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## Modelling yields at the lower bound through regime shifts

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**Abstract**

We propose a regime-switching approach to deal with the lower bound on nominal interest rates in dynamic term structure modelling. In the "lower bound regime", the short term rate is expected to remain constant at levels close to the effective lower bound; in the "normal regime", the short rate interacts with other economic variables in a standard way. State-dependent regime switching probabilities ensure that the likelihood of being in the lower bound regime increases as short rates fall closer to zero. A key advantage of this approach is to capture the gradualism of the monetary policy normalization process following a lower bound episode. The possibility to return to the lower bound regime continues exerting an influence in the early phases of normalization, pulling expected future rates downwards. We apply our model to U.S. data and show that it captures key properties of yields at the lower bound. In spite of its heavier parameterization, the regime-switching model displays a competitive out-of-sample forecasting performance. It can also be used to gauge the risk of a return to the lower bound regime in the future. As of mid-2018, it provides a more benign assessment than alternative measures.

Keywords: zero lower bound; term premia; term structure of interest rates; monetary policy rate expectations; regime switches.

JEL Codes: E31, E40, E44, E52, E58, E62, E63.

**Non-technical summary**

After the global financial crisis of 2008-09, short-term nominal interest rates have reached levels close to their lower bound (henceforth, LB) in many advanced countries. This experience has taught us two lessons. The first one is that recessions accompanied by binding LB experiences are much more prolonged than normal recessions. The second lesson is that once the economy exits from the LB period, the pace of monetary policy normalisation is very slow.

The aforementioned two lessons suggest that a regime-switching approach may be well-suited to model interest rates at, and away from, the lower bound. This is the approach adopted in this paper. Yield dynamics are explicitly described as a function of two possible policy-induced stochastic regimes: a normal regime and a LB regime. In the normal regime, interest rates are an affine function of a Gaussian state vector, which follows an unrestricted VAR process. In the LB regime, interest rates are again an affine function of a Gaussian state vector, but the short-term rate is restricted to be a white noise process around a constant, low level. Stochastic regime changes are assumed to occur with probabilities that are state-dependent, i.e. such that a switch to the LB regime is increasingly more likely, the closer short term rates fall towards zero.

Consistently with the aforementioned two lessons, the model explicitly allows for different macro and yield dynamics in the LB period compared to normal situations. It can also account for very gradual recoveries from the LB: as soon as the system switches to the LB regime, the short-term rate is expected to remain at low levels as long as the economy stays in this regime.

We apply a version of the model to monthly U.S. data and show that it captures well the evolution of short and long-term interest rates over the past 20 years. The estimated regimes are in line with intuition. They suggest that the U.S. economy was in the LB regime with a high probability from October 2008 until the end of 2015 and started moving towards the normal regime thereafter. In spite of its richer parameterization, the regime-switching model is also competitive with more parsimonious alternatives in out-of-sample forecasts.

The nonlinearity of our framework has implications for the decomposition of yields into expectations and risk premia. When the short rate is at the LB and this regime is persistent, future short rates are expected to persistently remain at the lower bound. As a result, even if observed 10-year yields are low, estimated risk premia remain positive, which is arguably a plausible feature. By contrast, models with a single regime tend to generate mean reversion in short rates and, as a result, they suggest that risk premia are negative when observed long-term yields are low.

We employ our model to gauge the risk of a return to the LB regime in the U.S.. We specifically measure the probability to switch back to the lower bound over the next three years. At the end of our estimation sample, in mid-2018, this probability is around 5%, which represents a more benign assessment than suggested from alternative measures.

# 1 Introduction

After the global financial crisis of 2008-09, short-term nominal interest rates reached levels close to their lower bound (henceforth, LB) in many advanced economies – see Figure 1 for the case of the United States.[1] Over time, two new lessons have been learned from these LB experiences.

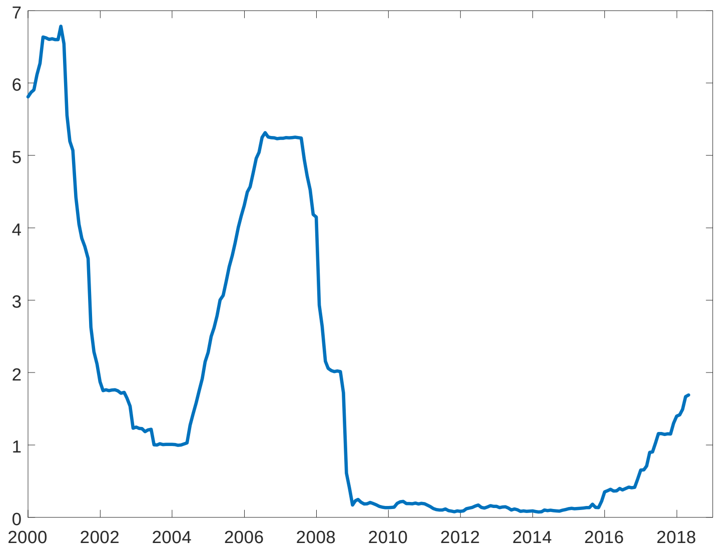
The first lesson is that recessions accompanied by binding LB episodes are much more prolonged than normal recessions. Interest rates (and inflation rates) are more persistently low after a recession in which they reach the LB, than after normal recessions. One-month money market rates remained below the 25 basis points level for 7 years in the United States before the Federal Reserve (Fed) began raising rates again. This evidence appears consistent with theoretical arguments suggesting that periods when policy rates reach their LB result in deeper recessions, and that such period are characterized by different economic dynamics compared to normal situations – i.e. recessions where policy interest rates can be reduced without constraints (e.g. Eggertsson and Woodford, 2003). As a result, a yield curve model should ideally allow for the law of motion of the state variables to change when interest rates are close to the LB.

The second lesson is that once the economy exits from the LB period, the pace of monetary policy normalization is very slow. After seven years at the LB range of $0 - 0.25\%$, the FOMC raised the target range for the federal funds rate by 25 basis points in December 2015. It took the Committee one more year to raise it by another 25 basis points, and two years after the initial rate hike the target range was only 125 basis points above the LB. By contrast, historical US tightening cycles in the two decades before the financial crisis had on average seen the target rate rise by 256 basis points within one year of the initial rate hike. A well-specified term structure model should be able to capture such differences in the speed of policy normalization in normal times and following an LB episode, and it should also be able to account for the ensuing differences in longer-term yields.

The aforementioned two lessons suggest that a regime-switching approach may be well-suited to model yields at, and away from, the lower bound. This is the approach we adopt

[1] The exact level of the LB is in principle unknown. In theory, a negative interest rate level on short-term bonds would give rise to an arbitrage opportunity – going short in the bond yielding a negative interest rate and long in cash. The zero level should therefore represent a lower bound for interest rates. In practice, storage costs have discouraged banks from transforming central bank reserves into banknotes. In many constituencies, including the euro area, Sweden and Switzerland, key policy interest rates have became negative and have brought down with them short-term market rates.

Figure 1: US one-month interest rate




in this paper. Yield dynamics are explicitly described as a function of two possible policy-induced stochastic regimes: a normal regime and a LB regime. In the normal regime, interest rates are an affine function of a Gaussian state vector, which follows an unrestricted VAR process. In the LB regime, interest rates are again an affine function of a Gaussian state vector, but the short-term rate is assumed to be a white noise process around the constant, lower bound level. We finally assume that stochastic regime changes occur with probabilities that are state-dependent. We model the state-dependence to imply that a switch to the LB regime, conditional on being in the normal regime, is increasingly more likely as short-term rates move lower and lower.

Consistent with the first new lesson, the model explicitly allows for different macro and yield dynamics in the LB period compared to normal situations. The model can also account for the very high persistence of LB episodes. As soon as the system is in the LB regime, the short-term policy rate will be expected to remain at low levels for as long as the economy stays in this regime. If this regime is persistent, longer-term yields will also remain low, reflecting such expectations. Moreover, by letting the probability that the economy switches from the normal regime to the LB regime depend on the level of interest rates, the model can capture the slow normalization of policy rates following an LB episode, in line with the second lesson. Specifically, with rates still low after exiting the LB regime, the model assigns a high probability to a possible switch back to the LB regime. With this probability remaining high, interest rates rise slowly and yields on longer-dated bonds remain low to reflect it. As a result, the model is able to account for very slow recoveries from the LB, including "secular stagnation" scenarios of extremely persistent low growth, low inflation, and low nominal interest rates (Summers, 2013).

Of course, the added flexibility of the regime-switching model comes at a cost. Compared to an affine specification, it leads to an increase in the size of the parameter vector to be estimated. This can be problematic in standard applications to data from recent decades, where only one LB episode is observed. We mitigate this problem in two ways. First, we rely on a model in which the state vector is made up solely of observable variables. Second, we specify the functional dependence of regime-switching probabilities on the state vector on economically motivated, a priori grounds, rather than estimating its features from the data.

To illustrate our approach, we apply a version of the model on monthly U.S. data where the state vector consists of the term spread, a measure of curvature, and the short rate. Moreover, we allow two macro variables, inflation and industrial production growth, to influence expectations about future short-term interest rates as well as term premia. We show that this simple specification fits the data well. Across six nominal bond maturities, the estimated standard deviation of the measurement error is on average 8 basis points. We also show that, in forecasting, the model effectively rules out the possibility of rates deeply below the lower bound.

Our estimated regimes are in line with intuition. They suggest that the U.S. economy was in the LB regime with a high probability from October 2008 until the end of 2015, and that it started moving towards the normal regime thereafter. At the end of our estimation sample, in April 2018, the probability of being in the normal regime is estimated to be essentially 100%. Nonetheless, forecasts beyond April 2018 indicate that the small probability to switch back to the LB regime continues exerting an influence on model dynamics, pulling expected future yields downwards.

The nonlinearity of our framework has implications for the decomposition of yields into expectations and risk premia. When the short rate is at the lower bound but yields remain in positive territory, expectations of low future short rates can be the result of the model's nonlinearity and need not be associated with deeply negative term premia, as is often the case in affine models. Our estimated risk premia on 10-year yields do in fact remain positive for most of the LB period and its aftermath. This is arguably a plausible feature. Nonlinear macro models, capable of fitting relevant conditional covariances when the lower bound constraint binds, suggest that nominal term premia at long maturities should be smaller, but only under peculiar specifications would they turn negative (Nakata and Tanaka, 2016; Gourio and Ngo, 2016).

Regime-switching probabilities also have an effect on the interpretation of risk premia. At any point in time, risk premia do not only compensate investors for the risk of unexpected changes in the state of the economy along a linear path, but also for regime-switching risk. Regime-shift risk premia are estimated to be near-zero before the Great Recession, but at times during the lower bound period and its aftermath they rise to up to 25 basis points in absolute monthly excess return terms. Specifically, regime-shift risk premia tend to be elevated at times when the probability to transition from the current regime to the other is high.

To assess the performance of our model, we compare our results to those of a shadow rate model, which is a popular alternative to deal with the LB in term-structure modelling. Two key differences between the two approaches emerge when interest rates leave the LB ("lift-off"). First, above the LB, the shadow rate model becomes a Gaussian affine model and it forecasts that yields return to their long-run mean at the same speed as in normal times. In contrast, the regime-switching model can remain consistent with a much slower normalization of interest rates, if the probability to return to the LB is non-negligible after lift-off. Second, the shadow rate model imposes identical steady state levels of the underlying term structure factors in LB episodes and in normal times. This means that the implied long-run interest rate level will be very sensitive to the sample used to estimate the model parameters. In particular, if the sample is relatively short and includes a prolonged LB spell, the model's steady state interest rate level will be implausibly low. Instead, if the researcher estimates the shadow rate model on pre-LB data only, then the model will imply a very quick return to high pre-LB average levels once lift-off takes place, which is clearly inconsistent with the U.S. experience in recent years. By contrast, our regime-switching approach allows for different long-run mean levels in LB and in normal times. This, combined with the aforementioned non-negligible probability to switch back to the LB following a lower bound episode, makes the regime-switching model able to capture the gradualism of monetary policy normalization towards a reasonable long-run level.

These key differences are important to understand our forecasting results. If the shadow rate model is estimated on pre-crisis data only, its forecasting performance is not very good. After lift-off, all interest rates are expected to quickly return to pre-crisis unconditional means, which are implausibly high after a prolonged period at the LB. If instead the shadow rate model is estimated on the full sample, which includes a long stretch of near-zero interest rates, then its near-term forecasting performance improves as the estimated unconditional mean of the short rate drops. But its forecasting performance over longer horizons suffers from the fact that this estimated long-run mean tends to be unrealistically low. The advantage of the regime-switching model is that it can accommodate both a relatively high mean of interest rates in the normal, pre-crisis regime, and a lower future mean of interest rates if the probability to return to the LB regime remains non-negligible. We show that, in spite of its richer parameterization, our regime-switching model is also competitive with the shadow rate model in out-of-sample forecasts.

Our paper is related to the applied literature studying the term structure of interest rates at the LB. This literature has developed over the past decade. The shadow rate model has proven to be the most popular empirical approach (see Bomfim, 2003, Ueno et al., 2006, Ichiue and Ueno, 2007, Kim and Singleton, 2012, and Christensen and Rudebusch, 2016, for estimates using Japanese yield data; Krippner, 2013, Ichiue and Ueno, 2013, Priebsch, 2013, and Wu and Xia, 2016, for applications to U.S. data). Compared to standard single-regime affine models, it has two advantages (see also the discussion in Christensen and Rudebusch, 2016): it rules out negative interest rates (or rates below some other specified lower bound); and it can account for the observed reduction in the volatility of shorter-term yields when the policy rate is at the LB (see Swanson and Williams, 2014). The shadow rate model is also relatively parsimonious; away from the LB, it boils down to the standard affine formulation, which has been studied in an extensive literature.

A paper closer to our approach is Koeda (2013), which adopts a regime-switching set-up in an application to Japan's experience with the lower bound. Nevertheless, some important differences characterize our approaches. First, Koeda (2013) assumes that the state dynamics under the risk-neutral measure are identical in the normal and the LB regimes, while we relax this assumption. Second, Koeda (2013) assumes that the regimes are observable not only to market participants, but also to the econometrician – an assumption we do not impose.

Our paper is organized as follows. Section 2 presents a simple structural model to motivate our analysis. We then move on to specify our term structure model with normal and lower-bound regimes in Section 3. We derive approximate bond-pricing equations and then characterize the model likelihood. An application of the model to U.S. data is described in Section 4. This section describes the fitting performance of the model and derives its implications for risk premia and for forecasting. The results of a comparison to the shadow rate model are also presented here. Section 5 concludes.

## 2 Motivation

In the applied literature two main approaches have been used to deal with the lower bound constraint: the shadow rate model and a quadratic term structure model with parameter restrictions. Kim and Singleton (2012) provide an early assessment of these approaches based on the early lower bound experience in Japan and show that they can both overcome the limitations of affine models. More recently, Andreasen and Meldrum (2018) revisit the issue and conclude that the two approaches have comparable benefits, but also shortcomings. Specifically, the "time-series dynamics of yields during the ZLB period change, and simply modifying the functional form of the short rate does not fully capture this change."

