Event risk, contingent claims, and the temporal resolution of uncertainty*

Pierre Collin Dufresne[a,b,c] and Julien Hugonnier[a,b]

[a]Swiss Finance Institute
[b]Ecole Polytechnique Fédérale de Lausanne
[c]NBER

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Abstract

We study the pricing and hedging of contingent claims that are subject to *Event Risk* which we define as rare and unpredictable events whose occurrence may be correlated to, but cannot be hedged perfectly with standard marketed instruments. The super-replication costs of such event sensitive contingent claims (ESCC), in general, provide little guidance for the pricing of these claims. Instead, we study utility based prices under two scenarios of resolution of uncertainty for event risk: when the event is continuously monitored, or when it is revealed only the payment date. In both cases, we transform the incomplete market optimal portfolio choice problem of an agent endowed with an ESCC into a complete market problem with a state and possibly path dependent utility function. For negative exponential utility, we obtain an explicit representation of the utility based prices under both information resolution scenarios and this in turn leads us to a simple characterization of the early resolution premium. For CRRA utility functions we propose a simple numerical scheme and study the impact of size of the position, wealth and expected return on these prices.

Keywords. Event risk, incomplete markets, utility based valuation, certainty equivalent, temporal resolution of uncertainty, credit risk.

AMS Subject Classification (1991). 60H30, 90A09, 90A10, 93E20.

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Introduction

We study the pricing and hedging of contingent claims that are subject to *Event Risk*. We define event risk as rare and unpredictable events whose occurrence may be correlated with standard marketed securities. Typical examples of such events are default, natural catastrophies, death, prepayment of mortgages. Thus our analysis has implications for the pricing of a wide range of securities such as credit derivatives, vulnerable derivatives, Mortgage Backed Securities and catastrophe, life or unemployment insurance.

We assume that event risk is not hedgeable using market instruments. Formally we model the occurence of such events as the first jump of a point process whose intensity depends only on the information generated by marketed securities (e.g. stocks and bonds). We model the latter using a standard Brownian motion model of financial security markets (as presented in Karatzas and Shreve (1999)). Thus even though in our model the underlying market for traded securities is complete, contingent claims with event sensitive payoff (ESCC) are not perfectly hedgeable. In fact, absence of arbitrage only requires prices of such claims to fall within an arbitrage-free interval, whose lower and upper bound correspond to the sub-replication cost (lower hedging price) and the super-replication cost (upper hedging price) respectively (e.g., Karatzas and Shreve (1999)). We show that for ESCCs these upper and lower hedging prices are uninformative. Consider for example an event digital, that pays one dollar if the event occurs the terminal time and nothing otherwise. The lower hedging price is zero and the upper hedging price is that of a risk free zero bond. A more general result along these lines is provided below. It demonstrates that pure arbitrage arguments, in general, provide little guidance for the pricing of ESCCs.* We thus study an alternative pricing rule that embeds the pricing problem into the agent's global portfolio/consumption decision: utility-based pricing. Consider an agent who commits to sell an ESCC and chooses his optimal portfolio to maximize his expected utility of terminal wealth. The *utility-based selling price* of the contingent claim is defined as the smallest amount, which, when added to his initial capital, allows him to achieve at least as high a level of expected utility as he would have obtained without selling the claim.†

*This is reminiscent of Cvitanic, Shreve and Soner (1995) who show that, in the presence of transaction costs, the minimal amount necessary to dominate the pay-off of a call option is the value of the underlying itself. Other papers studying almost sure hedging in incomplete markets, albeit with different forms of incompleteness, include Cvitaniȅ and Karatzas (1993), El Karoui and Quenez (1995), Kramkov (1996) and Karatzas and Kou (1996).

†The *utility-based buying price* is defined similarly. *Utility-based pricing* has been used in the literature on transaction costs by Dumas and Luciano (1991), Hodges and Neuberger (1989), Davis and Panas (1994) and Davis et al. (1993) among others; and on incomplete/constrained markets by El Karoui and Rouge (2000), Delbaen et al. (2002), Henderson and Hobson (2002) and Malamud et al. (2013). See Henderson and Hobson (2009) for a survey.

Frequently, in real-world applications, the payment date does not coincide with the event date: the event usually happens prior to the payment date and investors do not monitor the event continuously, but it is revealed only at the payment date. For example, when investors purchase Principal Only (PO) or Interest Only (IO) Mortgage-Backed securities, the payments are contingent on the occurrence of prepayments prior to the payment date. While pool managers presumably monitor prepayments continuously, investors, in general, do not. To investigate the impact of the temporal resolution of uncertainty on the pricing of ESCC, we thus consider two alternative timing scenarios for event uncertainty. Under 'early resolution' the investor has continuous access to information about the event, i.e sees the event when it happens, whereas under 'late' resolution of uncertainty, the investor learns only at the payment date, whether the event has occurred previously or not.

Our results show that the temporal resolution of uncertainty has no impact on the arbitrage bounds and are therefore consistent with Ross (1989) who shows that in complete markets, the temporal resolution of uncertainty does not affect prices. However, early resolution does affect the utility-based prices because it allows the investor to re-optimize his portfolio at the event time. Such an impact of the temporal resolution of uncertainty on preference dependent prices was also brought to light, albeit in different models, by Robishek and Myers (1966) and Epstein and Turnbull (1980). We show that, for both information resolution scenarios, the incomplete market problem faced by an agent endowed with an ESCC can be recast into an equivalent complete market problem where the agent is endowed with a different, state and, for the early resolution case, path dependent utility function. The path-dependence of the modified utility function in the early resolution case, captures the uncertainty due to the non-hedgeable event risk. It basically acts as an endogenous liquidity constraint on the agent who, at any time, needs to be prepared to absorb the wealth impact of an event. In effect, the early release of information affects the dynamic trading strategy and thus the future wealth of the agent in a way that cannot be offset prior to the event (because markets are incomplete with respect to event risk). In this framework the agent is always better off with early resolution of uncertainty and is thus willing to pay a premium for early resolution. This implies, for example, that if we assume that mortgage prepayments are (at least partially) unhedgeable, pool managers could sell information about prepayments to their clients prior to the payment date.

To determine the magnitude of the early resolution premium and obtain some comparative statics on early and late resolution utility-based prices, we study two special cases of utility functions, namely negative exponential (CARA) and power (CRRA) utility functions. In the exponential case we obtain explicit representations for the utility-based prices under both resolution scenarios. Under early resolution of uncertainty, the utility-based prices are solutions to a nonlinear recursive equation whereas under late resolution, the utility based-prices are given by the present value of an event insensitive contingent claim which pays the certainty equivalent of the ESCC conditional on the market filtration. In both cases, utility-based prices are independent of wealth, a special feature of the negative exponential utility function. While wealth effects are absent for CARA agents, the early resolution premium is nevertheless positive due to the impact of the event on the hedging demand induced by the ESCC. Thus, sufficient conditions for CARA investors to be indifferent to the temporal resolution is that both the payoff of the ESCC and the probability of the event are independent of the market filtration. Note that similar results do not hold for typical utility functions such as power utility function, since holding the ESCC induces wealth effects for the investor.

In the CRRA case and under Markovian assumptions on the financial market model, we propose a simple numerical technique to compute both early and late resolution prices and provide numerical applications for several typical examples including defaultable bonds, vulnerable derivatives[‡] and credit derivatives. We study the impact of size of the position, wealth, expected return and volatility on utility based prices. Our results show that standard contingent claim pricing intuition does not apply in the presence of event risk. For example, vulnerable put option buy prices are decreasing in the drift of the underlying stock, while vulnerable call option buy price may be either increasing or decreasing depending on the moneyness of the option.

We find that for CRRA investors, the early resolution premium is typically very small and nonlinear in wealth. It vanishes both at zero where all prices, early and late resolution, approach the arbitrage bounds and at infinity where both prices approach the *'risk-neutral'* price which is obtained by risk adjusting the marketed sources of risk while keeping the same functional form for the intensity. Note that since it may be a function of traded assets, the probability of an event will, in general, be different under this risk neutral measure. As was shown in Collin Dufresne and Hugonnier (1999), this risk neutral price depends neither on the investor's initial capital nor utility function and agrees with the common limit of (i) the utility-based buying and selling prices as initial wealth increases and (ii) the utility-based unit prices as the size of the position decreases (under both resolution).[§] Our numerical results suggest that utility based prices converge

[‡]Vulnerable options have been studied by Johnson and Stulz (1987) and Hull and White (1993). However, in these papers the default event is modeled as the first passage time of the writer's assets value (assumed to be a diffusion) at some boundary. Defaultable contingent claims thus become default free knock-out barrier options and since every Brownian stopping time is predictable, default of the writer is a predictable process, i.e. does not come as a surprise.

[§]This price was also central to the analysis of Jarrow, Lando and Yu (2000) who obtain it as a result of an APT-like diversification argument quickly as a function of exposure (in percentage of wealth) to the risk neutral price and therefore provide some rational for the approach adopted by many practitioners, who (i) price and hedge credit derivatives using historical measure default probabilities estimates, and (ii) impose size based limits on the exposures to specific counterparties (Crouhy et al. (2000)).

In contemporaneous work, Liu, Longstaff and Pan (2001) also investigate Event Risk. However, their definition of Event risk is different from ours. In their paper, events are jumps in the underlying traded asset prices and they focus on optimal portfolio choice. Assuming an affine structure and CRRA utility they obtain closed-form solutions for the portfolio choice problem. In contrast, we define events as extraneous to the underlying traded security market. Our focus is on the incompleteness generated by such extraneous events and on the pricing of securities with payoff that are either explicitly (e.g. insurance) or implicitly (e.g. vulnerable derivatives) contingent on the event.

