issuance. The third and fourth columns report the average specialness spread, in bps, for off- and on-the-run bonds, respectively. The last column reports the on-the-run average price residual (actual minus model-implied), in percent of par, estimated from the GSW model. Numbers in parentheses are standard deviations.

Maturity (years)	Avg # Bonds	Avg Spread (off-the-run)	Avg Spread (on-the-run)	Avg Premium (on-the-run)
All	220	4.79 (6.16)	19.7 (41.6)	0.167 (0.608)
2	11.2	5.37 (8.8)	20.5 (38.7)	0.0551 (0.361)
3	21.7	5.72 (7.6)	21.1 (36)	0.0733 (0.247)
5	46.9	3.84 (6.09)	24.5 (41.7)	-0.0223 (0.239)
7	47.8	4.85 (5.47)	6.18 (11)	-0.0631 (0.396)
10	35.2	3.44 (3.69)	35.4 (66.5)	0.885 (0.95)
30	58.6	5.86 (6.43)	8.97 (23.2)	0.0688 (0.522)

Table 2. The table reports regression results from estimating $\eta_{i,t} = \alpha_i + \beta_1 y_t^i + \beta_2 \eta_{i,t-1} + \xi_{i,t}$, where $\eta_{i,t}$ are price residuals from the GSW model, and the special spread y_t^i is defined in equation (7). Special spreads in the regression are annualized, and the price residuals are in percent of par value. The table reports t-statistics, clustered at the CUSIP level, in parentheses.

y_t^i	0.00280***	0.000340***	0.00231***	0.000382***
	(5.121)	(5.704)	(4.970)	(5.326)
$\eta_{i,t-1}$		0.902***		0.866***
		(105.7)		(69.47)
R^2	0.007	0.822	0.006	0.759
Observations	496,420	495,792	496,420	495,792
CUSIP FE	NO	NO	YES	YES
Number of CUSIP	628	628	628	628

Table 3. The table reports regression coefficients and R^2 between our measures of SC repo risk factors and proxies of demand and supply of money-like assets. The second column reports the number of observations in each regression; subsequent columns report the regression coefficient and R^2 for the (2,3)-year factor, the 10-year factor, and the 30-year factor. Each factor is an average special spread across on-the-run and first off-the-run bonds of the indicated maturity, after subtracting out the deterministic auction-cycle component plotted in Figure 2. Each row reports results for different proxies of demand/supply, which are: the 3-month GC Repo/T-Bill spread; the 1-month and 3-month OIS/T-Bill spread; the percentage of the Treasury security allocated to dealers and brokers (DB) at auction, with maturity matching that of the corresponding SC risk factor; amount outstanding (in millions) of primary-dealer (PD) reverse repos and repos collateralized by US Treasury securities excluding TIPS and reported in the FR2004 produced weekly by the Federal Reserve Bank of New York. Because of the auction frequency, the auction allotments are available at monthly frequency. t-statistics are beneath each regression coefficient in parentheses; *** denotes significance at the 1% level.

Series	# Obs	(2,3)-Year Coefficient	(2,3)-Year R^2	10-Year Coefficient	10-Year R^2	30-Year Coefficient	30-Year R^2
3-month GC Repo Spread	2236	67.03*** (39.62)	0.41	48.17*** (27.84)	0.26	41.31*** (23.18)	0.19
1-month OIS Spread	2239	40.11*** (33.93)	0.34	25.17*** (20.79)	0.16	27.94*** (23.9)	0.2
3-month OIS Spread	2239	36.08*** (37.98)	0.39	19.5*** (18.97)	0.14	26.47*** (28.05)	0.26
% of Auction Allotted to DB	107	-0.342*** (-6.167)	0.25	-0.2589*** (-5.038)	0.19	-0.389*** (-8.323)	0.40
PD Financing Reverse Repos	463	-167*** (-15.18)	0.33	-118.5*** (-10.71)	0.2	-158.7*** (-16.56)	0.37
PD Financing Repos	463	-99.52*** (-11.09)	0.21	-66.6*** (-7.62)	0.11	-120.8*** (-16.97)	0.38