The intuition explaining the change in the time-series dynamics of yields at the LB can be obtained from macro-economic models. For illustrative purposes, consider a simplified version of the model used by Svensson (1997)

$$\pi_t = \beta x_t + \pi_{t-1}, \tag{1}$$

$$x_t = -\frac{1}{\gamma}\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t\right), \tag{2}$$

$$r_{t+1} = \rho r_t + \sigma \varepsilon_{t+1}, \tag{3}$$

where $\pi_t$ is inflation, $x_t$ is a notion of output gap, $i_t$ is the short-term nominal interest rate and $r_t$ is an exogenous variable (which could be interpreted as the natural interest rate of the model). The first equation is a type of Phillips curve and the second equation shows that the output gap will contract when the real interest rate $i_t - \mathrm{E}_t \pi_{t+1}$ is above the level $r_t$ and it will expand when the real interest rate is below $r_t$. The parameters $\beta$, $\gamma$ and $\rho$ are such that $0 < \beta < 1$, $\gamma > 0$ and $0 < \rho < 1$. The model is then closed by a Taylor-type rule which reacts to inflation, subject to the zero bound constraint $i \geq 0$, or

$$i_t = \max\left(\alpha\pi_t, 0\right) \tag{4}$$

for $\alpha > 1$ (the so-called Taylor principle).

Following Guerrieri and Iacoviello (2015) and restricting attention to the case when the LB is only expected to bind for at most one period, Appendix A1 shows that the solution of this model can be written as

$$i_t = B_j \begin{bmatrix} \pi_{t-1} \\ r_t \end{bmatrix},$$

and

$$\begin{bmatrix} \pi_t \\ r_{t+1} \end{bmatrix} = M_j \begin{bmatrix} \pi_{t-1} \\ r_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} \varepsilon_{t+1},$$

where $B_j$ and $M_j$ assume different values depending on whether the ZLB is binding ($j = L$) or not ($j = N$). Specifically, when $i_t \geq 0$ the solution is such that

$$B_N = \begin{bmatrix} \frac{1}{2}\frac{\alpha}{\beta}\left(\gamma + \alpha\beta - \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}\right) & \frac{\alpha\beta}{\frac{1}{2}(\gamma+\alpha\beta)-\beta\rho-\frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}} \end{bmatrix},$$

$$M_N \equiv \begin{bmatrix} \frac{1}{2}\frac{1}{\beta}\left(\gamma + \alpha\beta - \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}\right) & \frac{\beta}{\frac{1}{2}(\gamma+\alpha\beta)-\beta\rho-\frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}} \\ 0 & \rho \end{bmatrix},$$

while if the zero bound is binding for only one period

$$B_L = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$M_L \equiv \begin{bmatrix} \frac{2\gamma}{\gamma-\alpha\beta+\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}} & \frac{\beta}{\frac{1}{2}(\gamma+\alpha\beta)-\beta\rho-\frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}}\frac{\gamma+\alpha\beta-\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}}{\gamma-\alpha\beta+\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}} \\ 0 & \rho \end{bmatrix}.$$

This simple model is obviously highly stylized, but it is sufficient to highlight two key properties of a world where the nominal interest rate can occasionally hit the effective lower bound constraint.

The first property is obvious: once the interest rate hits its lower bound (zero in this simple model) the standard relationship between interest rates and the state variables no longer holds. In terms of this stylized model, $B_N$ turns into $B_L$, which is a row of zeros. This is the property imposed by shadow rate models, i.e. models which explicitly replace the standard affine relationship between the interest rate and the state vector ($i_t = B_N [\pi_{t-1}, r_t]'$ in our simple model) with $i_t = \max\left(B_N [\pi_{t-1}, r_t]', 0\right)$.

The second property is also intuitively clear, but it is not typically taken into account in term structure applications. Not only does the ZLB constraint prevent nominal rates from becoming negative, but it also changes the law of motion of the endogenous state variables. Since the nominal interest rate is not allowed to fall as much as it should according to the Taylor rule, monetary policy is unable to stimulate output and inflation. As a result, the evolution of the economy in reaction to shocks is in general different at the zero bound: the law of motion of the state vector changes from $M_N$ to $M_L$. The law of motion of the state vector would only remain unchanged in the special case where all state variables are exogenous, which is not empirically plausible.

The illustrative example above suggests that, based on economic intuition, one would want to allow for a model of yields at the effective lower bound such that not only are yields prevented from falling deeply below the bound, but the dynamics of the yield factors are also allowed to change depending on whether the lower bound is binding or not. This is the main motivation for the approach we adopt in this paper.

A more empirical motivation for this assumption can also be directly obtained from recent U.S. data. Following up on the conclusion in Andreasen and Meldrum (2018), Table 1 reports autocorrelation coefficients for the short rate and yields before and after the LB period. Since the fed funds rate hit the $0-0.25\%$ range at the end of 2008, the cut-off point is set to December 2008. The sample corresponds to the one we will use in our empirical application of the model: January 1987 to April 2018. For each of the two periods, the table reports the sample correlation coefficient plus lower and upper bounds for a 95% confidence interval for each coefficient. The table shows that all yields, and especially those with maturity up to one year, were extremely persistent in the pre-LB period. The autocorrelation coefficients are above 0.99 for short-term yields and above 0.98 for yields up to 10-year. Once the LB sets in, yield persistence falls markedly, in a number of cases below 0.9. Only long-term yields remain more persistent, but nevertheless their serial correlation falls in a statistically significant manner to levels around 0.94. The fall is less striking, but present, also for the yield curve slope defined as the difference between the 10-year yield and the 1-month rate. Its autocorrelation coefficient falls from 0.97 to 0.94.

To summarize, theoretical considerations and a descriptive analysis of the data suggest that a model of yields when short-term rates are close to their lower bound should incorporate two features. The first one, which is widely recognized, is to account for the fact that further reductions in short-term rates are low-probability events. The second feature is to allow the law of motion of the factors driving yields to change. We propose a regime-switching model which incorporates these two features.

## 3 A regime-switching model of the lower bound

Our regime-switching model captures the nonlinearity of economic dynamics at the lower bound in a flexible fashion. Compared with a Gaussian model, it can explicitly allow for changes in the law of motion of the factors driving the yield curve when the short-term rate hits the LB. Compared with a shadow rate model, it relaxes the assumption that the state vector is expected to follow the same dynamic evolution, independently of whether the economy is at the LB or not. This assumption is likely to become more overly restrictive, the longer the length of the LB period – see also Svensson (2014).

Our starting point is a VAR model, under the objective probability measure $\mathbb{P}$, for the state vector $X_t$

$$X_t = K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} X_{t-1} + \Sigma_X^j \varepsilon_{X,t}^{\mathbb{P}}, \tag{5}$$

where the state at time $t$ is $s_t = j$, where either $j = N$ (for the Normal regime) or $j = L$ (for the LB regime). In line with the vast literature on the term structure of interest rates, the state vector is assumed to consist of three yield factors, namely the curvature ($c$), slope ($s$), and short rate ($r$) of the term structure, i.e. $X_t = [c_t, s_t, r_t]'$. The formulation in (5) can be easily generalized to accommodate additional discrete regimes. Moreover, the $X_t$ vector can be expanded to include lags of its variables. Thus this formulation does not restrict us to work with a VAR(1) specification. Note that the vector $K_{X,0}^{\mathbb{P}j}$ determining the conditional long-run mean of the state vector, the feedback matrix $K_{X,1}^{\mathbb{P}j}$ and the variance parameters $\Sigma_X^j$ are all indexed by the prevailing regime.

We can identify the LB regime by assuming that all entries in the row and column of $K_{X,1}^{\mathbb{P}L}$ corresponding to $r_t$ are equal to zero when the economy is at the LB (this corresponds to the last row and column, since we order the variables in $x_t$ such that the short-term rate $r_t$ is the last element of the state vector). This implies that, in the LB regime, the short-term rate does not affect the other variables of the system and is itself an i.i.d. variable around a constant mean. We also assume that, at the LB, the short rate is not affected by shocks to the other equations. It follows that, conditional on remaining in the LB state, the law of motion of the short rate can be written as

$$r_{t+1} = \mu_r^L + \sigma_r^L \varepsilon_{t+1}^r$$

where $\mu_r^L$ is its conditional mean and $\sigma_r^L$ is a scalar. In contrast, $K_{X,1}^{\mathbb{P}N}$, including the elements corresponding to the short rate, is unrestricted in normal times.

The model is complemented by a short-rate equation of the form

$$r_t = \delta_0 + \delta_x' X_t \tag{6}$$

where $\delta_0 = 0$ and $\delta_x$ is a vector of zeros, with the exception of a 1 (loading on the short rate) in the last position. Note that $\delta_0$ and $\delta_x$ are regime-independent, but that differences in the mean short rate between regimes can be accommodated by $K_{X,0}^{\mathbb{P}j}$.

Following Dai, Singleton and Yang (2007, henceforth DSY), assume finally for the stochastic discount factor (SDF), $\mathcal{M}_{t,t+1}$, that

$$\log \mathcal{M}_{t+1} = -r_t - \Gamma_{t,t+1} - \frac{1}{2}\Psi_t' \Psi_t - \Psi_t' \varepsilon_{t+1} \tag{7}$$

$$\Psi_t^j = \psi_0^j + \psi_1^j X_t \tag{8}$$

$$\Gamma_{t,t+1} = \log \gamma_t^{j,k}, \tag{9}$$

where $\Psi_t^j$ are regime-dependent market prices of factor risk and $\Gamma_{t,t+1}$ are market prices of regime shift risk. A non-zero $\Psi_t^j$ captures risk premia that compensate investors for unpredictable variation in the state variables, while a non-zero $\Gamma_{t,t+1}$ captures premia required for being exposed to unpredictable regime shifts.

The risk-neutral state dynamics are given by

$$X_t = K_{X,0}^{\mathbb{Q}j} + K_{X,1}^{\mathbb{Q}j} X_{t-1} + \Sigma_X^j \varepsilon_{X,t}^{\mathbb{Q}}, \tag{10}$$

where $K_{X,0}^{\mathbb{Q}j} = K_{X,0}^{\mathbb{P}j} - \Sigma_X^j \psi_0^j$ and $K_{X,1}^{\mathbb{Q}j} = K_{X,1}^{\mathbb{P}j} - \Sigma_X^j \psi_1^j$.

Note that DSY assumes that the market prices of factor risk are such as to produce a regime-independent feedback matrix under $\mathbb{Q}$, i.e. $K_{X,1}^{\mathbb{Q}L} = K_{X,1}^{\mathbb{Q}N}$. This assumption is unappealing for our application, because it would imply that bonds are priced *as if* the risk-neutral state vector dynamics in the lower bound regime were identical to those in the normal regime. We therefore allow for state-dependent matrices $K_{X,1}^{\mathbb{Q}j}$. A disadvantage with this approach is that closed-form solutions for arbitrage-free bond prices are unavailable. To overcome this, we rely on the approximate bond pricing approach of Bansal and Zhou (2002).

Specifically, denote bond maturity by $n$ and the price of an $n$-period bond at $t$ by $P_{t,n}$, and note that the no arbitrage condition $P_{t,n} = \mathrm{E}_t\left[\mathcal{M}_{t,t+1}P_{t+1,n-1}\right]$ can be rewritten as

$$1 = \mathrm{E}_t\left[\mathcal{M}_{t,t+1}\frac{P_{t+1,n-1}}{P_{t,n}}\right].$$

Appendix A2 shows that the bond price can be written as

$$P_{t,n} = \exp\left(-A_n^j - B_n^j X_t\right),$$

for

$$\begin{aligned} A_n^j &= \sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\left(\delta_0^j + A_{n-1}^k + B_{n-1}^k K_{X,0}^{\mathbb{Q}j} - \frac{1}{2}B_{n-1}^k\Sigma_X^j\Sigma_X^j\left(B_{n-1}^k\right)'\right), \\ B_n^j &= \sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\left(\delta_x' + B_{n-1}^k K_{X,1}^{\mathbb{Q}j}\right), \end{aligned}$$

starting from

$$\begin{aligned} A_1^j &= \delta_0, \\ B_1^j &= \delta_x'. \end{aligned}$$

It follows that yields are given by $y_{t,n} = \frac{1}{n}A_n^j + \frac{1}{n}B_n^j X_t$.

We also need to specify the regime-switching probabilities under both $\mathbb{P}$ and $\mathbb{Q}$. Denote the transition probabilities from regime $s_t = j$ to regime $s_{t+1} = k$ as $\Pr\left[s_{t+1} = k | s_t = j\right] = \pi_t^{\mathbb{P}jk}$, for $0 \leq \pi_t^{\mathbb{P}jk} \leq 1$ and $\sum_{k=0}^{S}\pi^{\mathbb{P}jk} = 1$. In order to keep bond pricing somewhat tractable, we follow the existing asset pricing literature and assume that the $\mathbb{Q}$-probabilities, $\pi^{\mathbb{Q}jk}$, are constant over time. As for the $\mathbb{P}$-probabilities, $\pi_t^{\mathbb{P}jk}$, they can in general be time-varying and state-dependent. More specifically, we model them using the cumulative probability of a multivariate normal distribution:

$$\pi_t^{\mathbb{P},NL} = \int^{\theta_x^{NL}} \frac{1}{\sqrt{(2\pi)^2 \left|\Sigma_X^N\right|}} \exp\left(-\frac{1}{2}\left(X - \mu_{t+1}^N\right)' \Sigma_X^{N^{-1}} \left(X - \mu_{t+1}^N\right)\right) dX,$$

where $\mu_{t+1}^N$ is the next-period conditional expectation of $X$, given that the economy is currently in state $N$, and where $\theta_x^{NL}$ is a vector of critical levels, or thresholds, that indicate at which point the sensitivity of the regime-switching probability is at its highest. In other words, these thresholds represent levels of the state variables where investors would start to become concerned that the economy would hit the LB. The probability to leave the lower bound, $\pi_t^{\mathbb{P},LN}$, is modelled symmetrically, for thresholds $\theta_x^{LN}$.

Finally, in order to link the $\mathbb{P}$ and $\mathbb{Q}$ transition probabilities, we follow DSY and assume that

$$\pi^{\mathbb{Q},NL} = \frac{\pi_t^{\mathbb{P},NL}}{\gamma_t^{NL}},$$

i.e. that the market prices of regime-switching risk are such that scaling the $\mathbb{P}$-probabilities by the risk price results in the $\mathbb{Q}$-probability. As shown by DSY, if regime shift risk is not priced, then the transition probabilities under $\mathbb{P}$ and $\mathbb{Q}$ would coincide.

## 3.1 Pricing consistency and JSZ form

Joslin, Singleton and Zhu (2011, JSZ henceforth) and Hamilton and Wu (2012) highlight an identification problem in affine term structure models, which arises from the property that the state vector $X_t$ is often specified in terms of linear combinations of yields. In this section we briefly highlight the problem in the context of our regime-switching model.

The problem has to do with the prices of risk parameters $\psi_0^j$ and $\psi_1^j$ and it is particularly clear when the state vector is composed of yields only. As an illustrative example, consider the simple case where the state vector consists of a slope factor $s_t$, given by the difference between a $n$-month maturity yield $y_{t,n}$ and the short rate $r_t$, and the short rate itself, so that in this simplified case $X_t = [s_t, r_t]'$.

To further simplify this illustrative example, consider the single-regime affine case. Given state $X_t$ and short rate $r_t = [0, 1] X_t$, bond prices will follow a recursion such that $P_{t,k} = \exp(-A_k - B_k X_t)$ for maturity $k$. It is then immediately clear that estimation must be carried out under an additional constraint requiring that the slope factor $s_t = y_{t,n} - r_t$ is consistent with the pricing parameters in $A_n$ and $B_n$. In this example, the constraints are $y_{t,n} = \frac{1}{n}\left(A_n + B_n\ x_t^f\right) = s_t + r_t$, or $\frac{1}{n}A_n = 0$ and $B_n = [n, n]$. This, in turn, implies non-linear constraints on the parameters governing the risk-neutral dynamics of $X$.