Collin Dufresne and Hugonnier (2007) is closest related to our work. They study the existence and qualitatite properties of utility based prices in the presence of extraneous event risk for general utility functions. Instead, we focus here on explicit computation and characterization of the utility based prices for a specific type of event sensitive contingent claims. We also investigate the relevance of temporal resolution of uncertainty for such claims. Duffie, Schroder and Skiadas (1996) analyze a reduced-form model of default risk and show that resolution of uncertainty may affect the prices of defaultable securities if the recovery process is such that the price solves a nonlinear recursive equation. Their result has some relation to our treatment of the CARA case, except that in our model the non-linearity is due to risk-aversion (in a sense endogenous), rather than to the assumed recovery scenario. Other related literature include the reduced-from, intensity-based models of default risk among which Jarrow, Lando and Turnbull (1992), Jarrow and Turnbull (1995) and Duffie and Singleton (1999). All use point processes to model the default event, but focus mainly on pricing defaultable bonds, taking the default intensity under some equivalent martingale measure as given (i.e. avoiding the question of market incompleteness and selection of a martingale measure).

The remaining of the paper is organized as follows. First we present the framework and recall some results on almost sure hedging. Then we give the reformulation of the incomplete market portfolio choice problem which allow to solve for Utility based prices of ESCC. Section two presents the continuous revelation of uncertainty, whereas section three presents the late resolution case. Section four applies our results to negative exponential utility and section five to power utility function. We conclude in section six.

1 The Economy

1.1 Information Structure

We consider a continuous time financial market model on the finite time span $[0, T]$. The uncertainty is represented by a probability space $(\Omega, \mathbb{H}, \mathcal{H}, P)$ on which are defined an $n-$dimensional standard Brownian motion B and a point process N. The Brownian motions represent innovations in traded securities prices while the point process models events whose occurrence may be influenced (through the intensity of the point process), but not completely determined, by market factors. The filtration

$$\mathbb{H} \triangleq \{\mathcal{H}(t) : t \in [0, T]\} \tag{1}$$

is the usual augmentation of the filtration generated by (B, N) and we assume that $\mathcal{H} = \mathcal{H}(T)$ so that the true state of nature is entirely determined by the paths of the process (B, N) up to the terminal time of the financial market. The usual augmentation of the natural filtration generated by B will be our reference filtration. It is denoted by $\mathbb{F} := \{\mathcal{F}(t)\}$ and we let $\mathcal{F} := \mathcal{F}(T)$.

All statements involving random variables and/or processes are understood to hold either almost surely or almost everywhere on $[0, T] \times \Omega$ depending on the context. In what follows, the point process will be assumed to admit a bounded $\mathbb{F}-$predictable *intensity process* under the objective measure. In other words, we shall from now on assume that there is a strictly positive $\mathbb{F}-$predictable and bounded process λ such that the compensated sum of jumps

$$M(t) \triangleq \left(N(t) - \int_0^t \lambda(s)ds\right) \tag{2}$$

is a uniformly integrable $(\mathbb{H}, P)-$martingale. Such a point process is sometimes referred to as a Cox Process or a doubly stochastic point process.

1.2 Traded Securities

There is a single perishable good (the numéraire) in units of which all quantities are expressed. The financial market, denoted by $\mathbb{M}$, consists in $n + 1$ long lived securities. The first security is locally riskless and pays no dividends. It is referred to as the *bank account* and its price S_0 has dynamics

$$dS_0(t) = r(t)S_0(t)dt, \qquad S_0(0) = 1 \tag{3}$$

for some non negative *interest rate* process r. The remaining n securities are risky. We shall refer to them as the *stocks* and assume that the vector S of stock price processes obeys the linear stochastic differential equation

$$dS(t) = \operatorname{diag}(S(t))\left(a(t)dt + \sigma(t)dB(t)\right), \qquad S(0) \triangleq S \in (0,\infty)^n \tag{4}$$

for some vector valued *appreciation rate* process a and some $(n \times n)-$matrix valued *volatility* process σ.

Assumption 1 *The coefficients are assumed to be* $\mathbb{F}-$*progressively measurable and uniformly bounded processes and furthermore, the volatility process* σ *is assumed to be invertible almost everywhere.*

The conditions imposed on the coefficients of the model by Assumption 1 imply that the *risk premium* process

$$\xi(t) \triangleq (\sigma(t))^{-1}\left(-r(t)\underline{1} + a(t)\right) \tag{5}$$

where $\underline{1} = (1,...,1)$ denotes a unit vector in $\mathbb{R}^n$, is uniformly bounded. The non negative exponential local martingale Z_0 defined by

$$Z_0(t) \triangleq \exp\left(-\int_0^t \xi(s)^* dB(s) - \frac{1}{2}\int_0^t \|\xi(s)\|^2 ds\right) \tag{6}$$

where $\|\cdot\|$ denotes the Euclidean norm, is thus a strictly positive and uniformly integrable $(\mathbb{H}, P)-$martingale and the formula $P^0(A) := E[1_A Z_0(T)]$ defines a probability measure which is equivalent to the objective probability measure and under which the process

$$W(t) \triangleq B(t) + \int_0^t \xi(s)ds \tag{7}$$

is an $n-$dimensional standard $\mathbb{H}-$Brownian motion by Girsanov theorem. Under this new probability measure, the stock price dynamics are given by

$$dS(t) = \operatorname{diag}(S(t))\left(r(t)\underline{1}\,dt + \sigma(t)dW(t)\right).$$

The discounted stock price process S/S_0 is thus an $(\mathbb{H}, P^0)-$martingale and this justifies the fact that the probability measure P^0 is referred to as a *risk neutral* or *equivalent martingale measure* for the financial market model.

Remark 2 Under our assumptions, the process Z_0 is not only an $\mathbb{H}-$martingale but also an $\mathbb{F}-$martingale under P so that P^0 is not only equivalent to P on $\mathcal{H}$ but also on $\mathcal{F}$ and the process W is both an $\mathbb{H}$ and an $\mathbb{F}-$Brownian motion under P^0. As a result, if the information available to agents was restricted to that generated by asset prices then P^0 would be the unique equivalent martingale measure and markets would be complete with respect to $\mathbb{F}$ (see Lemma 5 below). On the contrary, when dealing with the finer information structure $\mathbb{H}$ the measure P^0 *no longer constitutes the unique equivalent martingale measure* and markets are in general incomplete with respect to the enlarged filtration $\mathbb{H}$.

1.3 Event Sensitive Contingent Claims

In addition to being able to trade based on the extra information represented by the point process, we assume that agents can trade event sensitive contingent claims (ESCCs). These specify payoffs contingent on a non market event whose occurrence we model as the first jump time

$$\tau \triangleq \inf \{t \in [0,T] : N(t) \neq 0\} \tag{8}$$

of the point process. Let $p \in (1, \infty)$ be a fixed constant throughout the paper, introduce the space of (equivalence classes of) random variables

$$L \triangleq \mathbf{L}^p \left(\Omega, \mathcal{H}, P^0\right) \equiv \left\{X \text{ is } \mathcal{H}\text{−measurable with } E^0|X|^p < \infty\right\} \tag{9}$$

and denote by $\mathcal{E}$ the set of càdlàg semimartingales e whose bilateral supremum lies in the space L. We then define an ESCC as follows:

Definition 3 *An event sensitive contingent claim is defined by a pair $(A, R(\cdot))$ where A is a non negative $\mathcal{F}$−measurable random variable in L which specifies the payoff at time T on the set $\{\tau > T\}$ and $R(\cdot)$ is a non negative $\mathbb{F}$−adapted process in $\mathcal{E}$ which specifies the payoff at time τ on the set $\{\tau \leq T\}$.*

As is easily seen, the overall cumulative income process associated with a long position in an arbitrary ESCC is the càdlàg semimartingale $e \in \mathcal{E}$ defined by

$$e(t) \triangleq 1_{\{t \geq T\} \cap \{\tau > T\}} A + 1_{\{\tau \leq t\}} R(\tau). \tag{10}$$

ESCCs specify payoffs that are measurable with respect to the market filtration $\mathbb{F}$ but whose actual payment is conditional on the realization of a non market event modeled by an unpredictable stopping time τ. This definition encompasses a variety of event sensitive contingent claims such as:

(a) *Defaultable bonds.* A zero coupon bond with face value \$1 and maturity T issued by a firm which may default at time τ corresponds to $A = 1$ and either $R(t) = (1 - \delta)$

or ($D_0 := 1/S_0$ is the discount factor)

$$R(t) = (1-\delta) \cdot E^0 \left(\tfrac{D_0(T)}{D_0(t)} \middle| \mathcal{F}(t) \right)$$

for some $\delta \in [0,1]$ depending on whether the recovery is a fraction of the bond's face value or a fraction of an otherwise identical default free bond.

(b) *Vulnerable derivatives.* A vulnerable derivative is a contingent claim issued by a counterparty which may default at any time prior to maturity. Assuming that in bankruptcy the buyer receives a fraction of an otherwise identical default free contingent claim, a vulnerable call with strike K on the $n^{\text{th}}-$stock would correspond to $A = (S_n(T) - K)^+$ and

$$R(t) = (1-\delta) \cdot E^0 \left(\tfrac{D_0(T)}{D_0(t)} \left(S_n(T) - K\right)^+ \middle| \mathcal{F}(t) \right)$$

for some fractional loss quota $\delta \in [0,1]$. Johnson and Stulz (1987) study this type of defaultable options in a model where default is triggered when the seller's assets value reaches some fixed barrier.

(c) *Credit derivatives.* The buyer of a credit derivative pays an up front free in return for a contractual protection against default on some security. Termination values for such contracts come in multiple forms such as (i) digital cash payment, (ii) par value minus post-default market value, (iii) initial price minus post-default market value and (iv) normalized price (in general the price of an equivalent default-free security) minus post-default market value. See Tavakoli (1998) for a thorough description. Assuming that the post-default value is set to be a fraction of an equivalent default free security then case (iv) corresponds to

$$R(t) = (1-\delta) \cdot E^0 \left(\tfrac{D_0(T)}{D_0(t)} \bar{R} \middle| \mathcal{F}(t) \right) \tag{11}$$

and $A = 0$ for some $\delta \in [0,1]$ where $\bar{R}$ is the $\mathcal{F}-$measurable payoff of the corresponding default free security while case (iii) corresponds to

$$R(t) = \left(P - \delta \cdot E^0 \left(\tfrac{D_0(T)}{D_0(t)} \bar{R} \middle| \mathcal{F}(t) \right) \right)^+$$

and $A = 0$ for some $\delta \in [0,1]$ where P is the price at which the buyer purchased the underlying credit risky security at time 0.