Table 4. The table reports the estimated parameters of the benchmark 3- and 4-factor models discussed in the text. Panel A reports the 3-factor model parameters. The first three elements of μ^* are normalized to zero, and the top-left corner of Σ is normalized to be $I/365$ so that it is identity at an anual frequency. Panel B reports the repo-factor parameters and standard errors for the model including special spreads on 10-year on-the-run and first off-the-run notes. Panel B reports point estimates and standard errors for a risk-neutral version (columns 1 and 4), a model where the fourth element of μ^* is allowed to differ from μ (columns 2 and 5), and a model where the fourth row of Φ^* is allowed to differ from Φ (columns 3 and 6). Details on the computation of standard errors are in Appendix B.

Panel A: 3-Factor Model Parameters				
δ_0	δ_1	Φ^*_{TL}		
-6.6335e-06	0.0006079	0.99992	-0.0021584	
(3.9636e-08)	(4.5554e-07)	(5.6969e-07)	(2.1283e-06)	
	0.0010215	0.00012974	0.99964	
	(1.3918e-06)	(2.9422e-07)	(1.0411e-06)	
	0.00030939		0.00067582	1.0001
	(1.145e-07)		(1.1707e-06)	(5.4701e-08)

Panel B: 10-Year Only 4-Factor Model Parameters						
Parameter	Parameter Estimates			Standard Errors		
	risk-neutral	constant	time-varying	risk-neutral	constant	time-varying
ρ	0.795	0.795	0.795	(0.000864)	(0.000864)	(0.000864)
σ_x	0.000382	0.000382	0.000382	(7.68e-07)	(7.68e-07)	(7.68e-07)
μ^*	0.000192	0.00216	0.059		(0.00236)	(0.0026)
Φ^*_{BL}	0.00247	0.00247	0.0399			(0.00516)
	0.00823	0.00823	-0.533			(0.0244)
	0.000849	0.000849	-0.18			(0.0121)
Φ^*_{BR}	0.713	0.713	-0.98			(0.00329)
Σ_{21}	1.37e-05	1.37e-05	1.37e-05	(2.27e-05)	(2.27e-05)	(2.27e-05)
	1.64e-05	1.63e-05	1.63e-05	(6.5e-05)	(6.5e-05)	(6.5e-05)
	-1.05e-07	-1.07e-07	-1.07e-07	(2.44e-05)	(2.44e-05)	(2.44e-05)
Σ_{22}	0.000666	0.000666	0.000666	(6.26e-07)	(6.26e-07)	(6.26e-07)

Table 5. The table reports correlation coefficients and regression R^2 between the 10-year SC risk premium implied by three models and other Treasury price anomalies. The GSW on-the-run spread is defined as the difference between the off-the-run 10-year yield implied by the yield curve estimated by GSW (available on the Federal Reserve's website) and the on-the-run 10-year yield. The Off Note-Bond Spread is the average yield difference between notes and bonds of duration between 7 and 9 years. The TIPS-Treasury Puzzle is 10-year inflation swap rate minus the 10-year TIPS break-even rate. The TIPS liquidity premium is the difference between 10-year TIPS yield and the model-implied 10-year real risk-free rate as estimated by D'Amico, Kim and Wei (2018). The Nominal Fitting Errors are the average absolute pricing residual from equation (17), applied to all bonds, using the parameters in Panel A of Table 4.