In the regime-switching case, the constraints must correspondingly ensure that bond prices are consistent with the state vector for both states $N$ and $L$, i.e. for $A_n^L$, $A_n^N$, $B_n^L$ and $B_n^N$, which in turn implies non-linear restrictions on $K_{X,0}^{\mathbb{P}j}$ and $K_{X,1}^{\mathbb{P}j}$.

JSZ show that the restrictions can be applied much more directly and more efficiently in the affine case, by skillfully exploiting the Jordan decomposition. We apply this approach in each regime to ensure pricing consistency in our model. Specifically, given the regime-dependent $\mathbb{P}$-dynamics (5), $\mathbb{Q}$-dynamics (10) and the short-rate equation (6), the state variables are defined, in the language of JSZ, as "portfolios of yields" with weights $W$, i.e. $X_t = W y_t$.[2] Under the assumption that these yield portfolios are priced perfectly by the model, they can be viewed as observable pricing factors. In this case, JSZ show that the $\mathbb{Q}$ distribution of $X_t$ can be fully characterized by the parameters $\Theta^{\mathbb{Q}j} = \left(k_\infty^{\mathbb{Q}j}, \lambda^{\mathbb{Q}j}, \Sigma_X^j\right)$ , where $k_\infty^{\mathbb{Q}j}$ is a parameter that is proportional to $r_\infty^{\mathbb{Q}}$, the risk-neutral long-run mean of the short rate in regime $j$ (under $\mathbb{Q}$-stationarity), $\lambda^{\mathbb{Q}j}$ is the vector of eigenvalues of $K_1^{\mathbb{Q}j}$, and $\Sigma_X^j \Sigma_X^{j\prime}$ is the covariance of innovations to the portfolios of yields. The parameters of the $\mathbb{P}$ distribution of $X_t$ are $\Theta^{\mathbb{P}j} = \left(K_{X,0}^{\mathbb{P}j}, K_{X,1}^{\mathbb{P}j}, \Sigma_X^j\right)$ .

## 3.2 Unspanned macro factors

Thus far, we have assumed that the dynamics of yields in our regime-switching model can be adequately captured by a small number of yield factors, namely level, slope, and curvature. This is in line with a vast literature on the term structure of interest rates, which, starting with Litterman and Scheinkman (1991), shows that these three yield factors (or, alternatively, the first three principal components (PCs) of yields) explain almost all of the variation in yields.

However, Joslin, Priebsch and Singleton (2014, JPS henceforth) show that while a small number of yield factors can capture yield dynamics well, macroeconomic factors can nevertheless play an important role for bond risk premia and expected future interest rates. Since these macro factors are not spanned by the current yield curve, JPS denote them as 'unspanned' macro factors. These macro factors therefore represent risk factors that are distinct from yield

[2] In our case, with the factors specified as curvature, slope and short rate, we choose $W$ to select weights on individual yields that correspond to our definition of curvature, slope and short rate; see Section 4 for more details.

level, slope and curvature or yield PCs. Specifically, unspanned macro factors can affect risk premia without directly impacting yields (beyond their effect on the yield factors themselves) because, in the bond market pricing kernel, the market prices of yield factor risks are allowed to depend not only on the yield factors, but also on the unspanned macro factors.

Within our setting, it seems plausible that key macroeconomic variables, such as inflation and economic activity, could potentially play an important role for bond risk premia and expectations of future short-term interest rates. In fact, investors are likely to be keenly sensitive to the evolution of such macro variables precisely in periods where the policy rate is stuck at the lower bound, or when the economy appears to be heading towards the LB. We therefore adjust our model to allow for unspanned macro risks. Specifically, we include CPI inflation and industrial production growth as candidates for unspanned macro risk factors.

In terms of modeling the term structure, the inclusion of unspanned macro factors involves expanding the risk factors in the bond market from the yield factors $X_t$ to $Z_t = [X_t, M_t]$ that includes the macro factors $M_t$. Moreover, the $\mathbb{P}$-dynamics of $Z_t$ are assumed to follow a VAR where $X_t$ is allowed to depend on $M_t$ and vice versa:

$$Z_t = K^{\mathbb{P}j}_{Z,0} + K^{\mathbb{P}j}_{Z,1} Z_{t-1} + \Sigma^j_Z \varepsilon^{\mathbb{P}}_{Z,t}, \tag{11}$$

or

$$\begin{bmatrix} X_t \\ M_t \end{bmatrix} = \begin{bmatrix} K^{\mathbb{P}j}_{X,0} \\ K^{\mathbb{P}j}_{M,0} \end{bmatrix} + \begin{bmatrix} K^{\mathbb{P}j}_{XX,1} & K^{\mathbb{P}j}_{XM,1} \\ K^{\mathbb{P}j}_{MX,1} & K^{\mathbb{P}j}_{MM,1} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ M_{t-1} \end{bmatrix} + \begin{bmatrix} \Sigma^j_{XX} & \Sigma^j_{XM} \\ \Sigma^j_{MX} & \Sigma^j_{MM} \end{bmatrix} \begin{bmatrix} \varepsilon^{\mathbb{P}}_{X,t} \\ \varepsilon^{\mathbb{P}}_{M,t} \end{bmatrix}.$$

As before, we allow the dynamics to depend on the regime $j$. Under $\mathbb{Q}$, the yield factors $X_t$ evolve according to (10), while the short-rate dynamics remain as in (6). Given the pricing kernel (7), the market prices of risk are now implicitly given by the difference in the drift of $X_t$ in the $\mathbb{P}$-dynamics of $Z_t$ in (11) and the drift of $X_t$ under $\mathbb{Q}$ in (10):

$$\Psi^j_t = \left(\Sigma^j_{XX}\right)^{-1} \left( \left(K^{\mathbb{P}j}_{X,0} - K^{\mathbb{Q}j}_{X,0}\right) + \left( \begin{bmatrix} K^{\mathbb{P}j}_{XX,1} & K^{\mathbb{P}j}_{XM,1} \end{bmatrix} - \begin{bmatrix} K^{\mathbb{Q}j}_{X,1} & 0 \end{bmatrix} \right) \begin{bmatrix} X_{t-1} \\ M_{t-1} \end{bmatrix} \right).$$

As a result, although only $X_t$ are priced risks, investors' tolerance of these risks is allowed to depend on both $X_t$ and on the macro factors $M_t$.

## 3.3 Estimation

When estimating the model, we will include yields not only for the maturities needed to construct our observable yield factors. We assume that the additional yields are observed with some random pricing errors, and that the errors are normally distributed with covariance matrix $\Sigma_e \Sigma_e'$. In this setting, JSZ show that the conditional density of the collection of yields $y$ in our sample is given by the product of the $\mathbb{P}$ density of the observable yield factors $X$ (and observable unspanned macro factors, if any) times the conditional density of the pricing errors. This latter density is determined by the cross-sectional relationship among the yields, and is therefore only a function of the $\mathbb{Q}$ dynamics of $X$ (and $\Sigma_e$). As a result, the $\mathbb{P}$ density and the $\mathbb{Q}$ density are fully separate, which means that the conditional mean of $X$ under $\mathbb{P}$ can be estimated separately in a first step, while the pricing error density, which depends on the conditional mean of $X$ under $\mathbb{Q}$ and on the short-rate dynamics, can be estimated subsequently.[3]

In a single-regime case with state vector $Z_t$ that includes both observable yield factors and unspanned macro factors, the conditional density of $(Z_t, y_t)$ is (see JPS)

$$\begin{aligned} f\left(Z_t, y_t \mid Z_{t-1}; \Theta\right) &= f\left(y_t \mid Z_t, Z_{t-1}; \Theta\right) \times f\left(Z_t \mid Z_{t-1}; \Theta\right) \\ &= f\left(y_t \mid X_t; k_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_Z, \Sigma_e\right) \times f\left(Z_t \mid Z_{t-1}; K_{Z,0}^{\mathbb{P}}, K_{Z,1}^{\mathbb{P}}, \Sigma_Z\right). \end{aligned}$$

The states $Z$ as well as the measurement errors are assumed to be conditionally Gaussian.

In a regime-switching setting, the joint probability $f\left(Z_t, y_t, s_{t-1} = j, s_t = k \mid Z_{t-1}; \Theta\right)$ can be rewritten as

$$f\left(Z_t, y_t, s_{t-1} = j, s_t = k \mid Z_{t-1}; \Theta\right) = f\left(Z_t, y_t \mid Z_{t-1}, s_{t-1} = j, s_t = k; \Theta\right) \times f\left(s_{t-1} = j, s_t = k \mid Z_{t-1}\right).$$

If we define the probability of regime $s_{t-1} = j$ given $Z_{t-1}$ as $Q_{t-1}^j$, i.e. $Q_{t-1}^j \equiv f\left(s_{t-1} = j \mid Z_{t-1}\right)$, it follows that

$$\begin{aligned} f\left(s_{t-1} = j, s_t = k \mid Z_{t-1}\right) &= f\left(s_{t-1} = j \mid Z_{t-1}\right) \times f\left(s_t = k \mid Z_{t-1}, s_{t-1} = j\right) \\ &\equiv Q_{t-1}^j \pi_{t-1}^{\mathbb{P}jk}, \end{aligned}$$

[3] The only link between the two densities is the covariance of the innovations, but as emphazised by JSZ, this covariance does not affect the ML extimates of the conditional factor mean.

and

$$f\left(Z_t, y_t, s_{t-1}=j, s_t=k \mid Z_{t-1};\Theta\right) = f\left(Z_t, y_t \mid Z_{t-1}, s_{t-1}=j, s_t=k;\Theta\right) \times Q_{t-1}^{j} \pi_{t-1}^{\mathbb{P}jk},$$

so that the likelihood in period $t$ can be obtained by integrating out the discrete states

$$f\left(Z_t, y_t \mid Z_{t-1};\Theta\right) = \sum_{k=N,L}\sum_{j=N,L} f\left(Z_t, y_t \mid Z_{t-1}, s_{t-1}=j, s_t=k;\Theta\right) \times Q_{t-1}^{j} \pi_{t-1}^{\mathbb{P}jk}.$$

The sample log-likelihood is then

$$\log L = \frac{1}{T-1}\sum_{t=0}^{T-1} \log f\left(Z_t, y_t \mid Z_{t-1};\Theta\right).$$

Denoting the updated $Q_{t-1}^{j}$ as $Q_t^k \equiv f\left(s_t = k \mid Z_t\right)$, we thus have

$$Q_t^k = \frac{\sum_j f\left(Z_t, y_t \mid Z_{t-1}, s_{t-1}=j, s_t=k;\Theta\right) \times Q_{t-1}^{j} \pi_{t-1}^{\mathbb{P}jk}}{f\left(Z_t, y_t \mid Z_{t-1};\Theta\right)}.$$

## 4 An application to U.S. data

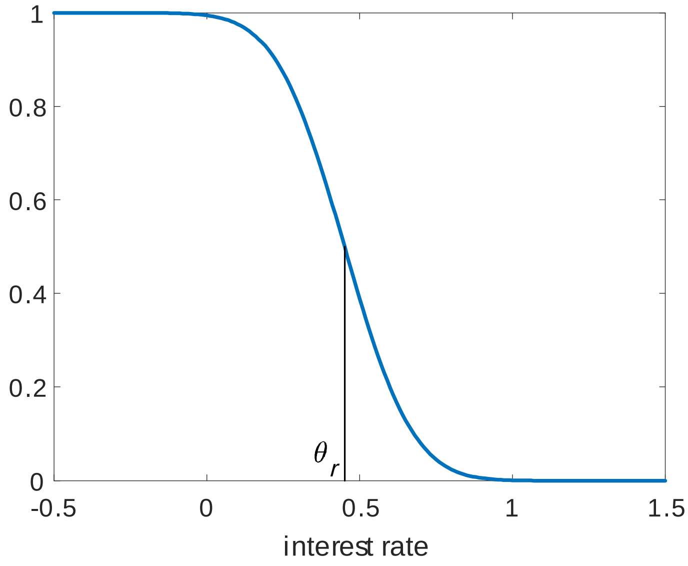
We test the performance of our regime-switching model in an application to U.S. data.

As mentioned above, the state vector is completely observable and includes three yield factors: a curvature factor $c_t$, defined as the sum of the 1-month nominal rate and the 10-year yield minus twice the 3-year yield; a slope factor $s_t$, defined as the spread between the 10-year and 1-month rate; and the 1-month nominal interest rate $r_t$. The state vector also includes two unspanned macro factors: CPI inflation and industrial production growth, both defined in terms of annual log-differences. We estimate the model using six additional maturities apart from the ones used to construct the yield factors: 3 and 6 months; 1, 2, 5 and 7 years. All the data is sampled at the monthly frequency (end-of-month values). The sample period runs from January 1987 to April 2018.

Given that the sample period is relatively short and that presumably few regime switches occurred in the data, we adopt two simplifying assumptions.

First, we pre-estimate the $\mathbb{P}$-parameters for the two regimes under the assumption that we can identify time periods when the economy was highly likely to have been in the normal

Figure 2: Transition probability $\pi^{NL}$



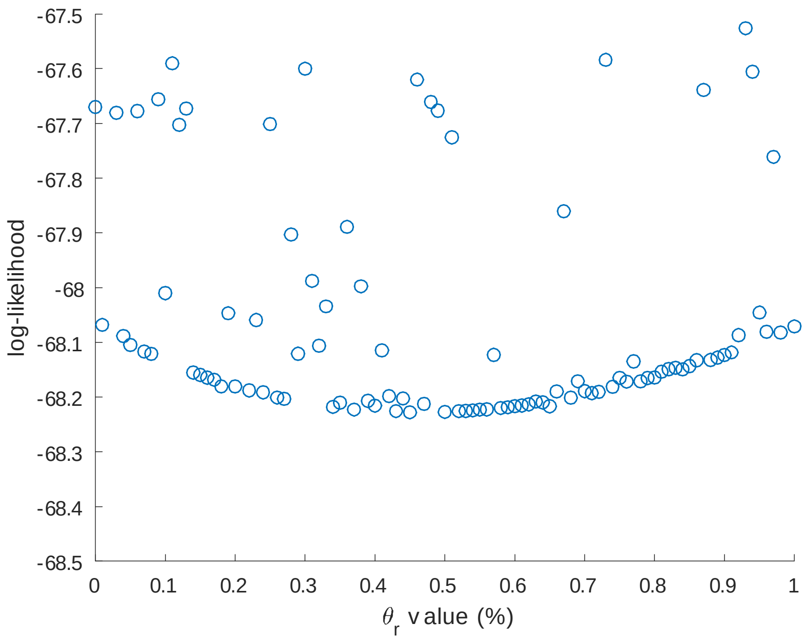

regime or in the lower bound regime. Specifically, we assume that the system was in the normal regime before June 2007 and estimate a VAR model on the state variables over this period. We then assume that the LB regime prevailed from end-December 2008 until October 2015 and reestimate the state vector VAR over this period. This yields parameter estimates for $K^{\mathbb{P}N}_{Z,0}, K^{\mathbb{P}N}_{Z,1}, \Sigma^N_Z, K^{\mathbb{P}L}_{Z,0}, K^{\mathbb{P}L}_{Z,1}$ and $\Sigma^{LN}_Z$. We keep these parameters fixed when estimating the remaining ones.