(d) *First to default contracts.* A first to default contract is virtually the same as a credit derivative except that termination is triggered by the occurrence of the first in a pre-specified list of credit events which are not necessarily default. See Duffie (1999) for a thorough discussion.

(e) *Credit swaps.* A credit swap is a variation of the basic credit derivative in which the protection buyer does not pay the protection seller up front but either continuously or at discrete points in time until the occurrence of the credit event.

(f) *Life insurance.* A life insurance contract gives right to a payment $R(\tau)$ which can be either fixed (as is the case for annuities) or random (as is the case for equity-linked contracts) at the stopping time τ if before the term of the contract and zero otherwise. The corresponding ESCC is thus of the form $(0, R(\cdot))$ for some arbitrary $\mathbb{F}-$measurable process $R \in \mathcal{E}$.

In all of the above the examples we assume that the fractional loss quota is constant. This is merely for simplicity of exposition since in each case we can assume that it is stochastic provided that it is measurable with respect to the filtration generated by the Brownian motion.

It is also worth noting that in most of the above examples the ESCC's payoffs are ordered in the sense that either, as in examples (a) and (b) we have

$$R(t) \leq E^0 \left(\tfrac{D_0(T)}{D_0(t)} A \middle| \mathcal{F}(t) \right) \tag{VD}$$

(condition VD for Vulnerable Derivative) or the reverse inequality as in examples (c) and (d) (condition CD for Credit Derivative). In what follows we shall always assume that either one of these conditions is satisfied. Also in most cases presented above, the recovery process R corresponds to the value process of some $\mathbb{F}-$measurable trading strategy which can be identified with an $\mathcal{F}-$measurable terminal value $R(T)$. In that case the conditions VD and CD are equivalent to the terminal value constraints $R(T) \leq A$ and $R(T) \geq A$, respectively.

1.4 Admissible Trading Strategies

Consider a *small* investor initially endowed with x units of the consumption good and a cumulative income process e. A trading strategy is an $\mathbb{H}-$predictable and almost surely square integrable process $\{\theta_k : 1 \leq k \leq n\}$ of amounts invested in each of the available stocks. Let X denote the corresponding wealth process. If the strategy is used in a self-financing way, then $\theta_0 := X - \theta^* \underline{1}$ is invested in the bank account and, in accordance with the model set forth in the previous sections, the wealth process obeys the linear stochastic differential equation

$$dX(t) = r(t)X(t)dt + \theta(t)^*\sigma(t)\left(dB(t) + \xi(t)dt\right) + de(t) \tag{12}$$

with initial condition x. In what follows the self-financing condition will always be in force and so we may associate to any triple (x, e, θ) a value process given by the solution $X = X^{x,e,\theta}$ to equation (12).

Definition 4 *A trading strategy is said to be admissible and we write* $\theta \in \Theta$ *if the minimal element* $\inf X^\theta(t)$ *of the wealth process* $X^\theta := X^{0,0,\theta}$ *associated with the triple* $(0, 0, \theta)$ *lies in the space* L *defined by* (9)*.*

Given an arbitrary endowment pair (x, e) and an arbitrary trading strategy θ, Itô's lemma, (7) and (12) show that we have

$$D_0(t)X(t) - \int_0^t D_0(s)de(s) = x + \int_0^t D_0(s)\theta(s)^*\sigma(s)dW(s). \tag{13}$$

As is easily seen, the process on the righthand side of (13) equals $x + D_0X^\theta$ and is an $(\mathbb{H}, P^0)-$local martingale. If moreover $\theta \in \Theta$ then this local martingale is also uniformly bounded from below by a P^0-integrable random variable, hence a supermartingale by Fatou's lemma and consequently, we obtain that

$$E^0\left(D_0(T)X^{x,e,\theta}(T) - \int_0^T D_0(t)de(t)\right) \leq x \tag{14}$$

holds for all $(x, e, \theta) \in \mathbb{R} \times \mathcal{E} \times \Theta$. Equation (14) is referred to as a *static budget constraint* and excludes any arbitrage opportunity from the market (see Dybvig and Huang (1988) for the original argument).

1.5 Arbitrage Pricing of Contingent Claims

In this section, we shortly discuss arbitrage bounds for value of an arbitrary cumulative income process e. As is well known (see for example Karatzas and Kou (1996)), the assumption of absence of arbitrage imposes that the value of such a contingent claim falls in an interval $I(e)$ whose endpoints correspond to the so-called hedging prices. The upper bound is the upper hedging price $\hat{u}(e)$ which corresponds to the smallest initial wealth endowment necessary to cover a short position in the claim. Symmetrically, the lower bound corresponds to the lower hedging price $\check{u}(e)$ which is the largest initial debt that can be contracted along with a long position in the claim without going bankrupt.

In mathematical terms, the upper and lower hedging price of a cumulative income process $e \in \mathcal{E}$ are respectively defined by

$$\hat{u}(e) \triangleq \inf \left\{ x \in \mathbb{R} : \exists\, \theta \in \Theta \text{ such that } X^{x,-e,\theta}(T) \in \mathbb{R}_+ \right\} \tag{15}$$

and $-\check{u}(e) := \hat{u}(-e)$ with the convention that $\inf \emptyset = \infty$. For *event insensitive contingent claims* these two are equal and coincide with the expectation of the discounted cumulative income process under the risk neutral probability P^0 and as a result markets are complete with respect to $\mathbb{F}$.

Lemma 5 *Let e denote an arbitrary $\mathbb{F}-$adapted process in $\mathcal{E}$. Then we have*

$$\check{u}(e) = \hat{u}(e) = u_0(e) \triangleq E^0 \left(\int_0^T D_0(t) de(t) \right) \tag{16}$$

and there exists an admissible trading strategy $\theta \in \Theta$ such that the terminal value of the wealth process associated to the triple $(u_0, -e, \theta)$ is equal to zero.

Consider now an event sensitive contingent claim $(A, R(\cdot))$ with corresponding cumulative income process e defined by (10) and assume that

$$R(t) = E^0 \left(\tfrac{D_0(T)}{D_0(t)} \bar{R} \middle| \mathcal{F}(t) \right) \tag{17}$$

for some $\mathcal{F}-$measurable random variable $\bar{R} \in L^+$ with $\bar{R} \leq A$ (condition VD). It is easy to see that the arbitrage free price of the event insensitive contingent claims with respective payoff A and $\bar{R}$ at the terminal time provide upper and lower bounds for the hedging prices of the ESCC:

$$u_0(\bar{R}) \triangleq E^0 \left(D_0(T)\bar{R} \right) \leq \check{u}(e) \leq \hat{u}(e) \leq E^0 \left(D_0(T) A \right) \triangleq u_0(A). \tag{18}$$

The following result establishes that, under the present assumptions, the above price bounds are tight in the sense that the outer inequalities in (18) actually hold as equalities.

Proposition 6 *Let $(A, R(\cdot))$ be an* ESCC*, denote by $e(\cdot)$ the corresponding cumulative income process and assume that $R(\cdot)$ is given by* (17) *for some $\mathcal{F}-$measurable random variable $\bar{R} \in L^+$. Under condition* VD *we have*

$$u_0(\bar{R}) = \check{u}(e) \leq u_0(e) \leq \hat{u}(e) = u_0(A). \tag{19}$$

If on the contrary condition CD *holds true, then equation* (19) *remains valid if we interchange the role of the $\mathcal{F}-$measurable random variables $\bar{R}$ and A.*

Proof. The proof follows from results in Collin Dufresne and Hugonnier (1999) and is available upon request. ■

The above proposition points to the weaknesses of the almost sure hedging criterion when dealing with event sensitive contingent claims and is reminiscent of the well-known fact that in a model with proportional transaction costs the cheapest way to hedge a call option is to buy the underlying and hold it until the option's maturity (see Cvitaniĉ et al. (1995)). Specializing this result to the different types of ESCC discussed in Section 1.3, we obtain:

(a) *Vulnerable derivatives.* Consider the case of a European derivative (possibly a bond) with $\mathcal{F}-$measurable payoff $A \in L^+$ at the terminal time and settlement payment

$$R(\tau) = (1-\delta) \cdot E^0 \left(\frac{D_0(T)}{D_0(t)} A \middle| \mathcal{F}(t) \right) \Big|_{t=\tau}$$

for some fractional loss quota $\delta \in [0,1]$ if the seller defaults before the maturity of the contract. Applying the result of Proposition 6 to this contract we obtain the following arbitrage free interval

$$I\left((A, R(\cdot)\right) = \left(u_0\left((1-\delta)A\right), u_0(A)\right).$$

In particular, in the case of zero recovery ($\delta \equiv 1$) the arbitrage free interval becomes trivial: zero on the one hand and the complete market, default free price $u_0(A)$ on the other.

(b) *Credit derivatives and First to default contracts.* If we assume that the post default market value is set to a constant fraction of an otherwise equivalent default free security as in (11), then the recovery process $R(\cdot)$ satisfies the requirement of Proposition 6 and the corresponding arbitrage free interval is given by

$$I\left((0, R(\cdot))\right) = \left(0, R(0)\right) = \left(0, u_0\left((1-\delta)\bar{R}\right)\right).$$

In particular, it follows from the above expression that the maximal price that a protection buyer can afford to pay for the contract at time 0 while being sure to end up solvent at the terminal time is zero.

The proposition and examples show that the typical price interval obtained from arbitrage consideration only can be very wide. Almost sure hedging is therefore not a reasonable criterion for the pricing of event sensitive contingent claims in a framework with extraneous risk. Instead, we study in the next section an alternative pricing rule: utility based pricing.