Anomaly	10-Year Only		10-Year (pooled)		10-Year (split)	
	Corr	R^2	Corr	R^2	Corr	R^2
GSW 10-Year On-the-Run	0.91	0.83	0.9	0.81	0.75	0.56
Off Note-Bond Spread	0.79	0.62	0.73	0.53	0.68	0.46
TIPS-Treasury Puzzle	0.82	0.68	0.78	0.61	0.71	0.5
TIPS Liq Premium	0.76	0.58	0.78	0.62	0.63	0.4
Nominal Fitting Errors	0.6	0.36	0.52	0.27	0.6	0.36

Table 6. The table reports the estimated repo-factor parameters and standard errors of the "pooled" 4-factor model discussed in the text, in which 4th factor incorporates special spreads of on-the-run and first off-the-run maturities of 2, 3, 10, and 30 years. Panel A of Table 4 reports the values of δ_0, δ_1, and Φ^*_{TL} for this model. The table reports point estimates and standard errors for a risk-neutral version (columns 1 and 4), a model where the fourth element of μ^* is allowed to differ from μ (columns 2 and 5), and a model where the fourth row of Φ^* is allowed to differ from Φ (columns 3 and 6). Details on the computation of standard errors are in Appendix B.

Parameter	Parameter Estimates			Standard Errors		
	risk-neutral	constant	time-varying	risk-neutral	constant	time-varying
ρ	0.781	0.781	0.781	(0.000888)	(0.000888)	(0.000888)
σ_x	0.000379	0.000379	0.000379	(7.61e-07)	(7.61e-07)	(7.61e-07)
μ^*	0.000192	0.0019	0.0368		(0.00133)	(0.00254)
Φ^*_{BL}	0.00552	0.00552	0.0672			(0.00357)
	0.0118	0.0118	-0.29			(0.0225)
	0.00209	0.00209	-0.0906			(0.011)
Φ^*_{BR}	0.625	0.625	-0.993			(0.00117)
Σ_{21}	1.67e-06	1.67e-06	1.67e-06	(1.66e-05)	(1.66e-05)	(1.66e-05)
	-1.61e-06	-1.63e-06	-1.63e-06	(4.68e-05)	(4.68e-05)	(4.68e-05)
	-7.38e-06	-7.38e-06	-7.38e-06	(1.72e-05)	(1.72e-05)	(1.72e-05)
Σ_{22}	0.000488	0.000488	0.000488	(3.36e-07)	(3.36e-07)	(3.36e-07)

Table 7. The table reports estimated repo-factor parameters and standard errors for the 6-factor model, in which y_t^S is a 3×1 vector of average special spreads obtained by averaging over 30-year special bonds, 10-year special notes, and 2- and 3-year special notes, separately. Panel A of Table 4 reports the values of δ_0, δ_1, and Φ^*_{TL} for this model. The table reports point estimates and standard errors for a risk-neutral version (columns 1 and 4), a model in which the 4th, 5th, and 6th elements of μ^* are allowed to differ from μ (columns 2 and 5), and a model in which the4th, 5th, and 6th rows of Φ^* are allowed to differ from Φ (columns 3 and 6). Details on the computation of standard errors are in Appendix B.