Second, we assume that the transition probability from $N$ to $L$, $\pi^{\mathbb{P},NL}_t$, is only a function of the short-term rate. This implies, intuitively, that the likelihood to switch to the lower bound regime is higher, the lower the level of the short-term rate. Specifically, we set

$$\pi^{\mathbb{P},NL}_t = \int^{\theta_r} \frac{1}{\sigma^N_r \sqrt{(2\pi)^2}} \exp\left(-\frac{1}{2}\left(\frac{r-\mu^{N,r}_{t+1}}{\sigma^N_r}\right)^2\right) dr,$$

where $\mu^{N,r}_{t+1}$ is the next-period conditional expectation of $r$, given that the economy is currently in state $N$, and $\sigma^N_r$ is the standard deviation of $r$ in state $N$. In this case, the threshold $\theta_r$ could be seen as representing the rate at which the transition probability is most sensitive to changes in the short-term interest rate. Figure 2 provides a stylized illustration of the transition probability to switch from $N$ to $L$ as a function of the short rate. For simplicity, we also assume symmetry in the probabilities to switch regime, so that $\pi^{\mathbb{P},LN}_t = 1 - \pi^{\mathbb{P},NL}_t$.

Figure 3: Log-likelihood as a function of $\theta_r$

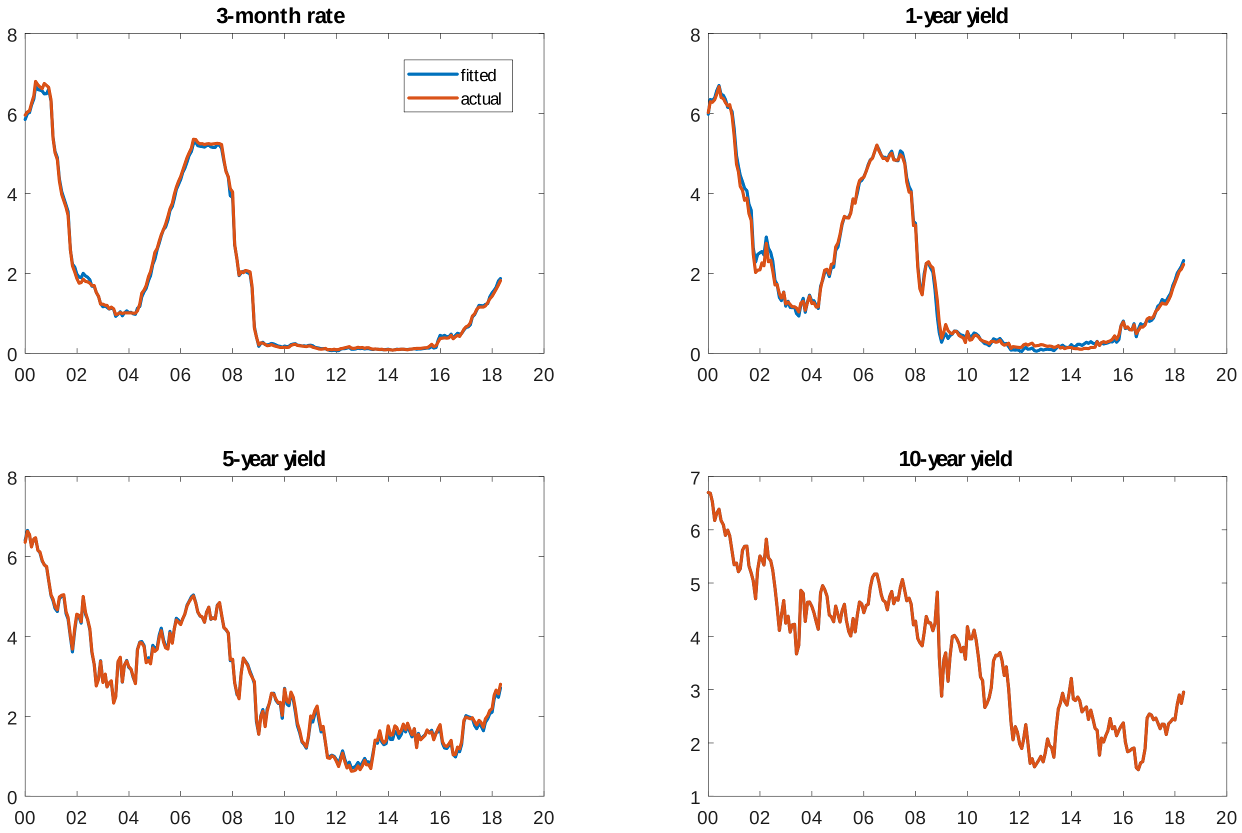



Given these two simplifying assumptions, we proceed to estimate the remaining parameters of our model. Specifically, we jointly estimate $k_\infty^{\mathbb{Q}j}, \lambda^{\mathbb{Q}j}, \Sigma_X^j$ for $j = N, L$, as well as $\Sigma_e$ and the transition probabilities $\pi^{\mathbb{Q},NN}$ and $\pi^{\mathbb{Q},LL}$. We further simplify the estimation problem by assuming that $\Sigma_e$ is diagonal and that the diagonal elements are identical.

Given our short sample period and the very few regime switches within it, we are unlikely to be able to accurately estimate $\theta_r$ as a free parameter. We therefore implement a grid-search, whereby we fix $\theta_r$ at different values within an interval between 0 and 100 basis points and estimate the free parameters for each value of $\theta_r$. Figure 3 displays the (negative of the) log-likelihood values obtained against the various $\theta_r$ values. The highest log-likelihood value is attained for $\theta_r = 45$ basis points, which we therefore use as our threshold for the remainder of our analysis.

Table 2 reports the parameter values that maximize the likelihood function. The common standard deviations of the yield measurement errors is around 8 basis points, indicating that the model is able to fit yields well. This is confirmed by Figure 4, which displays actual and fitted values of bond yields across various maturities. The lower right-hand panel shows that the 10y-yield is fitted perfectly by construction.

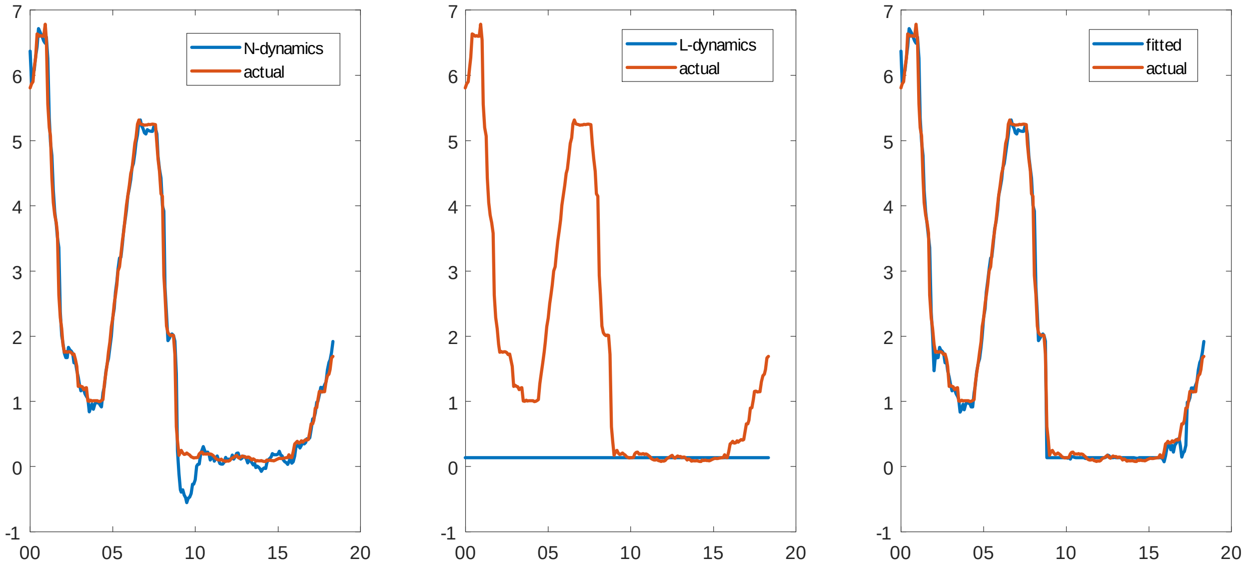
Figure 4: Actual and fitted yields




One notable feature of the parameter vector is the different steady state values conditional on each of the two regimes. In the lower bound regime, the steady state short rate is more than 4 percentage points lower, at 0.14 percent, than in the normal regime (4.47 percent). The slope is around 130 basis points higher in the $L$ regime, at 2.48% versus 1.16% percent, and the curvature factor is around 60 basis points higher (1 percent vs 39 basis points in the $N$ regime). The macro factors also display important steady-state differences across the regimes. The long-run mean of inflation in the $N$ regime is 2.74%, whereas it is 1.09% in the $L$ regime. And steady state industrial production growth in the $N$ regime is 2.67%, but only 1.09% in the $L$ regime. These estimates show that the model dynamics imply a substantially lower long-run growth rate and steady-state inflation far below the Fed's target, when the economy is in the lower bound regime.

Dynamics in the $N$ regime are much more persistent than in the $L$ regime. The largest eigenvalue of the $K_{Z,1}^{\mathbb{P}N}$ matrix is 0.987 compared to 0.948 for $K_{Z,1}^{\mathbb{P}L}$. Four of the eigenvalues of

Figure 5: Actual and model short rate

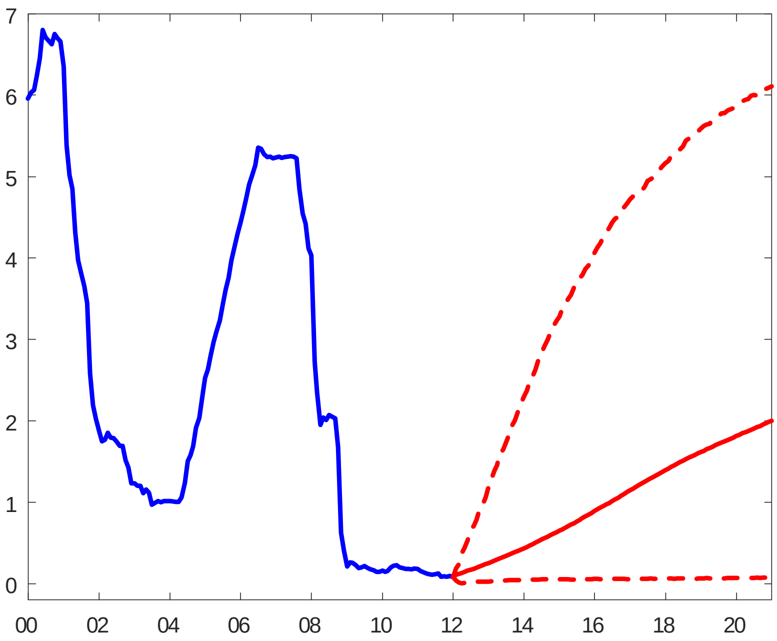



$K^{\mathbb{P}N}_{Z,1}$ are above 0.9 compared to two eigenvalues of $K^{\mathbb{P}L}_{Z,1}$. All in all, factor dynamics are less strongly attracted by their conditional long-run mean in the $L$ regime. Consistent with the descriptive statistics of Section 2, yield movements are also less persistent in the $L$ regime.

Figure 5 shows actual and fitted values of the short-term interest rate separately in the two regimes and for the full model. The figure highlights that the Gaussian model would not provide a terrible fit of the short-term interest rate over the binding lower bound period. For given $N$-regime parameters, the short-term rate would only dip into mildly negative territory. Hence, fitting errors over the period 2009-2016 would be small, albeit highly persistent (see the left panel). By contrast, the ultra-low policy rate period can be better captured by the $L$-regime parameters. As highlighted by the middle panel in the figure, however, the $L$ regime quickly becomes inconsistent with the data, once policy rates are above the level consistent with $K^{\mathbb{P}L}_{Z,0}$. The right panel in figure 5 shows that the regime switching model effectively combines the dynamics of the $N$ and $L$ regimes to ensure that the evolution of the short rate is consistent with the data over the whole sample period.

Figure 6 illustrates the model's ability to ensure that nominal interest rates do not attain deeply negative values. It shows the in-sample forecast distribution (90% confidence bands) of the 3-month interest rate in December 2011, a time when the economy is in the lower bound regime with a probability close to one. Going forward, the probability of moving to the normal

Figure 6: 3-month interest rate forecast as of end-2011

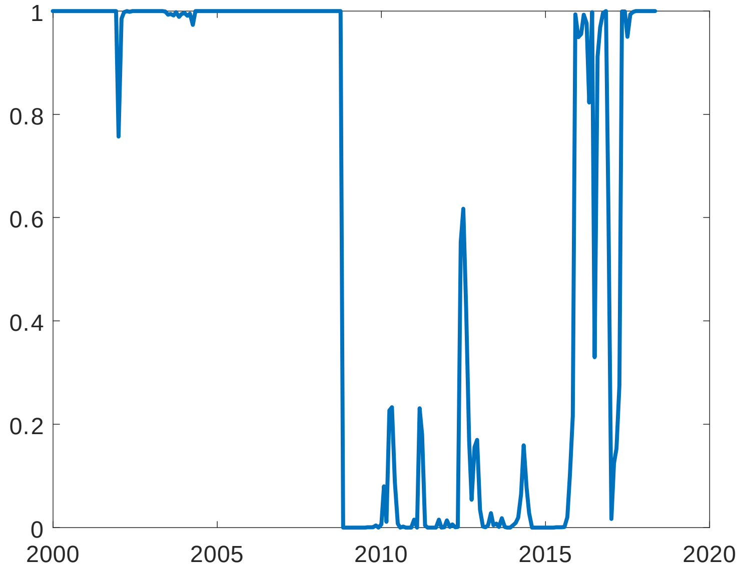



regime is expected to slowly rise over time. Consequently, interest rates are slowly expected to increase too. Nevertheless, the forecast distribution is wide, and near-zero values remain possible for the entire horizon. However, the forecast distribution effectively rules out values significantly below zero.