2 Utility Based Pricing

As shown by the previous result, the arbitrage free interval associated with an event sensitive contingent claim can be very wide and is even trivial in some cases. In such a situation, the problem becomes that of selecting a *price and corresponding hedging strategy*. Dybvig (1992) shows that when a particular contingent claim is not marketed one *cannot* assume, except in some special cases (e.g. negative exponential utility, see Section 4), that the pricing and hedging problem can be separated from the rest of the agent's portfolio. Motivated by this insight, we now study a pricing rule that *embeds* the pricing and hedging problem into the agent's portfolio choice problem.

2.1 Price Definition and General Properties

We consider an economic agent endowed with an initial capital x and whose preferences over terminal consumption are represented by an expected utility functional $X \mapsto EU(X)$. The strictly increasing, strictly concave and continuously differentiable function $U : (\alpha, \infty) \to \mathbb{R}$ with $\alpha \in \{-\infty, 0\}$ is referred to as the agent's *utility function* and will be assumed to satisfy the following:

Assumption 7 *The utility function U satisfies the Inada conditions at both α and infinity and has reasonable asymptotic elasticity (see Schachermayer (2000) for details on this condition).*

Suppose that in addition to his initial capital $x \in \mathbb{R}$, the agent is endowed with an arbitrary ESCC and denote by $e \in \mathcal{E}$ the corresponding cumulative income process. Given this endowment pair, the agent's portfolio choice problem is to find an admissible trading strategy θ which maximizes his expected utility of terminal wealth. The agent's value function is thus given by:

$$V_e(x) \triangleq \sup_{\theta \in \Theta} EU\left(X^{x,e,\theta}(T)\right). \tag{20}$$

Without the ESCC, the agent's value function is $V_0(x)$. If he could use a certain amount $k \in \mathbb{R}$ to purchase the ESCC at the initial time, then the agent would only do so as long as this trade allows him to improve on his utility index and this naturally leads to the following:

Definition 8 *For an agent with initial capital $x \in \mathbb{R}$ and utility function U, the utility based buying price of an arbitrary* ESCC *is defined by*

$$u_b(x, e) \triangleq \sup\left\{k \in \mathbb{R} : V_0(x) \leq V_e(x-k)\right\} \tag{21}$$

where e denotes the corresponding cumulative income process. Symmetrically, the utility based selling price of an ESCC *is defined by* $-u_s(x,e) := u_b(x,-e)$.

Remark 9 It follows from Collin Dufresne and Hugonnier (1999), Schachermayer (2000), Cvitanič et al. (2000), Owen (2001) and Hugonnier and Kramkov (2001) among others that the conditions imposed on the utility function are sufficient to guarantee the existence of unique solution to the agent's utility maximization problem (20) for every initial capital $x > -\check{u}(e) + \alpha E^0 D_0(T)$.

A natural question to address before pursuing the study of utility based prices any further is that of existence and consistency of the derived pricing rule with the absence of arbitrage opportunities. As mentioned in the above remark, the conditions that we impose on the utility function are sufficient to guarantee the existence of a unique solution to the agent's portfolio choice problem. The corresponding value functions being strictly concave, they are continuous on the interior of their domain and existence of the utility based prices follows. Our next result establishes consistency of the pricing rule and is a mild generalization of Theorem 5.9 in Collin Dufresne and Hugonnier (1999).

Theorem 10 *Let $e \in \mathcal{E}$ be an arbitrary cumulative income process. Then its utility based prices exist and satisfy the consistency condition $\check{u}(e) \le u_b(x,e) \le u_0(e) \le u_s(x,e) \le \hat{u}(e)$ for every initial capital $x \in (\alpha, \infty)$.*

Proof. The proof follows from results in Collin Dufresne and Hugonnier (1999) and is available upon request. ■

On the contrary to Collin Dufresne and Hugonnier (1999) who mainly focus on theoretical properties of the utility based pricing rule, we are interested in the actual computation of the prices associated with an arbitrary ESCC. To this end, we first show how to solve the utility maxmization problem by transforming the objective function and recasting it into a standard, complete markets, portfolio choice problem with state, time and path-dependent utility function.

2.2 Reduction to a Complete Market Problem

In order to study the agent's portfolio choice problem, we start by defining his value function at some intermediate date and establish that it verifies the principle of dynamic programming. Let us define Θ^t to be the set of time$-t$ admissible trading strategies, that is the set of $\mathbb{H}-$predictable and almost surely square integrable processes θ such that the process

$$X_t^\theta(u) \triangleq X^\theta(u) - \frac{D_0(t)}{D_0(u)} X^\theta(t) \equiv \int_t^u \frac{D_0(s)}{D_0(u)} \theta(s)^* \sigma(s) dW(s) \tag{22}$$

is uniformly bounded from below by a random variable in the space L of (9). The value function of an agent endowed with a contingent claim e and having initial capital x at time t is now defined by

$$V_e(t,x) \triangleq \operatorname*{ess\,sup}_{\theta\in\Theta^t} E_t U\left(X_t^{x,e,\theta}(T)\right) \tag{23}$$

where E_t is the time$-t$ conditional expectation operator under the objective probability measure and where we denote by $X \equiv X_t^{x,e,\theta}$ the unique solution to (12) with initial condition x at time t. The following result establishes the dynamic programming principle for the agent's utility maximization problem.

Proposition 11 *Let $(x,e) \in \mathbb{R}\times\mathcal{E}$ denote an arbitrary endowment pair. The value function of the agent's utility maximization problem satisfies the equation*

$$V_e(\check{\tau},x) = \operatorname*{ess\,sup}_{\theta\in\Theta^{\check{\tau}}} E_{\check{\tau}} V_e\left(\hat{\tau}, X_{\check{\tau}}^{x,e,\theta}(\hat{\tau})\right) \tag{24}$$

of dynamic programming on the stochastic interval $[\check{\tau},\hat{\tau}]$ for every pair $(\check{\tau},\hat{\tau})$ of $\mathbb{H}-$stopping times such that $\check{\tau}\le\hat{\tau}\le T$.

Proof. Let (x,e) be given and consider an arbitrary pair $(\check{\tau},\hat{\tau})$ of stopping times satisfying the above conditions. Using the definition of the agent's value function in conjunction with the law of iterated expectations we have

$$V_e(\check{\tau},x) = \operatorname*{ess\,sup}_{\theta\in\Theta^{\check{\tau}}} E_{\check{\tau}} U\left(X_{\check{\tau}}^{x,e,\theta}(T)\right) \le \operatorname*{ess\,sup}_{\theta\in\Theta^{\check{\tau}}} E_{\check{\tau}} V_e\left(\hat{\tau}, X_{\check{\tau}}^{x,e,\theta}(\hat{\tau})\right). \tag{25}$$

In order to establish the reverse inequality, let us assume that the agent's initial capital is such that $V_e(\check{\tau},x)$ is finite for otherwise there is nothing to prove. According to Remark 9 there exists an optimal trading strategy for the agent starting from (x,e) at the stopping time $\check{\tau}$. Denoting this trading strategy by $\hat{\theta}$ it is now easily seen that we have

$$\begin{aligned} V_e(\check{\tau},x) &= E_{\check{\tau}} U\left(X_{\check{\tau}}^{x,e,\hat{\theta}}(T)\right) = E_{\check{\tau}}\left(E_{\hat{\tau}}\, U\left(X_{\check{\tau}}^{x,e,\hat{\theta}}(T)\right)\right) \\ &= E_{\check{\tau}} V_e\left(\hat{\tau}, X_{\check{\tau}}^{x,e,\hat{\theta}}(\hat{\tau})\right) \end{aligned} \tag{26}$$

by the law of iterated expectations and the optimality of $\hat{\theta}$. Comparing (25) with (26) we conclude that (24) holds and our proof is complete. ■

Let now $h = (A, R(\cdot))$ be an arbitrary event sensitive contingent claim as in Definition 3 and denote by e the corresponding cumulative income process. Since after the event time the agent has no endowment, it should be the case that on the set $\{\tau \le t\}$ his value function coincides with that of a complete markets utility maximization problem with no contingent claim. The next proposition makes this intuition precise.

Proposition 12 *Let $x \in \mathbb{R}$ denote an arbitrary initial capital and consider an agent endowed with one unit of the event sensitive contingent claim h. Then*

$$V_0(t,x) \triangleq \operatorname*{ess\,sup}_{\theta \in \Theta^t} E_t U\left(X_t^{x,0,\theta}(T)\right) = V_e(t,x) \tag{27}$$

holds for every t in the stochastic interval $[\tau \wedge T, T]$ where $e(\cdot)$ denotes the cumulative income process associated with the event sensitive contingent claim.

Proof. Straightforward application of the dynamic programming equation and the definition of the event sensitive contingent claim. ■

In view of the above proposition, it is now clear that the optimal trading strategy for the agent's portfolio choice problem (20) is of the form:

$$\hat{\theta}(t) = \theta_\tau(t)1_{\{t<\tau\}} + \theta_0(t)1_{\{t\geq\tau\}} \tag{28}$$

where θ_0 is the optimal strategy for a no-contingent claim utility maximization problem where the agent starts from the initial capital

$$X^{x,e,\theta_\tau}(\tau) = R(\tau) + X^{x,0,\theta_\tau}(\tau-)$$

at the event time τ. Since the latter can be computed using standard techniques, we are only left with the problem of determining the optimal trading strategy to be used prior to the event time. To this end we use a well-known result from Dellacherie (1970) (see also Jeanblanc and Rutkowski (1999)):

Lemma 13 *Let k denote an arbitrary $\mathbb{H}-$predictable process. Then there exists a unique $\mathbb{F}-$predictable process k_τ such that $k(t) = k_\tau(t)$ holds almost surely for every time $t \in [0,T]$ on the set $\{t < \tau\}$.*

As a result of the above lemma, the trading strategy θ_τ in (28) may be chosen to be predictable with respect to the filtration generated by the Brownian motion and combining this with the definition of the agent's value function we obtain:

Lemma 14 *Let $x \in \mathbb{R}$ denote an arbitrary initial capital and consider an agent endowed with one unit of the event sensitive contingent claim h. Then*

$$V_e(t,x) = \operatorname*{ess\,sup}_{\theta \in \Theta_1^t} E_t U\left(X_t^{x,e,\theta}(T)\right) \tag{29}$$

on the set $\{t < \tau\}$ *where* $\theta \in \Theta_1^t$ *denotes the set of admissible trading strategies which admit a decomposition of the form* (28) *for some* $\mathbb{F}-$*predictable* $\theta_\tau \in \Theta^t$.