Parameter	Parameter Estimates			Standard Errors		
	risk-neutral	constant	time-varying	risk-neutral	constant	time-varying
ρ	0.779	0.779	0.779	(0.000891)	(0.000891)	(0.000891)
σ_x	0.000375	0.000375	0.000375	(7.53e-07)	(7.53e-07)	(7.53e-07)
μ^*	0.000457	-0.000724	0.113		(0.204)	(0.00956)
	0.000336	0.00134	0.129		(14.6)	(0.0132)
	0.0002	0.000517	0.0107		(6.58)	(0.00142)
Φ^*_{BL}	0.00361	0.00361	0.263			(0.0328)
	0.00443	0.00443	0.293			(0.112)
	0.00655	0.00655	0.0241			(0.0298)
	0.0132	0.0132	-1.26			(0.351)
	0.0134	0.0134	-1.43			(0.305)
	0.0128	0.0128	-0.118			(0.489)
	0.000758	0.000758	-0.351			(0.0395)
	0.00143	0.00143	-0.404			(0.154)
	0.00248	0.00248	-0.0332			(0.0423)
Φ^*_{BR}	0.663	0.663	3.55			(0.45)
	-0.153	-0.153	2.94			(0.385)
	-0.14	-0.14	0.249			(0.548)
	-0.0181	-0.0181	-2.66			(0.00385)
	0.792	0.792	-1.98			(0.0162)
	-0.0145	-0.0145	-0.243			(0.00461)
	-0.0527	-0.0527	-3.06			(0.048)
	-0.0759	-0.0759	-3.4			(0.0395)
	0.755	0.755	0.727			(0.054)
Σ_{21}	1.84e-05	1.84e-05	1.84e-05	(0.000692)	(0.000692)	(0.000692)
	1.02e-05	1.02e-05	1.02e-05	(0.00163)	(0.00163)	(0.00163)
	-9.89e-06	-9.89e-06	-9.89e-06	(0.00092)	(0.00092)	(0.00092)
	-3.44e-05	-3.44e-05	-3.44e-05	(0.0211)	(0.0211)	(0.0211)
	1.83e-06	1.83e-06	1.83e-06	(0.0147)	(0.0147)	(0.0147)
	-1.14e-06	-1.14e-06	-1.14e-06	(0.016)	(0.016)	(0.016)
	-1.23e-05	-1.23e-05	-1.23e-05	(1.85e-05)	(1.85e-05)	(1.85e-05)
	-3.25e-06	-3.25e-06	-3.25e-06	(5.23e-05)	(5.23e-05)	(5.23e-05)
	-7.94e-06	-7.94e-06	-7.94e-06	(1.92e-05)	(1.92e-05)	(1.92e-05)
Σ_{22}	0.000504	0.000504	0.000504	(3.59e-07)	(3.59e-07)	(3.59e-07)
	0.000476	0.000476	0.000476	(4.01e-07)	(4.01e-07)	(4.01e-07)
	0.000657	0.000657	0.000657	(6.1e-07)	(6.1e-07)	(6.1e-07)
	0.000394	0.000394	0.000394	(3.15e-07)	(3.15e-07)	(3.15e-07)
	0.000451	0.000451	0.000451	(4.11e-07)	(4.11e-07)	(4.11e-07)
	0.000544	0.000544	0.000544	(4.19e-07)	(4.19e-07)	(4.19e-07)

Table 8. The table reports the share of the variation of on-the-run price residuals explained by the three estimated models, the first ("10-Year") using a factor that incorporates special spreads from 10-year on-the-run and first off-the-run bonds; the second ("Pooled") using special spreads from on-the-run and first off-the-run bonds of four different maturities in a single average y_t^S factor, and the third ("Split") using three separate y_t^S factors: one for 30-year bonds, one for 10-year notes, and one for 2- and 3-year notes. The first four columns report the sum of squared residuals for four versions of each model: one ignoring repos completely (i.e. a 3-factor model), one in which special spreads are priced risk-neutral, one with a constant risk premium, and one with time-varying risk premia. The last three columns convert the sum of squares into R^2 using equation (19).

Maturity	Model	Sum of Squared Residuals				R^2		
		no repo	risk -neutral	constant	time -varying	risk -neutral	constant	time -varying
10	10-year	0.379	0.37	0.238	0.0423	0.024	0.374	0.889
	pooled	0.37	0.365	0.253	0.0996	0.0134	0.317	0.731
	split	0.37	0.362	0.228	0.037	0.0205	0.383	0.9
30	pooled	0.107	0.107	0.14	0.139	-0.000303	-0.307	-0.296
	split	0.107	0.108	0.101	0.0563	-0.00883	0.052	0.474
2	pooled	0.03	0.0285	0.0273	0.0445	0.0525	0.0897	-0.48
	split	0.03	0.0281	0.0254	0.0263	0.0645	0.153	0.124
3	pooled	0.015	0.0142	0.0138	0.0344	0.0536	0.0808	-1.29
	split	0.015	0.014	0.0125	0.0135	0.067	0.166	0.103