### 4.1 Inference on the regimes and implications for risk premia

Figure 7 reports the filtered probability of the economy being in the normal regime. The probability remained very close to one until late 2008, then it quickly dropped to zero in October 2008, two months before the Fed established a target range for the federal funds rate of 0 to 1/4 percent. The probability of being in the normal regime registered some fluctuations in subsequent years, but generally stayed very low until 2013, at the time of the taper tantrum, when it temporarily spiked above 50%. Finally, the probability increased sharply again in November 2015, a month ahead of the FOMC's decision to begin raising the target range for the federal funds rate from nearly zero. Nevertheless, the probability thereafter dipped significantly below one a number of times over the next few months. This is arguably consistent with Fed communication after the initial hike, which indicated that the FOMC expected economic conditions to evolve in a manner consistent with *only gradual* increases in

Figure 7: Filtered $N$-regime probability



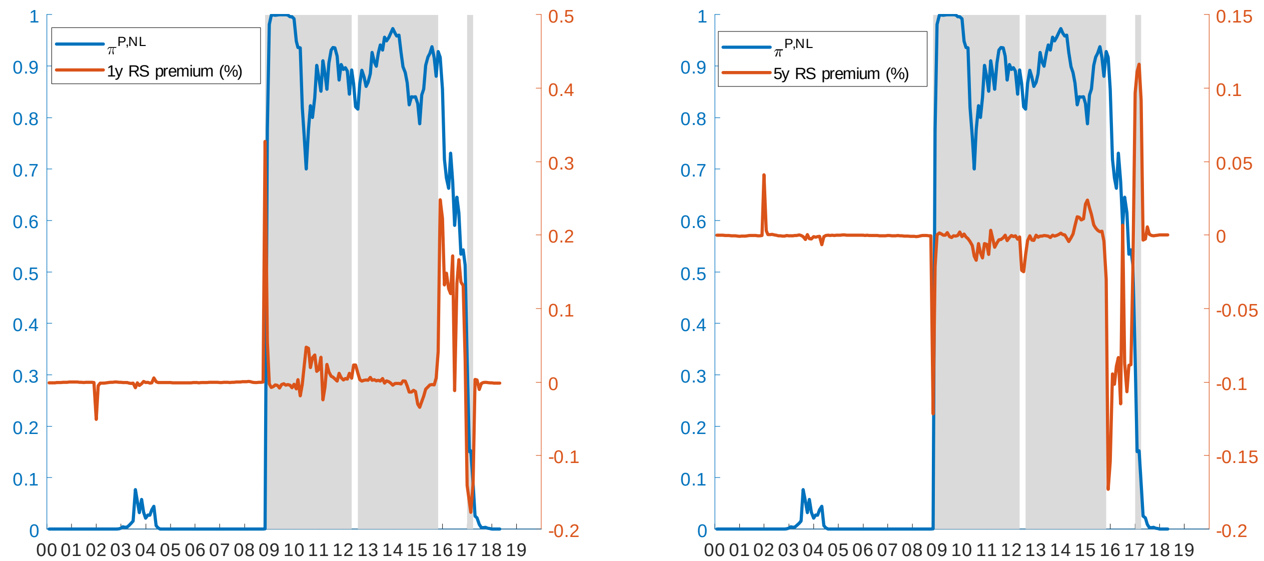

interest rates. By early 2017, after the target range had been raised three times, the probability of being in the normal regime had increased more decisively to its pre-crisis levels.

Our model is also informative with regard to whether investors require compensation for being exposed to regime shift risk. We investigate this issue by examining one-period ahead expected excess returns conditional on assuming that factor risk is not priced. In other words, we set the market prices of factor risk ($\Psi_t^j$) to zero, so that $K_{X,0}^{\mathbb{Q}j} = K_{X,0}^{\mathbb{P}j}$ and $K_{X,1}^{\mathbb{Q}j} = K_{X,1}^{\mathbb{P}j}$, and examine how model-implied expected excess returns behave over time. As shown in Appendix 2, any remaining non-zero expected excess returns in this scenario must (apart from a negligible Jensen's inequality term) be due to regime shift risk being priced, i.e. non-zero $\Gamma_{t,t+1}$.

As shown in Figure 8, the results from this exercise indicate that regime shift risk premia (RS premia) are typically close to zero, but that they at times can be non-negligible. In particular, RS premia are essentially zero when the $\mathbb{P}$ probability to transition from the current regime to the other regime is low. This is the case, for example, up until late 2008 when the economy was in the normal regime and $\pi^{\mathbb{P},NL}$ was tiny. The same goes for late 2008-2010 or so, when the economy was in the $L$ regime (shaded area) and $\pi^{\mathbb{P},LN}$ was close to zero.[4] The premium becomes more sizeable, however, in situations where these conditions no longer hold. Take, for example, late 2015 when the economy had just switched from $L$ to $N$, but

[4] Recall, $\pi^{\mathbb{P},LN} = 1 - \pi^{\mathbb{P},NL}$ by construction.

Figure 8: Regime-shift risk premium (monthly return) and $\mathbb{P}$ transition probability




the probability to switch back to $L$ was high. At this point, the left panel shows that the regime-shift premium on short bonds (1-year) reached 0.25% in monthly excess return terms. Because the probability to switch back is so high, short-term bonds price in a high likelihood that the expected average short rate will be very low over the near term. As a result, the main risk that holders of short-term bonds face is that the economy actually does not switch, even if $\pi^{\mathbb{P},NL}$ is high, and that the yield will end up rising as a result. Consequently, investors demand a positive premium to be exposed to this risk. As the probability to switch to $L$ progressively drops in subsequent months, the bond yield will increasingly reflect expectations of a higher average interest rate, and the risk of a negative surprise (i.e. an unexpected failure to switch back to $L$) will drop, and with it the RS risk premium.

What about longer-term bonds? The right panel of Figure 8 shows that the regime-shift premium was negative for 5-year bonds after the economy had switched to the normal regime in late 2015. We surmise that this is because, while a high likelihood of switching back to $L$ results in lower short-term interest rates, over longer horizons this effect dissipates. As a result, long-term bonds only partially price in the high near-term probability to switch to $L$, and yields are comparably higher than for short bonds. Consequently, the main 'risk' for holders of long bonds is that the switch back to $L$ is much more persistent than expected, so that long yields would drop significantly. This would clearly be beneficial to holders of long-term bonds, and the regime-shift premium is therefore negative. Again, as the probability to switch from $N$ to $L$ drops in the course of 2016, this effect is gradually reversed and the premium rises towards zero (it very briefly becomes positive in 2017 as the model deems the economy to temporarily have entered the $L$ regime).

All in all, these results suggest that regime-shift premia can be non-negligible around times when the probability to transition to another state is high and investors become particularly concerned (or exuberant) about the risk of a possible change in regime. Regime-shift premia are not a quantitatively crucial factor to explain yield dynamics over the time period we analyze, but they of course contribute at times. Compared to the standard premia required as compensation for changes in the factors within a given regime, regime-shift premia are much more volatile and can therefore account for some high-frequency movements in yields.

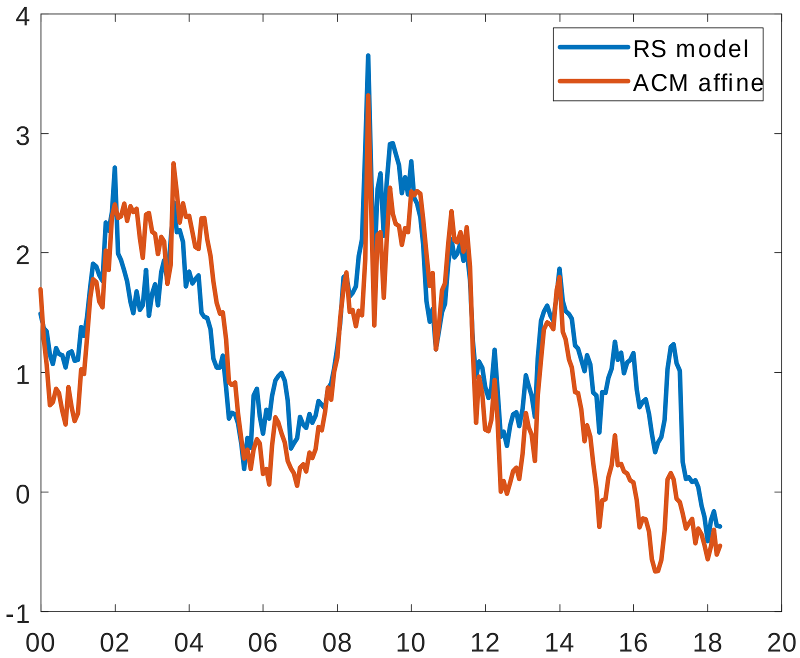
We also perform a standard decomposition of yields between an expectations-hypothesis (EH) component, defined as the average expected short-term interest rate over the maturity of the bond, and a risk premium component. Our results suggest that fluctuations in EH yields explain most of the variation of actual yields for most maturities in the early 2000s, while fluctuations in risk premia are the main driving factor for observed yields during the period when the economy is in the lower bound regime.[5]

To understand the impact of different regimes on the decomposition of observed yields, it is instructive to compare total risk premia in our model with those obtained through an affine specification. We perform this comparison for 10-year premia in Figure 9, relying on a commonly used benchmark, i.e. the model by Adrian, Crump and Moench (2013, ACM henceforth).

In spite of the different specifications, the two models deliver estimates of risk premia that are quite similar to each other over the period until 2014. However, noticeable differences emerge after "lift-off". The non-zero probability to switch back to the LB regime continues exerting an influence on yield dynamics in our model. It implies that expected future short rates are lower than in an affine specification. As a result, the regime-switching model produces higher risk premia than the ACM model – notably risk premia that are not deeply negative. Our estimated risk premia on 10-year yields do in fact remain positive for almost the entire sample period.

[5] These results are available on request from the authors.

Figure 9: 10-year yield risk premia in the RS model and in ACM




While negative term premia are theoretically possible, positive term premia on long-term bonds are arguably more plausible from the perspective of a structural model. A few recent nonlinear macro models have shown that they are capable of fitting relevant conditional covariances when the lower bound constraint binds. Even if such models are not estimated, they suggest that nominal term premia at long maturities should on average be lower when short rates are at the lower bound, but they should mostly remain positive (Nakata and Tanaka, 2016; Gourio and Ngo, 2016). The intuition for negative term premia comes from the inflation premium component. Nominal bonds are a hedge when the main driver of economic developments are demand shocks (which is arguably the case at the effective lower bound). In response to demand shocks, adverse (recession) states are associated with deflation, hence surprisingly high real bond returns. However, Nakata and Tanaka (2016) point out that an implication of the lower bound constraint is to reduce to near-zero levels the sensitivity to shocks of the short-term rate. As a result, the pressure towards negative term premia tends to be small. Moreover, the nominal term premium does not only include a compensation for inflation risk. Longer-term nominal bonds are also subject to duration risk. Compared to a shorter-term nominal bond – which is also a hedge against deflation risk – they expose investors to the risk of an unexpected policy tightening which would lead to capital losses.

## 4.2 Forecasting implications

We have argued that the regime-switching model provides sensible estimates of risk premia. However this model includes many parameters, notably more parameters than shadow rate models. There is an obvious risk of overparameterization. As a specification test, this section presents results of an out-of-sample forecasting exercise. More specifically, we first compare forecasts generated by our model to two versions of the shadow rate model at a specific point in time. This comparison highlights a particular advantage of the regime switching approach: it does not force us to take a stance *ex ante* on the future long-run level of interest rates. In a second step, we run an out-of-sample forecasting horse race between our model to two alternatives: the random walk and the shadow rate model. The comparison suggests that, in spite of its less parsimonious specification, our model has competitive forecasting properties.

### 4.2.1 Implications for the long run

We start with a comparison of interest rate forecast paths at a specific point in time. For the shadow rate model, we rely on the specification in Wu and Xia (2016, WX henceforth), which uses forward rates (rather than yields) in estimation. All forecasts are out-of-sample, in the sense that we only use data up to mid-2017 to estimate model parameters.[6] We focus on forecasts generated at mid-2017, which is an interesting point in time, because the Fed had raised the fed funds target range four times by then, and markets were pricing in further increases going forward.

Note that the choice of the estimation sample is of crucial importance for shadow rate models. WX estimate their model on a sample in which interest rates were at the lower bound for a significant portion of the sample period. As a result, the model interprets this as an indication that the long-run mean of yields is very low, and the parameter estimates reflect this low level. From a forecasting perspective, this implies that short-term rates need not increase much after lift-off to reach their low unconditional mean.

[6] For the WX model, we use the Matlab code available on Cynthia Wu's webpage https://sites.google.com/site/jingcynthiawu/.

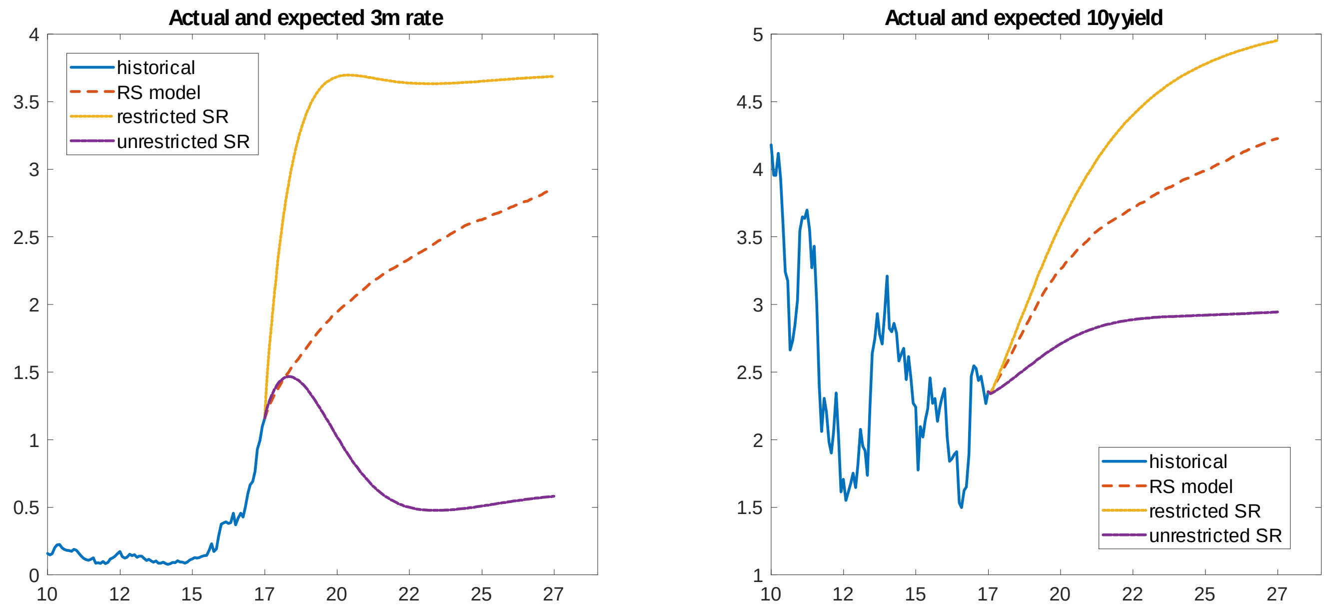
By contrast, in their implementation of a shadow rate term structure model, Bauer and Rudebusch (2016) estimate an affine model over the pre-2008 period and then complement it with a shadow rate specification without re-estimating the parameter values for the lower bound period. This approach is very efficient and illustrates a crucial advantage of the shadow rate model: it is very parsimonious, as it requires no additional parameters than an affine model. The Bauer and Rudebusch (2016) approach also ensures that the long-run mean of interest rates is consistent with pre-crisis values. The paper shows that this approach is not problematic for the model's ability to fit yields during the lower bound period. When forecasting yields after lift-off, however, this model will have the same properties as a standard affine model. More specifically, interest rates will need to increase to higher levels as quickly as after a normal recession to reach their unconditional pre-LB mean.

To illustrate these properties of shadow rate models and contrast them with those of the regime-switching model, Figure 10 reports interest rate forecasts over a horizon of up to 10 years for 3-month and 10-year yields as of end-June 2017. Apart from the regime-switching model, denoted by "RS", we show forecasts based on two versions of the Wu and Xia (2016) implementation of the shadow-rate model: a restricted version, denoted by "restricted SR", where the parameter values are fixed at their pre-crisis estimates (i.e. based on the period January 1990 - December 2007), and an unrestricted version, denoted by "unrestricted SR", where all parameters are estimated up until the forecast date.