Proof. This follows from the observation that the optimal trading strategy has to lie in Θ_1^t. See the proof of Lemma 116 in Hugonnier (2000) for details. ■

Using standard results on Cox processes (see Lando (1998) or Jeanblanc and Rutkowski (1999) for a survey), we now define a strictly positive $\mathbb{F}-$progressively measurable, bounded process Λ by setting

$$\Lambda(t) \triangleq P\{\tau > t | \mathcal{F}(t)\} \equiv \exp\left(-\int_0^t \lambda(s) ds\right). \tag{30}$$

and let Λ^t be the $\mathbb{F}-$progressively measurable process given by $\Lambda^t := \Lambda/\Lambda(t)$. Combining the previous results, we now obtain the main theorem of this section. It provides a reformulation of the agent's portfolio choice problem before the event time as an equivalent complete markets utility maximization problem with a modified state, time and path-dependent utility function.

Theorem 15 *Let* $x \in \mathbb{R}$ *denote an arbitrary initial capital and consider an agent endowed with one unit of the event sensitive contingent claim* h*. Then*

$$\begin{aligned} V_e(t,x) = \operatorname*{ess\,sup}_{\theta \in \Theta_0^t} E_t \Big[& \Lambda^t(T) U\left(A + X_t^{x,\theta}(T)\right) \\ & - \int_t^T V_0\left(s, R(s) + X_t^{x,\theta}(s)\right) d\Lambda^t(s) \Big] \end{aligned} \tag{31}$$

for every t *on the set* $\{t < \tau\}$ *where* Θ_0^t *denotes the set of admissible trading strategies* $\theta \in \Theta^t$ *that are predictable with respect to the Brownian filtration* $\mathbb{F}$.

Proof. Let (x, h) denote an arbitrary endowment pair as in the statement and fix an initial time $t \in [0, T]$. Placing ourselves on the set $\{t < \tau\}$ and writing the dynamic programming equation for the value function of the agent's utility maximization problem on the stochastic interval $[t, \tau \wedge T]$ we get

$$\begin{aligned} V_e(t,x) &= \operatorname*{ess\,sup}_{\theta \in \Theta^t} E_t V_e\left(\tau \wedge T, X_t^{x,e,\theta}(\tau \wedge T)\right) \\ &= \operatorname*{ess\,sup}_{\theta \in \Theta^t} E_t \left(U\left(X_t^{x,e,\theta}(T)\right) 1_{\{\tau > T\}} + E_t V_e\left(\tau, X_t^{x,e,\theta}(\tau)\right) 1_{\{\tau \le T\}} \right). \end{aligned}$$

Let now θ denote an arbitrary trading strategy in the set Θ_1^t and consider separately the two terms appearing in the expectation on the right hand side of the above equation. For the first term we have

$$E_t U\left(X_t^{x,e,\theta}(T)\right)1_{\{\tau>T\}} = E_t U\left(A + X_t^{x,\theta_\tau}(T)\right)1_{\{\tau>T\}} \tag{32}$$
$$= 1_{\{t<\tau\}} E_t \Lambda^t(T) U\left(t, A + X_t^{x,\theta_\tau}(T)\right)$$

for some strategy $\theta_\tau \in \Theta_0^t$ where the first equality follows from the definition of the cumulative income process associated with the event sensitive contingent claim h and the second from Jeanblanc and Rutkowski (1999, Proposition 3.2) in conjunction with (30) and the fact that the càdlàg process

$$V(t) \triangleq E_t \Lambda(T) U\left(A + X_t^{x,\theta_\tau}(T)\right)$$

is a $P-$martingale in the filtration generated by the Brownian motion. Similarly, placing ourselves on the set $\{t < \tau\}$ we have that the second term satisfies

$$E_t V_e\left(\tau, X_t^{x,e,\theta}(\tau)\right)1_{\{\tau\le T\}} = E_t V_e\left(\tau, R(\tau) + X_t^{x,\theta_\tau}(\tau)\right)1_{\{\tau\le T\}} \tag{33}$$
$$= -E_t \int_t^T V_0\left(s, R(s) + X_t^{x,\theta_\tau}(s)\right) d\Lambda^t(s)$$

for some $\theta_\tau \in \Theta_0^t$ where the first equality follows from the definition of the value process and the second from Duffie et al. (1996, Proposition 1) (see also Hugonnier (2000, Proposition 99)) in conjunction with Proposition 12 and the fact that the càdlàg process

$$Y(t) \triangleq E_t \int_t^T V_0\left(s, R(s) + X_t^{x,\theta_\tau}(s)\right) d\Lambda(s)$$

is a $P-$martingale in the filtration generated by the Brownian motion. Plugging (32) and (33) back into the dynamic programming equation (32), we obtain the reformulation (31) and our proof is complete. ■

Remark 16 Taking into account the measurability of the different processes involved, it easily seen that we may replace the $\mathcal{H}(t)-$conditional expectation operator E_t in (31) by the $\mathcal{F}(t)-$conditional expectation operator. This is consistent with the fact that, as implied by Lemma 13, every $\mathbb{H}-$predictable process can be chosen to be $\mathbb{F}-$predictable before the event time.

Remark 17 The previous theorem deals with the case of an agent endowed with a long position in the ESCC, but a similar reformulation can be established for the case where the agent sells the claim instead of buying it. In order to simplify the exposition of our results, we omit the details.

The above result illustrates the wealth effects induced by the dependence of the agent's endowment process on the event time. In order to maximize his expected utility, the agent's starts by weighting the utility indexes that he would obtain (i) at the terminal time if the event does not to happen and (ii) at the event time if happens prior to the maturity, by the respective probability of each of these scenarios and *then* chooses a trading strategy so as to maximize this modified path-dependent criterion. Consider an agent whose utility function is only defined on the positive real line ($\alpha = 0$). Since he is unable to hedge the jump in his endowment process, the agent must be prepared to absorb it at any time and therefore faces an *endogeneous* liquidity constraint of the form $X + R \in \mathbb{R}_+$ all along the path (in the classical complete markets utility maximization framework, such liquidity constraints have been studied by Cuoco (1997), Detemple and Serrat (1998) and El Karoui and Jeanblanc (1999)). In a slightly different, albeit related, setting the presence of such wealth effects in the agent's utility maximization problem was foreseen, although not demonstrated explicitly, by Dybvig (1992).

3 Late Resolution of Uncertainty

An interesting point is that the presence of such wealth effects in the agent's utility maximization problem is entirely due to the way in which the uncertainty induced by the event time τ is resolved and *not* to the actual payment of the cash flows. By separating both, e.g. studying the case where the information about the event gets revealed at the payment and not continuously through time, we can thus analyze the impact of the temporal resolution of uncertainty on both the arbitrage prices and the utility based prices of ESCCs.

3.1 Arbitrage Pricing under Late Resolution

In order to model such a late resolution of uncertainty let us assume (i) that the information available to agents at any time is no longer represented by the filtration $\mathbb{H}$ generated by the paths of both the Brownian motion and the point process but by the filtration

$$\mathbb{G} = \{\mathcal{G}(t)\} \triangleq \left\{ \begin{array}{ll} \mathcal{F}(t) & \text{if } t \in [0, T), \\ \mathcal{H}(T) & \text{if } t \equiv T. \end{array} \right\} \tag{34}$$

and (ii) that agents are constrained to choose their trading strategies in the set Θ_0 of admissible trading strategies that are predictable with respect to the filtration $\mathbb{F}$ generated by the Brownian motion.

Remark 18 Let ϕ be an almost surely square integrable, $\mathbb{R}^n-$valued process which is progressively measurable with respect to the filtration $\mathbb{G}$ and define a real valued, $\mathbb{G}-$adapted process X by setting

$$X(t) \triangleq \int_0^t \phi(s)^* dB(s) = \int_0^t \sum_{k=1}^n (\phi_k(s) dB_k(s)).$$

As is easily seen, the process B is still a Brownian motion in the enlarged filtration $\mathbb{G}$ and it follows that the process X is a $(\mathbb{G}, P)-$local martingale with continuous paths. In particular, such a process does not jump at time T and it follows that we may replace the integrand by the $\mathbb{F}-$progressively measurable process $1_{[0,T)}\phi$ without affecting the paths of the process X. This justifies the fact that even though the information available to agents is represented by the enlarged filtration $\mathbb{G}$ we restrict ourselves to the set of $\mathbb{F}-$measurable admissible trading strategies.

The particular class of ESCCs that we shall consider throughout this section consists of those claims for which the recovery is paid at the terminal time and is formally defined as follows:

Definition 19 *An event sensitive contingent claim with terminal settlement is represented by a pair $(A, \bar{R})$ of non negative $\mathcal{F}(T)-$measurable random variables in $L \times L$ satisfying either condition* VD *or condition* CD.

As is easily seen, the overall cumulative income process associated with a long position in an event sensitive contingent claim with settlement at the terminal date is the càdlàg semimartingale $e \in \mathcal{E}$ defined by

$$e(t) \triangleq 1_{\{\tau > T = t\}} A + 1_{\{\tau \le t = T\}} R. \tag{35}$$

For the case in which the information about the realization of the event time is revealed continuously through time, our results concerning arbitrage bounds and the reformulation of the agent's portfolio choice problem hold true if we define the settlement process by

$$R(t) \triangleq E^0 \left(\tfrac{D_0(T)}{D_0(t)} \bar{R} \middle| \mathcal{F}(t) \right). \tag{36}$$

In other words, if information about the event time is revealed continuously the agent is indifferent between receiving the amount $R(\tau)$ at the event time or the amount $\bar{R} \equiv R(T)$ at the terminal time.

Remark 20 Taking the above fact into account we shall from now on use the notation $\hat{u}(e)$ and $u_s(x, e)$ to designate, respectively, the early resolution upper hedging price and utility based selling price of both the ESCC $(A, \bar{R})$ with terminal settlement and cumulative income process given by (35) or the ESCC $(A, R(\cdot))$ with cumulative income process given by (10).

Let now $(A, \bar{R})$ be an arbitrary event sensitive contingent claim with terminal settlement and cumulative income process e given by (35). For such a claim, we define the upper and lower hedging prices under late resolution as

$$\hat{v}(e) \triangleq \inf \left\{ x \in \mathbb{R} : \exists \theta \in \Theta_0 \text{ such that } X^{x,\theta}(T) - e(T) \in \mathbb{R}_+ \right\} \tag{37}$$

and $-\check{v}(e) := \hat{v}(-e)$. Note that the only difference between early and late resolution hedging prices is the set of allowed trading strategies. Our first result shows that the temporal resolution of the uncertainty associated with the event time has no impact on the arbitrage pricing of the event sensitive contingent claim in the sense that its hedging prices are the same under late resolution and early resolution of the extraneous uncertainty.

Proposition 21 *Let $(A, \bar{R})$ be an arbitrary event sensitive contingent claim with cumulative income process e given by* (35). *Then we have $\hat{u}(e) = \hat{v}(e)$.*

Proof. Observing that $\Theta_0 \subset \Theta$ we deduce from (15) and (37) that $\hat{u}(e) \leq \hat{v}(e)$. To establish the reverse inequality and thus complete the proof, let us assume that the claim satisfies condition CD (the case where the claim satisfies condition VD is treated similarly). By Proposition 6 we have that $\hat{u}(e) = u_0(\bar{R})$. Applying Lemma 5 we know that starting from this amount there exists a trading strategy in Θ_0 whose terminal value is equal to $\bar{R}$ and observing that $\bar{R} - e(T) \in \mathbb{R}_+$ we conclude that $\hat{v}(e) \leq u_0(\bar{R}) = \hat{u}(e)$ holds. ■

3.2 Utility Based Pricing under Late Resolution

Consider now an agent whose preferences over terminal consumption bundles are represented by a utility function $U : (\alpha, \infty) \to \mathbb{R}$ as in the previous section and who is endowed at time zero with some initial capital x as well as with one unit of an event sensitive contingent claim $(A, \bar{R})$ with terminal settlement as in Definition 19. Let e defined as in (35) denote the associated cumulative income process. In accordance with the model set forth in the previous section, the agent's *late resolution* portfolio choice problem is to find an admissible trading strategy $\theta \in \Theta_0$ which maximizes his expected utility of terminal wealth. The corresponding value function is thus given by

$$V_e^\ell(x) \triangleq \sup_{\theta\in\Theta_0} EU\left(X^{x,\theta}(T)+e(T)\right) \tag{38}$$
$$\equiv \sup_{\theta\in\Theta_0} EU\left(X^{x,\theta}(T)+A1_{\{\tau>T\}}+\bar{R}1_{\{\tau\le T\}}\right)$$

where $X^{x,\theta}$ denotes the solution to the linear stochastic differential equation (12) corresponding to the initial capital $x\in\mathbb{R}$, the trading strategy θ and the cumulative endowment process $e=0$. We define the utility based prices of an arbitrary event sensitive contingent claim with settlement at the terminal date under late resolution of uncertainty as follows:

Definition 22 *For an agent with initial capital $x\in\mathbb{R}$ and utility function U the late resolution utility based buying price of an arbitrary event sensitive contingent claim with terminal settlement is defined by*

$$p_b(x,e) \triangleq \sup\left\{k\in\mathbb{R} : V_0^\ell(x)\le V_e^\ell(x-k)\right\} \tag{39}$$

where e denotes the corresponding cumulative income process. Symmetrically, the utility based selling price of an event sensitive contingent claim with terminal settlement is defined by $-p_s(x,e):=p_b(x,-e)$.

Our next result confirms the intuition that, even though it has no impact on the arbitrage prices of the ESCC, the temporal resolution of event-uncertainty has an impact on its utility based prices.

Proposition 23 *Let $(A,\bar{R})$ be an arbitrary event sensitive contingent claim with terminal settlement as in Definition* 19 *and denote by e the corresponding cumulative income process. Then we have the inequalities $p_b(x,e)\le u_b(x,e)$ and $u_s(x,e)\le p_s(x,e)$ for every initial capital $x\in(\alpha,\infty)$.*

Proof. Recall from the previous section, that for the event sensitive contingent claim under consideration the agent's early resolution value function is given by

$$V_e(x)=\sup_{\theta\in\Theta} EU\left(X^{x,\theta}(T)+e(T)\right).$$

Comparing this expression with (38) and observing that $\Theta_0\subset\Theta$ by definition we conclude that $V_e^\ell(x)\le V_e(x)$ holds for every initial capital. On the other hand, using Proposition 12 in conjunction with Lemma 13, Remark 18 and the fact that the wealth process $X^{x,\theta}$

has continuous paths, we obtain that

$$V_0(x) \equiv V_0^\ell(x) \triangleq \sup_{\theta\in\Theta_0} EU\left(X^{x,\theta}(T)\right)$$

holds for every initial capital. The result now follows from the definition of the utility based prices and the fact that both the late and the early resolution value functions are increasing in wealth. ■

As is easily checked, all the properties of the early resolution utility based pricing rule also hold for the late resolution price. In particular, provided that they exist the above prices are consistent with one another and with the absence of arbitrage opportunities. Combining these observations, we obtain that

$$\begin{aligned} \check{u}(e) = \check{v}(e) \le p_b(x,e) \ &\le \ u_b(x,e) \le u_0(e) \\ &\le \ u_s(x,e) \le p_s(x,e) \le \hat{v}(e) = \hat{u}(e) \end{aligned} \tag{40}$$

holds for every initial capital $x \in (\alpha, \infty)$. Thus, late resolution utility based prices (which are much easier to compute, see the next section) provide an upper bound on the early resolution utility based pricing interval.

3.3 The Utility Maximization Problem

We now turn to the study of the agent's late resolution utility maximization problem. Rather than treating the existence and characterization of the optimal policy separately as we did for the early resolution problem in Section 2, we shall attack both problems at the same time and actually compute the optimal policy. Our first result in this direction provides a reformulation of the late resolution utility maximization problem (38) into an equivalent complete market problem with a modified state dependent but path independent utility function.

Theorem 24 *Let $(A, \bar{R})$ denote an arbitrary event sensitive contingent with terminal settlement as in Definition* 19*. Then we have*

$$V_e^\ell(x) = \sup_{\theta\in\Theta_0} E\left(\Lambda(T)U\left(X^{x,\theta}(T) + A\right) + \Gamma(T)U\left(X^{x,\theta}(T) + \bar{R}\right)\right) \tag{41}$$

for every initial capital $x \in \mathbb{R}$ where Λ is the non negative, strictly decreasing process defined by (30) *and where we have set $\Gamma := 1 - \Lambda$*

Proof. Let $\theta \in \Theta_0$ and fix an arbitrary initial capital $x \in \mathbb{R}$. Observing that by definition of the set Θ_0 the triple $(A, \bar{R}, X^{x,\theta}(T))$ is measurable with respect to the filtration generated by the Brownian motion and using the law of iterated expectations in conjunction with the definition of Λ we get

$$\begin{aligned} EU\left(X^{x,\theta}(T)+e(T)\right) &= EU\left(X^{x,\theta}(T)+A1_{\{\tau>T\}}+\bar{R}1_{\{\tau\leq T\}}\right) \\ &= E\left(\Lambda(T)U\left(X^{x,\theta}(T)+A\right)+\Gamma(T)U\left(X^{x,\theta}(T)+\bar{R}\right)\right) \end{aligned}$$

where we have used the fact that the set $\{\tau=0\}$ is a $P-$null set of $\mathbb{G}$. Taking the supremum over admissible trading strategies $\theta\in\Theta_0$ on both sides of the above expression we get (41) and our proof is complete. ■

Let us now turn to the agent's equivalent utility maximization problem (41) and consider the $\mathcal{F}(T)-$measurable state dependent utility function defined by

$$U_e(x)\triangleq\Lambda(T)\cdot U\left(x+A\right)+\Gamma(T)\cdot U\left(x+\bar{R}\right). \tag{42}$$

Recalling the definition of the set Θ_0 of admissible trading strategies and using the result of Lemma 5, it is easily checked that an arbitrary random variable X is feasible for the initial capital $x\in\mathbb{R}$ if and only if it is $\mathcal{F}(T)-$measurable and satisfies the budget constraint

$$u_0(X)\triangleq E^0\left(D_0(T)X\right)\equiv E\left(H_0(T)X\right)\leq x \tag{43}$$

where $H_0:=D_0Z_0$ denotes the risk neutral state price density. Using this fact we may now regard the dynamic problem (41) as the static problem of maximizing the agent's modified expected utility functional over the set of $\mathcal{F}(T)-$measurable random variables that satisfy the budget constraint (43). In order to obtain a simple characterization of the solution to this variational problem, we start by observing that the function of (42) is almost surely strictly increasing, strictly concave and continuously differentiable in x on the open stochastic domain

$$D_e\triangleq(C_e,+\infty)=\left(\alpha-A\wedge\bar{R},+\infty\right) \tag{44}$$

and that the corresponding marginal utility function admits a state dependent, continuous and strictly decreasing inverse function $I_e(\cdot)$ which maps $(0,\infty)$ onto the stochastic interval of (44). Because the static problem is subject to a single budget constraint and because the objective functional is strictly concave, its solution may be found through the associated first order conditions which require that the optimal terminal wealth be given by

$$X_e(x)\triangleq I_e\left(\hat{y}(x)\cdot H_0(T)\right) \tag{45}$$

for some strictly positive Lagrange multiplier $\hat{y}(x)$ chosen in such a way that the budget constraint (43) is saturated. In order to make this statement precise, let us impose the following additional condition:

Condition *Let* $(A, \bar{R})$ *denote an event sensitive contingent claim with terminal settlement as in Definition* 19*. Then the pair* $(A, \bar{R})$ *is said to satisfy condition* FV *if the map* $\mathcal{X}_e$ *defined by* (47) *below is finitely valued.*

The following theorem provides a complete solution to the agent's modified utility maximization problem (41) under condition FV and constitutes the main result of this section.