Based on the regime-switching model estimates, the probability of being in the normal regime (not shown in the figure) is forecast to fall slowly initially from the near-1 values of April 2018, reaching a trough of around 0.85% after 2 years, and then increase very slowly over subsequent years. At the end of the forecast horizon, in 2028, the probability remains close to 0.9. As a result, the model also forecasts a very drawn-out policy normalization process. The 3-month rate rises quite slowly, reaching 2% by early 2020. The mean prediction is that the short rate will reach values around 2.9% percent in 2028. With the short rate increasing at a gradual pace, the RS model forecasts that 10-year yields will rise slowly as well, just exceeding the 4% mark at the end of the forecast horizon in 2028.

By contrast, forecasts from the *restricted* shadow rate model show that short-term yields increase very quickly. The 3-month rate is forecasted to exceed 3% within a year, by May 2019, and then continue to quickly rise to the long-run mean level of 3.65%. On the other hand, the *unrestricted* version of the shadow rate model produces a completely different forecast path. In

Figure 10: Yield forecasts as of end-June 2017




this case, following a few months of tiny increases, the 3-month rate is expected to quickly fall towards its estimated long-run mean level around 0.5%. This value appears to be exceptionally low by historical standards, albeit perhaps not unreasonable in a potential secular stagnation scenario. The 10-year yield forecasts reflect these very different short-rate paths, with the restricted shadow-rate model producing long-term yield projections substantially higher than the regime-switching model, and the unrestricted shadow rate model resulting in a much lower path.

All in all, the comparison of these forecasts based on the regime-switching and shadow rate models suggests that the former model can more flexibly capture the special nature of post-lower bound recoveries. The shadow rate model forces the researcher to specify at the outset whether the recovery will never be full – consistent with the secular stagnation hypothesis – or if it will be as fast as those following standard recessions. The regime-switching model weighs these two scenarios by the regime probabilities, which following a LB episode exhibit a persistent non-negligible likelihood to switch back to the lower bound.

### 4.2.2 Forecasting: a comparison to the shadow rate model

We evaluate the out-of-sample forecasting performance of our regime-switching model relative to two alternatives: the random walk and the unrestricted shadow rate model.[7] The models are first estimated until December 2012 and evaluated over the period January 2013 - April 2018. For each month within this forecast period, we then update the information set, reestimate the models, and generate new forecasts.

The root mean squared errors (RMSEs) of yields for selected maturities are presented in Table 3. Lower RMSE values mean better forecasts, and the best forecast at each horizon is highlighted in bold. The table displays the well-known difficulty of econometric models to beat the random walk forecast. Especially at the 1-month ahead horizon, the random walk almost always provides the best forecast over our evaluation period. As the forecast horizon becomes longer, however, the shadow rate model and the regime-switching model become more competitive. In particular, the regime-switching model becomes increasingly superior when the forecast horizon extends towards 2 years and beyond, and for medium-maturity yields.

The RMSEs tell us that, in spite of its heavier parameterization, the regime-switching model appears to display a competitive forecasting performance. To test this result more formally, Table 4 presents the results of a Diebold and Mariano (1995) test for equality of forecast accuracy of the regime-switching and the shadow rate models. Negative values indicate a superior performance of the regime-switching model. The point results of the test include both positive and negative values, although a majority are negative. The associated $p$-values (in parentheses) indicate that a number of these negative values are statistically significant. In fact, across all 36 forecast horizons and 9 maturities considered, we observe that in 40 cases the regime-switching model's performance is significantly superior at the 10% confidence level, while the shadow rate model is statistically superior in 12 cases. Consistently with the RMSFEs results, the performance of the regime-switching model is particularly strong for medium-term maturities (3 to 7 years).

Finally, we apply White's (2000) "reality check" test. This test examines whether the expected value of the forecast loss of a model is significantly greater than the forecast loss of a benchmark model. In other words, it tests for superior predictive ability rather than equal predictive ability.

[7] We use the unrestricted shadow-rate model as it performs substantially better than the restricted shadow-rate model in terms of out-of-sample forecasting.

We implement the test in two ways. First, we take our model as the benchmark and ask whether any of the two alternatives − the WX shadow rate model or the random walk − is able to produce forecasts that are significantly superior to our model across all 36 forecast horizons and each of the 9 bond maturities considered. In 98 out of the 324 cases we can reject the null hypothesis that the random walk model is not superior to the regime-switching model. These cases are mostly related to forecasts of bonds with very short maturities (up to 6 months). For none of the maturities and forecasting horizons we consider can we reject the null that the shadow rate model is not superior to the regime-switching model. We then turn the null hypothesis around and test for superior predictive ability of our model vis-à-vis each of the two alternatives. In 219 out of 324 cases we reject the null against the shadow rate model, meaning that for most of the combinations of maturities and forecast horizons considered here, the forecasting performance of our model is significantly better than this benchmark. In 142 cases we also reject the null against the random walk model. Looking at the results in more detail, we again see that our model is especially competitive at medium-term maturity yields. To illustrate these properties, Table 5 shows results for a selection of forecast horizons in the case where we test for superior forecast ability of our regime-switching model.

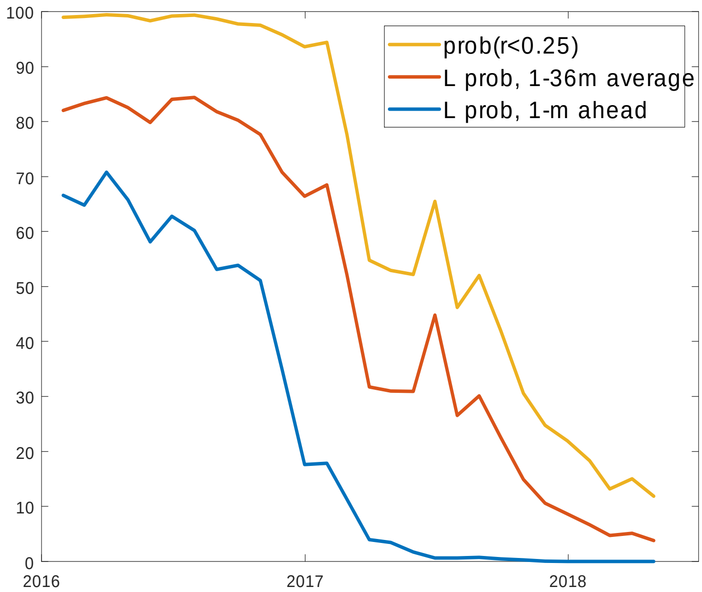
To summarize, in spite of its heavier parameterization the regime-switching model displays a competitive forecasting performance.

### 4.3 The risk of returning to the LB

Given its reasonably good forecasting performance, the regime-switching model can be useful to gauge the risk of a return to the LB regime at the end of the sample. This variable is likely to be of interest in the future, due to the persistently low level of all interest rates. It is particularly interesting in 2019 because of the prolonged nature of the U.S. economic expansion – see Nakata (2017) or Christensen (2019).

In a standard framework, the probability to hit the LB at some point in the future can be defined based on a simulation of future possible paths of the short-term interest rate, starting at a given point in time. The LB risk can then be measured as the fraction of the simulated paths in which the short rate reaches the LB level, say for three consecutive months, as suggested by Christensen (2019). This approach is followed to construct the yellow line in Figure 11, which counts the frequency of 3-year-ahead simulated paths in which the short-term interest rate is below 0.25% at least three months in a row. The resulting LB risk is extremely high

Figure 11: Probability of a binding ELB in the future




until early 2017, then it falls rapidly. However, even in mid-2018 the probability to touch the LB is larger than 10% – a level which appears high enough to be a reason for concern.

An obvious disadvantage of this approach is that it does not distinguish whether the LB episodes that it captures are persistent. Especially if interest rates remain at relatively low levels, it is natural to observe an increase in the number of times in which the short rate reaches the LB. This increased frequency of LB levels, however, need not imply an increased frequency of a binding LB. The short-term rate could just touch the LB for one quarter and then bounce back immediately.

Our model provides a natural way to assess the likelihood of a renewed, persistent LB episode. The LB risk can simply be defined as the probability to switch to the LB regime. As a result, occasional episodes in which the short rate touches the LB would not be counted as LB episodes. "True" LB episodes would only be episodes when the LB becomes a constraint and the economy switches to the LB regime.

The other lines in Figure 11 explore the implications of this different notion of LB risk. More specifically, the blue line measures the probability to switch to the LB in the next quarter. Given that policy rates are normally adjusted in small steps, this probability remains around 50% in 2016, when the federal funds rate hovers at very low levels. As the stance of monetary policy is tightened, however, the probability keeps falling and it becomes essentially zero in the second half of 2017, as the federal funds rate moves towards the 1% level from below.

Finally, the red line in Figure 11 measures the probability to switch to the LB at some point over a longer horizon. It counts the average of the 1-step-ahead to 36-step-ahead probabilities of switching to the LB regime over the next 3 years. Its time pattern is close to the one in the heuristic measure captured by the yellow line, but its level is lower over the whole period shown in the figure. Intuitively, even if occasional dips of the short rate to the LB level remain possible going forward, the probability that they correspond to a binding LB regime is much lower. In mid-2018 this is assessed to be below 5%. Its implications are therefore more benign than those of the heuristic measure.

## 5 Conclusions

We propose and analyze a dynamic term structure model with stochastic regime switches to deal with the lower bound on nominal interest rates. We allow for separate laws of motion for the evolution of the state vector in normal times and at the lower bound. State-dependent regime switching probabilities ensure that the likelihood of being in the lower bound regime increases as short rates fall closer to zero. A key advantage of this approach is that it allows for a very gradual normalization of monetary policy following a lower bound episode.

We apply our model to U.S. data and show that it does well in fitting key properties of yields at the lower bound. Moreover, after “lift-off”, the model is able to capture the slow pace of monetary policy normalization evident from the recent U.S. experience. In particular, the possibility to return to the lower bound regime continues exerting an influence in the early phases of normalization, pulling expected future rates downwards.

This mechanism also has implications for term premia. In particular, after exiting the lower bound, the regime-switching model implies lower average expected short-term interest rates than standard one-regime affine term structure model. As a result, the regime-switching model also implies higher term premia than alternative models, and notably risk premia that are not deeply negative.

In spite of its relatively heavy parameterization, the regime-switching model displays a competitive out-of-sample forecasting performance. The model’s ability to account for different state dynamics in and out of the lower bound period, coupled with its ability to allow for a high probability to switch back to the lower bound regime after having exited it, contributes to the model's reliable forecasting performance. The model also allows for a straightforward way to gauge the risk of a return to the lower bound regime in the future. In our application to U.S. data in mid-2018, it provides a more benign assessment than some alternative measures.

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## A1 An illustrative model

This section describes in some detail the illustrative model reported in Section 2. Recall the model equations

$$\pi_t = \beta x_t + \pi_{t-1}, \tag{12}$$

$$x_t = -\frac{1}{\gamma}\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t\right), \tag{13}$$

$$r_{t+1} = \rho r_t + \sigma \varepsilon_{t+1}, \tag{14}$$

$$i_t = \max\left(\alpha \pi_t, 0\right), \tag{15}$$

where $\pi_t$ is inflation, $x_t$ is a notion of output gap, $i_t$ is the short term nominal interest rate and $r_t$ is an exogenous variable. We impose the following parameter restrictions: $0 < \beta < 1$, $\gamma > 0$, $0 < \rho < 1$ and $\alpha > 1$.

To solve the model, start from the case where $i > 0$ and monetary policy follows the rule $i_t = \alpha \pi_t$. In this case, one can guess that the solution will be linear in the two state variables $r_t$ and $\pi_{t-1}$, or

$$\pi_t = A_\pi r_t + B_\pi \pi_{t-1},$$

$$x_t = A_x r_t + B_x \pi_{t-1},$$

$$i_t = A_i r_t + B_i \pi_{t-1},$$

for coefficients $A_\pi$, $A_x$, $A_i$, $B_\pi$, $B_x$ and $B_i$ to be determined.

Use the guesses in equations (12), (13) and (15) and use equation (14) to obtain

$$\left(A_\pi - \beta A_x\right) r_t = \left(\beta B_x + 1 - B_\pi\right) \pi_{t-1},$$

$$\left[\gamma A_x + A_i - A_\pi B_\pi - \left(1 + A_\pi \rho\right)\right] r_t = \left(B_\pi^2 - \gamma B_x - B_i\right) \pi_{t-1},$$

$$\left(A_i - \alpha A_\pi\right) r_t = \left(\alpha B_\pi - B_i\right) \pi_{t-1}.$$

These conditions give a system of six equations in the coefficient guesses. Since one equation is quadratic in $B_x$, we obtain six solutions:

$$B_x = \frac{1}{2}\frac{1}{\beta^2}\left[\gamma + \left(\alpha - 2\right)\beta \pm \sqrt{\left(\gamma + \alpha\beta\right)^2 - 4\beta\gamma}\right]$$

and

$$A_x = \frac{1}{\gamma + \left(\alpha - \rho\right)\beta - \beta\left(1 + \beta B_x\right)},$$

$$A_\pi = \beta A_x,$$

$$B_\pi = 1 + \beta B_x,$$

$$A_i = \alpha A_\pi,$$

$$B_i = \alpha B_\pi.$$

Note first that for $B_x$ to be real we need $(2\beta - \gamma - \alpha\beta)^2 \geq 4\beta^2(1-\alpha)$. Assuming $\alpha > 0$ this requires $\alpha > 2\sqrt{\frac{\gamma}{\beta}} - \frac{\gamma}{\beta}$. As a result, the law of motion of the two state variables is

$$\begin{bmatrix} \pi_t \\ r_{t+1} \end{bmatrix} = M_N \begin{bmatrix} \pi_{t-1} \\ r_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} \varepsilon_{t+1}$$

for

$$M_N \equiv \begin{bmatrix} \frac{1}{2}\frac{1}{\beta}\left(\gamma + \alpha\beta \pm \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}\right) & \frac{\beta}{\frac{1}{2}(\gamma+\alpha\beta) - \beta\rho \pm \frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}} \\ 0 & \rho \end{bmatrix}.$$

The eigenvalues of $M_N$ are $\frac{1}{2\beta}\left(\alpha\beta + \gamma \pm \sqrt{(\alpha\beta+\gamma)^2 - 4\beta\gamma}\right)$ and $\rho$. Note that both need to be less than 1 in modulus to ensure a stationary equilibrium. We have already assumed $\rho < 1$. For the other root, this condition requires

$$(\alpha\beta + \gamma) \pm \sqrt{(\alpha\beta+\gamma)^2 - 4\beta\gamma} < 2\beta.$$

Solving the associated equation $(\alpha\beta + \gamma) \pm \sqrt{(\alpha\beta+\gamma)^2 - 4\beta\gamma} = 2\beta$ for $\alpha$, we obtain that $\alpha = 1$, so that under our maintained assumption of $\alpha > 1$ the unique stationary eigenvalue is $\frac{1}{2\beta}\left(\alpha\beta + \gamma - \sqrt{(\alpha\beta+\gamma)^2 - 4\beta\gamma}\right)$.