Theorem 25 *Let* (A, R) *be such that condition* FV *holds and fix an arbitrary initial capital such that* $x > u_0(C_e)$*. Then the random variable of* (45) *where* $\hat{y}(x) \in (0, \infty)$ *is chosen such that the budget constraint* (43) *holds as an equality, is the unique optimal terminal wealth for the utility maximization problem* (41)*.*

Proof. Let (x, A, R) be as in the statement and observe that by concavity of the agent's modified utility function we have

$$U_e(z) - U_e\left(I_e(y)\right) \leq y \cdot \left(z - I_e\left(y\right)\right) \tag{46}$$

for every $(z, y) \in D_e \times \mathbb{R}_+$. Let us assume for the moment that a Lagrange multiplier has been found such that the budget constraint holds as an equality and consider the random variable of (45). By construction this random variable is feasible for the given initial capital and using (46) we have that

$$\begin{aligned} EU_e\left(X_e(x)\right) &\geq EU_e(X) + \hat{y}(x) \cdot E\left(H_0(T)\left(X_e(x) - X\right)\right) \\ &= EU_e(X) + \hat{y}(x) \cdot \left(x - E\left(H_0(T)X\right)\right) \geq EU_e(X) \end{aligned}$$

holds for every $x-$feasible random variable X. The above string of inequalities shows that the random variable $X_e(x)$ is optimal for the utility maximization problem (41) and since uniqueness of the optimum follows from the strict concavity of the modified utility function all there remains to prove is that we can indeed find a strictly positive constant fix such that (43) holds as an equality. To this end consider the function defined by

$$\mathcal{X}_e(y) \triangleq E\left(H_0(T)I_e\left(y \cdot H_0(T)\right)\right). \tag{47}$$

Under condition FV, this is a real valued, continuous and strictly decreasing function which takes the value infinity at zero and whose limit at infinity equals the risk neutral price $u_0(C_e)$. In particular, the above function admits a strictly decreasing inverse and it follows that for every $x > u_0(C_e)$ there exists a unique strictly positive Lagrange multiplier such that the budget constraint holds as an equality. ■

Remark 26 The validity of condition FV may in general be difficult to check since in most cases the inverse marginal utility function cannot be computed explicitly. Nevertheless, for an agent with log preferences we have

$$\begin{aligned} 2\kappa \cdot I_e(\kappa) &= 1 - \kappa\left(A + \bar{R}\right) \\ &+ \left(\left(\kappa\left(\bar{R} - A\right) - \Gamma(T) + \Lambda(T)\right)^2 + 4\Gamma(T)\Lambda(T)\right)^{\frac{1}{2}} \end{aligned} \tag{48}$$

and it is a matter of straightforward (albeit messy) calculations to verify that the inverse marginal utility function satisfies the prescribed limits and that the integrability condition FV holds for every ESCC with settlement at the terminal date. Similarly, for the case of an agent with negative exponential utility given by $-U(x) = \exp(-\gamma x)$ for some non negative γ we have

$$I_e(\kappa) = \gamma^{-1} \log\left(\gamma\kappa^{-1}\left(\Lambda(T)e^{-\gamma A} + \Gamma(T)e^{-\gamma\bar{R}}\right)\right) \tag{49}$$

and it is straightforward to verify that the inverse marginal utility function satisfies the prescribed limits and that the integrability condition FV holds for every ESCC with settlement at the terminal date.

4 Negative Exponential Utility

In this section we study the special case where the agent has constant absolute risk aversion. For this case we can compute explicitly late resolution utility prices and, under additional assumptions, early resolution utility based prices.

The negative exponential utility function has two interesting characteristics: (i) it corresponds to constant absolute risk-aversion and (ii) it allows for negative terminal wealth (because it is defined over the whole real line). As a result, we show below that the associated utility based prices are independent of the agent's initial capital, and that the endogeneous liquidity constraint discussed in Section 2.2 never binds. Nevertheless, we find that early resolution buy (resp. sell) prices are, in general, higher (resp. lower) than late resolution prices due to the impact of information on the hedging demand of the investor. When the hedging demand is zero, which is for example the case when the payoffs $(A, \bar{R})$ are constant, then late and early resolution prices coincide. In other words, when the CARA investor buys an ESCC with state independent payoffs he is not willing to pay a premium for receiving information early.

4.1 The No-Contingent Claim Problem

For simplicity of exposition we shall throughout this section restrict ourselves to the subset $\Pi \subset \Theta$ of admissible strategies which are such that

$$EU^{-}\left(\inf_{t\in[0,T]}\frac{X^{x,\theta}(t)}{D(t)}\right)=E\left[\exp\left(-\gamma\cdot\inf_{t\in[0,T]}\frac{X^{x,\theta}(t)}{D(t)}\right)\right]<\infty \tag{50}$$

where $X^{x,\theta}$ is the value process associated with the initial capital x at time zero, the trading strategy θ and the endowment process $e \equiv 0$. Before studying the early and late resolution utility maximization problems, we start by describing the solution to the no-contingent claim problem

$$V_0(t,x)=-\operatorname*{ess\,inf}_{\theta\in\Pi_0^t} E_t \exp\left(-\gamma X_t^{x,\theta}(T)\right). \tag{51}$$

Setting $A = R \equiv 0$ in (49) and using (45) we obtain that for an arbitrary initial capital x at time t, the agent's optimal terminal wealth is given by

$$X_t^0(x) \triangleq -\gamma^{-1}\log\left((\hat{y}(t,x)/\gamma)\cdot H_0^t(T)\right) \tag{52}$$

where we have set $H_0^t := H_0/H_0(t)$ and where the $\mathcal{F}(t)-$measurable Lagrange multiplier $\hat{y}(t,x)$ is chosen so as to saturate the agent's budget constraint:

$$x = E_t\left(H_0^t(T)X_t^0(x)\right). \tag{53}$$

In order to simplify the exposition of our results, we now introduce the two dimensional process (D,Z) with coordinates $D(t) := E_t H_0^t(T)$ and

$$Z(t) \triangleq E_t\left(H_0^t(T)\log H_0^t(T)\right). \tag{54}$$

As is easily seen, the non negative process D satisfies $D(T) = 1$ and represents the price process of a default free zero-coupon bond with maturity T. Using these definitions in conjunction with (52) and (53) we obtain that

$$\hat{y}(t,x)/\gamma \triangleq \exp-\left(\frac{\gamma x+Z(t)}{D(t)}\right) \tag{55}$$

and plugging this back into the definition of the optimal terminal wealth we get that the agent's no-contingent claim value function is given by

$$\begin{aligned} V_0(t,x) &\equiv V_0^{\ell}(t,x) = -E_t \exp\left(-\gamma X_t^0(x)\right) \\ &= -D(t)\cdot \exp -\left(\frac{\gamma x + Z(t)}{D(t)}\right) \end{aligned} \tag{56}$$

where the first equality follows from the results of the previous section (see in particular the proof of Proposition 23).

4.2 Late Resolution Utility Based Prices

Let $(A, \bar{R})$ be an arbitrary ESCC with terminal settlement and denote by e the corresponding cumulative income process. As is easily seen from (38), the modified utility function is of the form $U_e(x) = U(x + \mathcal{C}_e)$ where

$$\mathcal{C}_e \triangleq U^{-1}\left(E\left[e(T)|\mathcal{F}\right]\right) = -\gamma^{-1}\log\left(\Lambda(T)e^{-\gamma A} + \Gamma(T)e^{-\gamma R}\right) \tag{57}$$

is the $\mathcal{F}-$conditional certainty equivalent of the event sensitive contingent claim. Combining this definition with Theorem 25 and the results of the previous section, we now obtain an explicit representation of the late resolution utility based prices for an agent with negative exponential utility function.

Theorem 27 *Assume that the random variable $\mathcal{C}_e$ of* (57) *is integrable under the risk neutral measure. Then the late resolution utility based buying price corresponds to the price of receiving the conditional certainty equivalent:*

$$p_b(x,e) \equiv p_b(e) = u_0(\mathcal{C}_e) \triangleq E^0\left(D_0(T)\mathcal{C}_e\right) \tag{58}$$

In particular, it is independent from the agent's initial capital and decreasing in the agent's absolute risk aversion parameter γ with $\lim_{\gamma\to 0} u_0(\mathcal{C}_e) = u_0(e)$.