To summarize, for the case when $i_t > 0$ we obtain

$$i_t = \frac{\alpha\beta}{\frac{1}{2}(\gamma+\alpha\beta) - \beta\rho - \frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}} r_t + \frac{1}{2}\frac{\alpha}{\beta}\left(\gamma + \alpha\beta - \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}\right)\pi_{t-1}$$

and

$$\begin{bmatrix} \pi_t \\ r_{t+1} \end{bmatrix} = M_N \begin{bmatrix} \pi_{t-1} \\ r_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} \varepsilon_{t+1}$$

for

$$M_N \equiv \begin{bmatrix} \frac{1}{2}\frac{1}{\beta}\left(\gamma + \alpha\beta - \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}\right) & \frac{\beta}{\frac{1}{2}(\gamma+\alpha\beta) - \beta\rho - \frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}} \\ 0 & \rho \end{bmatrix}.$$

For the ZLB, we rely on the logic in Guerrieri and Iacoviello (2015). More specifically, consider the case where the zero bound only binds in period $t$, but it is expected not to bind again in $t+1$. In this case, we know that $i_t = 0$. This value of the short rate can be used in the output gap equation, where we can also compute $\mathrm{E}_t\pi_{t+1}$ from the solution for the case in which the ZLB does not bind to compute

$$x_t = \frac{1}{2}\frac{1}{\beta\gamma}\left(\gamma + \alpha\beta - \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}\right)\pi_t + \frac{1}{\gamma}\frac{\frac{1}{2}(\gamma+\alpha\beta) - \frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}}{\frac{1}{2}(\gamma+\alpha\beta) - \beta\rho - \frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}} r_t.$$

This equation can be used in the inflation equation to finally obtain

$$\pi_t = \frac{\beta}{\frac{1}{2}(\gamma+\alpha\beta) - \beta\rho - \frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}}\frac{\gamma + \alpha\beta - \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}}{\gamma - \alpha\beta + \sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}} r_t + \frac{2}{1 - \frac{\alpha\beta}{\gamma} + \frac{1}{\gamma}\sqrt{(\gamma+\alpha\beta)^2 - 4\beta\gamma}}\pi_{t-1}$$

Hence for the period in which the ZLB does bind we have

$$M_Z = \begin{bmatrix} \frac{2\gamma}{\gamma-\alpha\beta+\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}} & \frac{\beta}{\frac{1}{2}(\gamma+\alpha\beta)-\beta\rho-\frac{1}{2}\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}}\frac{\gamma+\alpha\beta-\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}}{\gamma-\alpha\beta+\sqrt{(\gamma+\alpha\beta)^2-4\beta\gamma}} \\ 0 & \rho \end{bmatrix}.$$

As shown Guerrieri and Iacoviello (2015), the same logic can be applied if the ZLB is expected to bind for many periods.

## A2 Bond pricing using a log-linear approximation

Postulate that bond prices are exponentially affine in $X_t$

$$P_{t,n} = \exp\left(-A_n^j - B_n^j X_t\right).$$

Using the expression for bond prices and the stochastic discount factor, the no-arbitrage condition can be rewritten as

$$1 = \mathrm{E}_t\left[\exp\left(\begin{array}{c} -\delta_0^j - \delta_x' X_t - \frac{1}{2}\left(\Psi_t^j\right)'\Psi_t^j - \Gamma_t^{j,k} - A_{n-1}^k + A_n^j - B_{n-1}^k K_{X,0}^{\mathbb{Q}j} - \left[\left(\Psi_t^j\right)' + B_{n-1}^k \Sigma^j\right]\varepsilon_{t+1} \\ -\left(B_{n-1}^k K_{X,1}^{\mathbb{Q}j} - B_n^j\right) X_t \end{array}\right)\right].$$

Note that the independence between normal and regime-switching shocks implies that

$$\mathrm{E}_t\left[\exp\left(-\left[\left(\Psi_t^j\right)' + B_{n-1}^k\Sigma^j\right]\varepsilon_{t+1}\right)\right] = \sum_{k=1}^{S}\pi_t^{\mathbb{P}jk}\exp\left(\frac{1}{2}\left(\Psi_t^j\right)'\Psi_t^j + B_{n-1}^k\Sigma^j\Psi_t^j + \frac{1}{2}B_{n-1}^k\Sigma^j\Sigma^j\left(B_{n-1}^k\right)'\right),$$

so that the no-arbitrage condition becomes

$$1 = \sum_{k=1}^{S}\pi_t^{\mathbb{P}jk}\exp\left(-\delta_0^j - \delta_x' X_t - \Gamma_t^{j,k} - A_{n-1}^k + A_n^j - B_{n-1}^k K_{X,0}^{\mathbb{Q}j} + \frac{1}{2}B_{n-1}^k\Sigma^j\Sigma^j\left(B_{n-1}^k\right)' - \left(B_{n-1}^k K_{X,1}^{\mathbb{Q}j} - B_n^j\right) X_t\right)$$

Finally use the assumption i.e. $\Gamma_t^{j,k} = \log \gamma_t^{j,k} = \log \frac{\pi_t^{\mathbb{P}jk}}{\pi^{\mathbb{Q}jk}}$ to obtain

$$1 = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-\delta_0^j - \delta_x' X_t - A_{n-1}^k + A_n^j - B_{n-1}^k K_{X,0}^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k K_{X,1}^{\mathbb{Q}j} - B_n^j\right) X_t\right).$$

At this point, take a first order approximation. Note that the right hand side must be 1 in steady state. The exponential can therefore be approximated around 0, using $\exp z \simeq 1 + z$. It follows that

$$0 = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(-\delta_0^j - \delta_x' X_t - A_{n-1}^k + A_n^j - B_{n-1}^k K_{X,0}^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k K_{X,1}^{\mathbb{Q}j} - B_n^j\right) X_t\right).$$

These equations are satisfied for all maturities as long as the $A$ and $B$ matrices follow the recursions

$$\begin{aligned} A_n^j &= \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(\delta_0^j + A_{n-1}^k + B_{n-1}^k K_{X,0}^{\mathbb{Q}j} - \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)'\right), \\ B_n^j &= \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(\delta_x' + B_{n-1}^k K_{X,1}^{\mathbb{Q}j}\right), \end{aligned}$$

starting from

$$\begin{aligned} A_1^j &= \delta_0^j, \\ B_1 &= \delta_x'. \end{aligned}$$

To compute the accuracy of the above approximation, we compare it to the exact solution for a few maturities. Given the bond pricing equation

$$1 = \mathrm{E}_t \left[M_{t,t+1} \frac{B_{t+1,n-1}}{B_{t,n}}\right]$$

and the our assumptions on $\delta_0$, $\delta_x'$, $M_{t+1}$, $\Lambda_t^j$ and $\Gamma_{t,t+1}$, $n$-period bonds can be written exactly as

$$B_{t,n} = \sum_{k=S} \pi^{\mathbb{Q}s_t=j,s_{t+1}=k} \sum_{l=S} \pi^{\mathbb{Q}s_{t+1}=k,s_{t+2}=l} \ldots \sum_{z=S} \pi^{\mathbb{Q}s_{t+n}=y,s_{t+n+1}=z} \exp\left(-A_n^{j,k,l,m,\ldots,z} - B_n^{j,k,l,m,\ldots,z} X_t\right)$$

for

$$A_n^{j,k,l,m,\ldots,z} = \delta_0 + A_{n-1}^{j,k,l,m,\ldots,y} + B_{n-1}^{k,l,m,\ldots,y} K_{X,0}^{\mathbb{Q}j} - \frac{1}{2} B_{n-1}^{k,l,m,\ldots,y} \Sigma^j \Sigma^j \left(B_{n-1}^{k,l,m,\ldots,y}\right)'$$

and

$$B_n^{k,l,m,\ldots,z} = \delta_x' + B_{n-1}^{k,l,m,\ldots,y} K_{X,1}^{\mathbb{Q}j}$$

The above expression involves $2^{n-1}$ terms for a bond of maturity $n$, so it quickly becomes computationally intractable. We can however compute it quickly for bonds of up to 18-month maturity. The approximation error implied by the log-linear approximation for these maturities is always smaller than one tenth of a basis point.

## A3 Regime shift risk premia

Recall, for $Z_t = [X_t, M_t]'$, the $\mathbb{P}$-dynamics of the model can be written as

$$\begin{bmatrix} X_t \\ M_t \end{bmatrix} = \begin{bmatrix} K^{\mathbb{P}j}_{X,0} \\ K^{\mathbb{P}j}_{M,0} \end{bmatrix} + \begin{bmatrix} K^{\mathbb{P}j}_{XX,1} & K^{\mathbb{P}j}_{XM,1} \\ K^{\mathbb{P}j}_{MX,1} & K^{\mathbb{P}j}_{MM,1} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ M_{t-1} \end{bmatrix} + \begin{bmatrix} \Sigma^j_{XX} & \Sigma^j_{XM} \\ \Sigma^j_{MX} & \Sigma^j_{MM} \end{bmatrix} \begin{bmatrix} \varepsilon^{\mathbb{P}}_{X,t} \\ \varepsilon^{\mathbb{P}}_{M,t} \end{bmatrix}.$$

Note that in this setting, with unspanned factors, the expectation of the *priced* factors $X$ is

$$\begin{aligned} E^{\mathbb{P}}_t [X_{t+1}] &= K^{\mathbb{P}j}_{X,0} + \begin{bmatrix} K^{\mathbb{P}j}_{XX,1} & K^{\mathbb{P}j}_{XM,1} \end{bmatrix} Z_t \\ &\equiv K^{\mathbb{P}j}_{X,0} + K^{\mathbb{P}j}_{X,1} Z_t. \end{aligned}$$

Next, re-write the $\mathbb{Q}$-dynamics of $X$ in the following way:

$$\begin{aligned} [X_t] &= K^{\mathbb{Q}j}_{X,0} + \begin{bmatrix} K^{\mathbb{Q}j}_{X,1} & 0 \end{bmatrix} Z_{t-1} + \Sigma^j_X \varepsilon^{\mathbb{Q}}_{X,t} \\ &\equiv K^{\mathbb{Q}j}_{X,0} + \tilde{K}^{\mathbb{Q}j}_{X,1} Z_{t-1} + \Sigma^j_X \varepsilon^{\mathbb{Q}}_{X,t}. \end{aligned}$$

The conditional expectation of $X$ under $\mathbb{Q}$ is then

$$E^{\mathbb{Q}}_t [X_{t+1}] = K^{\mathbb{Q}j}_{X,0} + \tilde{K}^{\mathbb{Q}j}_{X,1} Z_t.$$

Conditional on the current regime $s_t = j$, the expected return on an $n$-period bond $P^j_{t,n}$ is

$$E^{\mathbb{P}}_t [\log (P_{t+1,n-1}) \mid s_t = j] - \log \left( P^j_{t,n} \right).$$

The current bond price is

$$P^j_{t,n} = \exp \left( -A^j_n - B^j_n X_t \right),$$

so the log price is

$$
\begin{aligned}
\log\left(P_{t,n}^{j}\right) &= \log E_t^{\mathbb{Q}}\left[\exp\left(-r_t^j\right) P_{t+1,n-1} \mid s_t = j\right] \\
&= -r_t^j + \log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} E_t^{\mathbb{Q}}\left[P_{t+1,n-1}^{k} \mid s_t = j\right]\right) \\
&= -r_t^j + \log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-A_{n-1}^k\right) E_t^{\mathbb{Q}}\left[\exp\left(-B_{n-1}^k X_{t+1}\right) \mid s_t = j\right]\right) \\
&= -r_t^j + \log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-A_{n-1}^k\right) \exp\left(-B_{n-1}^k\left(K_{X,0}^{\mathbb{Q}j} + \tilde{K}_{X,1}^{\mathbb{Q}j} Z_t\right)\right)\right. \\
&\quad \left. E_t^{\mathbb{Q}}\left[\exp\left(-B_{n-1}^k \varepsilon_{t+1}\right) \mid s_t = j\right]\right) \\
&= -r_t^j + \log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{Q}j} + \tilde{K}_{X,1}^{\mathbb{Q}j} Z_t\right)\right) \exp\left(\frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j B_{n-1}^{k\prime}\right)\right) \\
&= -r_t^j + \log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{Q}j} + \tilde{K}_{X,1}^{\mathbb{Q}j} Z_t\right) + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j B_{n-1}^{k\prime}\right)\right),
\end{aligned}
$$

where, for notational simplicity, $\Sigma^j = \Sigma_{XX}^j$. The expected value ($\mathbb{P}$) of the $t+1$ log price is

$$
\begin{aligned}
& E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1}\right) \mid s_t = j\right] \\
=& \sum_{k=1}^{S} \pi^{\mathbb{P}jk} E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1}^k\right) \mid s_t = j\right] \\
=& \sum_{k=1}^{S} \pi^{\mathbb{P}jk}\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right).
\end{aligned}
$$

The expected excess return is then

$$
\begin{aligned}
xr_{t,n} &\equiv E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1}\right) \mid s_t = j\right] - \log\left(P_{t,n}^j\right) - r_t^j \\
&= \sum_{k=1}^{S} \pi^{\mathbb{P}jk}\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right) - \\
&\quad \log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{Q}j} + \tilde{K}_{X,1}^{\mathbb{Q}j} Z_t\right) + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j B_{n-1}^{k\prime}\right)\right).
\end{aligned}
$$

Suppose factor risk is unpriced, $\Psi_t = 0$, so that $K_{X,0}^{\mathbb{Q}j} = K_{X,0}^{\mathbb{P}j}$ and $\tilde{K}_{X,1}^{\mathbb{Q}j} = K_{X,1}^{\mathbb{P}j}$, then, ignoring the Jensen's inequality term, we get

$$\begin{aligned} xr_{t,n|\Lambda_t=0} &= E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1|\Lambda_t=0}\right) \mid s_t = j\right] - \log\left(P_{t,n|\Lambda_t=0}^{j}\right) - r_t^j \\ &= \sum_{k=1}^{S} \pi^{\mathbb{P}jk}\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right) - \\ &\quad \log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right)\right). \end{aligned}$$

We can use this expression to measure the impact of priced regime shift risk in a world where factor risk is not priced. Only to the extent that regime shift risk is priced will $\pi^{\mathbb{Q}jk}$ differ from $\pi_t^{\mathbb{P}jk}$, resulting in a non-zero $xr_{t,n|\Psi_t=0}$ (apart from a Jensen's inequality term). To see this, use the approximation

$$\log\left(\sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right)\right) \approx \sum_{k=1}^{S} \pi^{\mathbb{Q}jk}\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right),$$

and ignore the Jensen's inequality term to get

$$\begin{aligned} xr_{t,n} &\approx \sum_{k=1}^{S} \pi^{\mathbb{P}jk}\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right) - \sum_{k=1}^{S} \pi^{\mathbb{Q}jk}\left(-A_{n-1}^k - B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right) \\ &= \sum_{k=1}^{S}\left(\pi^{\mathbb{Q}jk} - \pi^{\mathbb{P}jk}\right)\left(A_{n-1}^k + B_{n-1}^k\left(K_{X,0}^{\mathbb{P}j} + K_{X,1}^{\mathbb{P}j} Z_t\right)\right). \end{aligned}$$

**Table 1**: US yield autocorrelation coefficients before and during the lower bound period

| | pre-LB | | | LB | | |
|---|---|---|---|---|---|---|
| maturity | $L_{95}$ | corr | $U_{95}$ | $L_{95}$ | corr | $U_{95}$ |
| 1m | 0.987 | 0.990 | 0.992 | 0.798 | 0.865 | 0.910 |
| 3m | 0.990 | 0.992 | 0.994 | 0.854 | 0.903 | 0.936 |
| 6m | 0.989 | 0.992 | 0.994 | 0.867 | 0.912 | 0.942 |
| 1y | 0.988 | 0.990 | 0.993 | 0.823 | 0.882 | 0.922 |
| 2y | 0.984 | 0.987 | 0.990 | 0.830 | 0.887 | 0.925 |
| 3y | 0.982 | 0.986 | 0.989 | 0.849 | 0.900 | 0.934 |
| 5y | 0.980 | 0.984 | 0.988 | 0.879 | 0.920 | 0.947 |
| 7y | 0.980 | 0.984 | 0.988 | 0.896 | 0.932 | 0.955 |
| 10y | 0.980 | 0.984 | 0.988 | 0.909 | 0.940 | 0.961 |
| slope (10y-1m) | 0.957 | 0.966 | 0.974 | 0.908 | 0.939 | 0.960 |

Note: pre-LB period is January 1987 - November 2008. $L_{95}$ and $U_{95}$ are lower and upper bounds, respectively, for a 95 per cent confidence interval for each coefficient.

**Table 2**: Parameter estimates

For the state vector $Z_t = [X_t, M_t]'$, with spanned yield factors $X_t = [c_t, s_t, r_t]'$ (curvature, slope, short rate level) and unspanned macro factors $M_t = [\pi_t, g_t]'$ (CPI inflation and industrial production growth), the model is

$$\begin{aligned} Z_t &= K_{Z,0}^{\mathbb{P}j} + K_{Z,1}^{\mathbb{P}j} Z_{t-1} + \Sigma_Z^j \varepsilon_{Z,t}^{\mathbb{P}}, \\ X_t &= K_{X,0}^{\mathbb{Q}j} + K_{X,1}^{\mathbb{Q}j} X_{t-1} + \Sigma_X^j \varepsilon_{X,t}^{\mathbb{Q}}, \\ r_t &= \delta_0 + \delta_x' X_t, \end{aligned}$$

for regimes $j = N, L$, $\mathbb{P}$ transition probabilities $\pi_t^{\mathbb{P},jk}$, and $\mathbb{Q}$ transition probabilities $\pi^{\mathbb{Q},jk}$. Imperfectly observed yields are subject to random pricing errors with covariance matrix $\Sigma_e \Sigma_e'$, where $\Sigma_e$ is diagonal with identical elements $\sigma_e$.The $\mathbb{Q}$ distribution of $X_t$ is fully characterized by the parameters $\Theta^{\mathbb{Q}j} = \left(k_\infty^{\mathbb{Q}j}, \lambda^{\mathbb{Q}j}, \Sigma_X^j\right)$ , where $k_\infty^{\mathbb{Q}j}$ is proportional to the risk-neutral long-run mean of the short rate in regime $j$ and $\lambda^{\mathbb{Q}j}$ is the vector of eigenvalues of $K_1^{\mathbb{Q}j}$. Figures in parentheses are asymptotic standard errors.

Panel A: $\mathbb{Q}$ parameter estimates

| parameter | $N$ regime | $L$ regime |
|---|---|---|
| $\lambda_1^{\mathbb{Q}j}$ | 0.994 (0.026) | 1.018 (0.017) |
| $\lambda_2^{\mathbb{Q}j}$ | 0.897 (0.000) | 1.017 (0.021) |
| $\lambda_3^{\mathbb{Q}j}$ | 0.896 (0.053) | 0.984 (0.000) |
| $k_\infty^{\mathbb{Q}j} \times 100$ | 0.035 | 0.014 |

$$\pi^{\mathbb{Q},jk} = \begin{array}{c} \\ j=N \\ j=L \end{array} \begin{array}{c} \quad k=N \qquad\quad k=L \\ \left[\begin{array}{cc} \underset{(0.053)}{0.998} & 1-\pi^{\mathbb{Q},NN} \\ 1-\pi^{\mathbb{Q},LL} & \underset{(0.006)}{0.959} \end{array}\right] \end{array}$$

$$\sigma_e \times 100 = \underset{(0.069)}{0.080}$$

Panel B: $\mathbb{P}$ parameter estimates

$$100 \cdot \left(I - \hat{K}^{\mathbb{P}N}_{Z,1}\right)^{-1} \hat{K}^{\mathbb{P}N}_{Z,0} = \begin{bmatrix} \underset{(0.023)}{0.385} & \underset{(0.024)}{1.160} & \underset{(0.020)}{4.465} & \underset{(0.011)}{2.741} & \underset{(0.045)}{2.671} \end{bmatrix}'$$

$$100 \cdot \left(I - \hat{K}^{\mathbb{P}L}_{Z,1}\right)^{-1} \hat{K}^{\mathbb{P}L}_{Z,0} = \begin{bmatrix} \underset{(0.015)}{1.019} & \underset{(0.023)}{2.475} & \underset{(0.005)}{0.136} & \underset{(0.030)}{1.094} & \underset{(0.116)}{1.086} \end{bmatrix}'$$

$$\hat{K}^{\mathbb{P}N}_{Z,1} = \begin{bmatrix} \underset{(0.041)}{0.812} & \underset{(0.026)}{-0.019} & \underset{(0.021)}{-0.010} & \underset{(0.032)}{0.019} & \underset{(0.010)}{0.004} \\ \underset{(0.037)}{0.191} & \underset{(0.023)}{1.002} & \underset{(0.019)}{0.028} & \underset{(0.029)}{-0.049} & \underset{(0.009)}{-0.028} \\ \underset{(0.032)}{-0.153} & \underset{(0.020)}{-0.015} & \underset{(0.016)}{0.963} & \underset{(0.024)}{0.042} & \underset{(0.007)}{0.028} \\ \underset{(0.035)}{0.026} & \underset{(0.022)}{0.031} & \underset{(0.018)}{0.044} & \underset{(0.027)}{0.908} & \underset{(0.008)}{0.007} \\ \underset{(0.074)}{-0.268} & \underset{(0.046)}{0.136} & \underset{(0.037)}{0.096} & \underset{(0.057)}{-0.208} & \underset{(0.017)}{0.932} \end{bmatrix}$$

$$\hat{K}^{\mathbb{P}L}_{Z,1} = \begin{bmatrix} \underset{(0.063)}{0.858} & \underset{(0.037)}{0.049} & 0 & \underset{(0.027)}{0.000} & \underset{(0.005)}{0.006} \\ \underset{(0.075)}{0.137} & \underset{(0.044)}{0.875} & 0 & \underset{(0.031)}{-0.066} & \underset{(0.006)}{-0.004} \\ 0 & 0 & 0 & 0 & 0 \\ \underset{(0.128)}{0.127} & \underset{(0.075)}{0.103} & 0 & \underset{(0.054)}{0.883} & \underset{(0.011)}{0.018} \\ \underset{(0.333)}{-0.701} & \underset{(0.194)}{0.701} & 0 & \underset{(0.140)}{0.247} & \underset{(0.028)}{0.971} \end{bmatrix}$$

$$100 \cdot \Sigma^N = \begin{bmatrix} \underset{(0.002)}{0.354} & & & & \\ \underset{(0.001)}{-0.219} & \underset{(0.001)}{0.236} & & & \\ \underset{(0.001)}{0.061} & \underset{(0.001)}{-0.172} & \underset{(0.001)}{0.203} & & \\ \underset{(0.001)}{-0.008} & \underset{(0.001)}{0.002} & \underset{(0.001)}{0.047} & \underset{(0.001)}{0.300} & \\ \underset{(0.002)}{-0.079} & \underset{(0.002)}{-0.023} & \underset{(0.002)}{0.029} & \underset{(0.002)}{-0.028} & \underset{(0.006)}{0.627} \end{bmatrix}$$

$$100 \cdot \Sigma^L = \begin{bmatrix} \underset{(0.001)}{0.206} & & & & \\ \underset{(0.001)}{0.030} & \underset{(0.001)}{0.241} & & & \\ 0 & 0 & \underset{(0.000)}{0.042} & & \\ \underset{(0.001)}{0.062} & \underset{(0.001)}{0.110} & 0 & \underset{(0.002)}{0.397} & \\ \underset{(0.002)}{0.004} & \underset{(0.003)}{0.110} & 0 & \underset{(0.005)}{0.197} & \underset{(0.017)}{1.064} \end{bmatrix}$$

Note: Figures in parentheses are asymptotic standard errors.

**Table 3**: Out-of-sample yield RMSFEs (in p.p.)

| | 1-month | 3-month | 6-month | 1-year | 2-year | 3-year | 5-year | 7-year | 10-year |
|---|---|---|---|---|---|---|---|---|---|
| Regime-switching model | | | | | | | | | |
| 1m ahead | 0.189 | 0.186 | 0.187 | 0.219 | 0.215 | 0.235 | 0.294 | 0.295 | 0.259 |
| 6m ahead | 0.313 | 0.332 | 0.329 | 0.359 | 0.351 | 0.365 | **0.389** | **0.395** | **0.433** |
| 12m ahead | 0.501 | 0.513 | 0.519 | 0.537 | 0.479 | **0.442** | **0.414** | **0.442** | 0.574 |
| 24m ahead | 0.686 | 0.700 | **0.704** | **0.708** | **0.628** | **0.542** | **0.424** | **0.473** | 0.735 |
| 36m ahead | **0.444** | **0.456** | **0.454** | **0.479** | **0.490** | **0.486** | **0.494** | 0.596 | 0.872 |
| Shadow rate model | | | | | | | | | |
| 1m ahead | 0.119 | 0.110 | 0.104 | 0.124 | 0.149 | 0.164 | 0.191 | **0.203** | **0.199** |
| 6m ahead | **0.154** | **0.155** | **0.186** | **0.278** | 0.344 | 0.382 | 0.434 | 0.465 | 0.496 |
| 12m ahead | **0.350** | **0.377** | **0.417** | 0.467 | 0.495 | 0.509 | 0.528 | 0.540 | 0.580 |
| 24m ahead | 0.834 | 0.824 | 0.811 | 0.792 | 0.754 | 0.735 | 0.722 | 0.741 | 0.836 |
| 36m ahead | 0.687 | 0.687 | 0.579 | 0.516 | 0.541 | 0.592 | 0.695 | 0.812 | 1.009 |
| Random walk | | | | | | | | | |
| 1m ahead | **0.057** | **0.053** | **0.058** | **0.084** | **0.115** | **0.148** | **0.190** | 0.204 | 0.201 |
| 6m ahead | 0.227 | 0.232 | 0.243 | 0.294 | **0.320** | **0.355** | 0.420 | 0.461 | 0.487 |
| 12m ahead | 0.403 | 0.415 | 0.430 | **0.466** | **0.443** | 0.443 | 0.481 | 0.513 | **0.546** |
| 24m ahead | **0.657** | **0.682** | 0.706 | 0.742 | 0.696 | 0.644 | 0.591 | 0.582 | **0.627** |
| 36m ahead | 0.872 | 0.917 | 0.964 | 1.067 | 0.967 | 0.809 | 0.576 | **0.518** | **0.627** |

Yield root-mean-squared-forecast-errors over Jan. 2013 - Apr. 2018 for (i) the regime switching model; (2) the WX shadow rate model; and (3) the random walk. The lowest RMSFE for each forecast horizon is highlighted in bold.

**Table 4**: Diebold-Mariano tests: regime-switching model vs shadow rate model

| | 1mth | 3mth | 6mth | 1yr | 2yr | 3yr | 5yr | 7yr | 10yr |
|---|---|---|---|---|---|---|---|---|---|
| 1m ahead | 2.274 (0.026) | 2.706 (0.009) | 3.247 (0.002) | 4.331 (0.001) | 2.678 (0.010) | 2.209 (0.031) | 2.977 (0.004) | 2.854 (0.006) | 2.358 (0.022) |
| 6m ahead | 1.120 (0.267) | 1.233 (0.223) | 1.027 (0.309) | 0.744 (0.460) | 0.066 (0.947) | −0.277 (0.783) | −1.122 (0.267) | −1.768 (0.082) | −1.241 (0.220) |
| 12m ahead | 0.521 (0.605) | 0.476 (0.636) | 0.367 (0.715) | 0.291 (0.772) | −0.092 (0.927) | −0.502 (0.618) | −2.139 (0.037) | −0.937 (0.353) | −0.035 (0.972) |
| 18m ahead | −0.024 (0.981) | −0.045 (0.964) | −0.141 (0.888) | −0.286 (0.776) | −1.573 (0.123) | −2.717 (0.009) | −3.558 (0.001) | −2.941 (0.005) | −2.153 (0.037) |
| 24m ahead | −0.528 (0.601) | −0.464 (0.645) | −0.424 (0.674) | −0.374 (0.710) | −0.701 (0.487) | −1.247 (0.220) | −2.077 (0.044) | −2.003 (0.052) | −0.784 (0.438) |
| 30m ahead | −0.222 (0.826) | −0.179 (0.859) | −0.138 (0.891) | −0.087 (0.931) | −0.122 (0.904) | −0.140 (0.890) | −0.297 (0.769) | −0.800 (0.429) | −0.437 (0.665) |
| 36m ahead | −0.066 (0.948) | −0.074 (0.942) | −0.054 (0.957) | −0.015 (0.988) | −0.015 (0.988) | −0.045 (0.964) | −0.168 (0.868) | −0.178 (0.860) | −0.098 (0.923) |

The table displays out-of-sample Diebold-Mariano (1995) forecast evaluation test statistics. The test's null hypothesis is that there is no difference in the forecasting accuracy of the regime switching model and the WX shadow rate model, where accuracy is measured in terms of squared forecast errors. The test is specified in terms of the mean of a loss differential defined as the squared forecast errors of the regime switching model minus the squared errors of the shadow rate model. Negative test statistics indicate that the regime switching model performs better than the shadow rate model in terms of this metric, while positive values indicate the opposite. Values in parenthesis are $p$-values for the null hypothesis. The forecast evaluation period is January 2013 - April 2018.

**Table 5**: Tests for superior out-of-sample predictive ability of the RS model

| | SR vs RS | | | | |
|---|---|---|---|---|---|
| | 1m rate | 1y yield | 3y yield | 5y yield | 10y yield |
| 1 month ahead | **0.103** | −0.071 | **0.042** | −0.023 | 0.001 |
| 6 months ahead | **0.037** | **0.194** | **0.292** | **0.323** | **0.559** |
| 12 months ahead | **0.585** | **0.903** | **1.002** | **0.994** | **0.585** |
| 24 months ahead | 3.500 | 2.060 | **2.194** | **2.272** | 1.288 |
| 36 months ahead | 1.456 | 0.199 | 0.607 | **1.264** | 1.363 |
| | RW vs RS | | | | |
| | 1m rate | 1y yield | 3y yield | 5y yield | 10y yield |
| 1 month ahead | −0.013 | −0.179 | −0.015 | −0.004 | 0.017 |
| 6 months ahead | −0.076 | −0.064 | **0.092** | **0.323** | **0.470** |
| 12 months ahead | −0.273 | 0.090 | **0.307** | **0.561** | 0.035 |
| 24 months ahead | −0.519 | **0.556** | **0.371** | **0.305** | −1.319 |
| 36 months ahead | **2.980** | **4.815** | **2.218** | 0.461 | −1.946 |

The table shows test statistics for superior ability of the regime switching model ("RS") compared to the WX shadow rate model (top panel, "SR") and compared to the random walk (bottom panel, "RW"), calculated according to White's (2000) "reality check". We use a squared forecast error loss function when implementing the test. The null hypothesis is that the expected differential between the forecast loss of the benchmark and that of the RS model is smaller than or equal to zero. Bold figures denote rejection of the null at the 10 per cent level, based on $p$-values generated by the stationary bootstrap approach, with 50,000 resamples of the loss differential series (using a smoothing parameter of 1/12). The forecast evaluation period is January 2013 - April 2018.

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