Proof. Comparing the definition of the conditional certainty equivalent with (49) and using the result of Theorem 25, we have that the optimal terminal wealth for the agent's late resolution problem is given by

$$X_e(x) \triangleq -\gamma^{-1}\log\left(\left(\hat{y}_e(x)/\gamma\right)\cdot H_0(T)\right) - \mathcal{C}_e$$

where the strictly positive Lagrange multiplier $\hat{y}_e(x)$ is chosen so as to saturate the agent's budget constraint (43). A straightforward computation then shows that $\hat{y}_e(x)$ is given by

$$\hat{y}_e(x) \triangleq \hat{y}\left(0, x + E^0 D_0(T)\mathcal{C}_e\right) \equiv \hat{y}\left(0, x + u_0(\mathcal{C}_e)\right)$$

in the notation of (55) and plugging this back in the definition of the optimal terminal wealth we obtain that for an arbitrary initial capital $x \in \mathbb{R}$, the agent's late resolution value function is given by

$$-V_e^\ell(x) = E \exp\left(-\gamma X_e(x)\right) = D(t) \cdot \exp - \left(\frac{\gamma \cdot [x + u_0(\mathcal{C}_e)] + Z(0)}{D(0)}\right).$$

Comparing this with the no-contingent value function given in the previous section and using the first equality in (56), it is easily seen that we have

$$V_e^\ell(x) = V_0^\ell\left(x + u_0(\mathcal{C}_e)\right) \equiv V_0\left(x + u_0(\mathcal{C}_e)\right)$$

and the first part of the statement now follows from the definition of the utility based buying price. The decrease of the price function with respect to γ being a consequence of the definition of the certainty equivalent, we are only left to prove the last part. Observing that

$$\lim_{\gamma \to 0} \gamma^{-1} \log\left(\Lambda(T)e^{-\gamma A} + \Gamma(T)e^{-\gamma \bar{R}}\right) = -\left(\Lambda(T)A + \Gamma(T)\bar{R}\right)$$

and using the decrease of the conditional certainty equivalent with respect to γ in conjunction with the monotone convergence theorem we obtain

$$\begin{aligned}\lim_{\gamma \to 0} -u_0(\mathcal{C}_e) &= E^0\left(D_0(T) \cdot \lim_{\gamma \to 0} \gamma^{-1} \log\left(\Lambda(T)e^{-\gamma A} + \Gamma(T)e^{-\gamma \bar{R}}\right)\right) \\ &= -E^0\left(D_0(T)\Lambda(T)A + D_0(T)\Gamma(T)\bar{R}\right) = -u_0(e)\end{aligned}$$

where the last equality follows from the law of iterated expectations, (30) and the definition of the cumulative income process associated with the ESCC under consideration. ■

Remark 28 Using the previous result in conjunction with Definition 22 and assuming that $\mathcal{C}_{-e}$ is integrable under the risk neutral probability measure, we obtain that the late resolution utility based selling price is given by

$$p_s(e) = -p_b(-e) \equiv E^0\left(D_0(T) \cdot \gamma^{-1} \log\left(\Lambda(T)e^{\gamma A} + \Gamma(T)e^{\gamma R}\right)\right). \tag{59}$$

and is an increasing function of the agent's absolute risk aversion parameter with $\lim_{\gamma \to 0} p_s(e) = \lim_{\gamma \to 0} p_b(e) = u_0(e)$. As easily seen, this limit price is computed by assigning a zero risk premium to the risk associated with the event time and it follows from Collin Dufresne and Hugonnier (1999, Section 6.2) that this price coincides with the *fair price* introduced by Davis (1994) and studied further by Karatzas and Kou (1996).

4.3 Early Resolution Prices

Let $(A, \bar{R})$ denote an arbitrary event sensitive contingent claim with settlement at the terminal date as in Definition 19 and denote by $R(\cdot)$ the value process of the event insensitive contingent claim with terminal pay-off given by $\bar{R}$. In order to simplify the presentation of our results, let us introduce the equivalent probability measure

$$\tilde{P}(C) \triangleq E^0\left(1_C \frac{D_0(T)}{D(0)}\right) = E\left(1_C \frac{H_0(T)}{D(0)}\right) \tag{60}$$

and denote by $\tilde{E}$ the associated expectation operator. In the terminology of term structure models this equivalent probability measure is referred to as the *forward measure* for the settlement date T and corresponds to taking the zero coupon bond as the numéraire instead of the savings account.

The following theorem constitutes the main result of this section. It provides an explicit characterization of the utility based prices in terms of the solution to a Backward Stochastic Differential Equation under the forward measure.

Theorem 29 *Assume that $(A, \bar{R})$ are bounded random variables. Then the early resolution utility based buying price is given by $u_b(x, e) = u_b(e) = Q(0)D(0)$ where Q is the maximal solution to the backward stochastic differential equation*

$$Q(t) = \tilde{E}_t\left(A + \frac{1}{\gamma}\int_t^T \lambda(s)\left\{1 - e^{\gamma\cdot[Q(s) - R(s)/D(s)]}\right\} ds\right) \tag{61}$$

In particular, it is independent from the agent's initial capital and decreasing in the agent's absolute risk aversion parameter γ with $\lim_{\gamma\to 0} u_b(e) = u_0(e)$.

Remark 30 Combining the result of Theorem 29 with the definition of the utility based prices, one can obtain an explicit characterization of the utility based selling price of the event sensitive contingent claim similar to that of the utility based buying price. We omit the details.

Sketch of the Proof. Theorem 29 is established by an argument similar in spirit to the separation of variables technique that was used by Merton (1971) to solve the utility maximization problem of an agent with negative exponential utility. Motivated by preliminary explorations of the Markov case, we start by guessing that, prior to the event time, the agent's value function is of the form

$$\begin{aligned} V_e(t,\omega,x) &= V_0(t,\omega,x+D(t)Q(t)) \\ &\equiv -D(t)\cdot\exp-\left(\gamma Q(t)+\frac{\gamma x+Z(t)}{D(t)}\right) \end{aligned}$$

for some non negative, bounded process Q with terminal value equal to A. Combining the result of Theorem 15 with the classical characterization of the value function (see e.g El Karoui (1981)) as the only adapted process such that

$$\Lambda(t)V_e\left(t,X^{x,\theta}(t)\right)-\int_0^t V_0\left(s,X^{x,\theta}(s)+R(s)\right)d\Lambda(s) \tag{62}$$

is a càdlàg supermartingale for every admissible trading strategy $\theta\in\Pi_0$ and a uniformly integrable martingale for the optimal trading strategy, we then obtain a backward stochastic differential equation for the unknown process Q as well as an explicit characterization of the optimal trading strategy in terms of the diffusion coefficient of the 3-dimensional process (D,Z,Q).

Finally, using recent results of Lepeltier and San Martin (1997) we establish the existence of a unique maximal solution to this recursive equation and conclude the proof by verifying that the constructed value function indeed coincides with the value function of the agent's utility maximization problem. Details of the proof are provided in the appendix. ■

4.4 The early resolution premium

Proposition 23 shows that the early resolution premium, as measured by the difference between early and late resolution utility based prices, is in general different from zero. Combining the expressions for the late and early resolution utility based prices prices given in Theorems 27 and 29 we get that for an investor with constant absolute risk aversion, the time-0 early resolution premium $\tau_b(e)$ on utility based buying prices is explicitly given by

$$\tau_b(e)\triangleq p_b(e)-u_b(e)=D(0)\cdot\left(\tilde{E}\mathcal{C}_e-Q(0)\right) \tag{63}$$

where the conditional certainty equivalent $\mathcal{C}_e$ of the event sensitive contingent claim under consideration is defined by (57) and Q is the maximal solution to the backward stochastic differential equation (61).

Inspection of this expression then shows that a sufficient condition for the time-0 early resolution premium $\tau_b(e)$ on the utility based buying prices to be equal to zero is that for every t we have the identity

$$\begin{aligned}\exp\left[-P(t)\right] &\triangleq \exp\left[-\tilde{E}_t \log\left(\Lambda^t(T)e^{-\gamma A} + \Gamma^t(T)e^{-\gamma \bar{R}}\right)\right] \\ &= \tilde{E}_t \exp\left[-\log\left(\Lambda^t(T)e^{-\gamma A} + \Gamma^t(T)e^{-\gamma \bar{R}}\right)\right].\end{aligned}$$

Indeed, if this condition holds then it is easily seen that the solution to the backward stochastic differential equation (61) is given by $DQ = -DP/\gamma$ and the result now follows by observing that at the initial time, this coincides with the late resolution utility based buying price given by (58).

A special case where this condition holds is that in which both the contingent claim's payoffs and the arrival intensity are deterministic. This corresponds, for example, to a default digital triggered by a deterministic intensity point process.

5 Constant Relative Risk Aversion

Let us now turn to the case where the agent's utility function exhibits constant relative risk aversion (CRRA) utility functions and is of the form:

$$u_\gamma(x) \triangleq \left\{\begin{array}{ll} x^{1-\gamma}/(1-\gamma), & \text{if } \gamma \in (0,\infty)\backslash\{1\} \\ \log x, & \text{if } \gamma = 1 \end{array}\right\}. \qquad (64)$$

Unlike in the CARA case that was studied in the previous section, utility based prices are in general functions of the wealth of the agent, as well as of his risk aversion. It is thus, in general, not possible to derive explicit results for utility based prices of ESCCs for CRRA investors. However, Propositions 15 and 24 lead to simple numerical procedures to compute both late and early resolution utility based prices under Markovian assumptions. Taking this into account, we specialize the economy to be of the Black and Scholes type and analyze some specific examples of contingent claims (credit derivatives, defaultable bonds and vulnerable options). We start by describing the setup and numerical resolution techniques before presenting the numerical results.

5.1 The model

The financial market model that we consider throughout this section is the standard Black and Scholes model of a financial market. In other words, we shall assume that there is only one stock price (or index) with constant coefficients r, a and σ. Thus we have:

$$\frac{S(t)}{S(0)} = \exp\left[\left(a - \tfrac{\sigma^2}{2}\right)t + \sigma B(t)\right] = \exp\left[\left(r - \tfrac{\sigma^2}{2}\right)t + \sigma W(t)\right] \qquad (65)$$

where the process W is the P^0-Brownian motion defined by (7). Further, we assume that the intensity of the point process under the historical probability measure is a strictly positive constant λ and consider contingent claims whose payoffs are at most deterministic functions of the terminal stock price.

5.2 No-contingent claim value function

The no-contingent claim value function and optimal trading strategy in this model are well known and were first derived by Merton (1971). For completeness we recall that they are respectively given by

$$V_0(t,x) = u_\gamma \left(x \cdot \exp\left[\left(r + \tfrac{\xi^2}{2\gamma} \right) \cdot (T-t) \right] \right) \tag{66}$$

and $\theta(\cdot) = \xi X(\cdot)/[\sigma\gamma]$ where ξ is the constant risk premium that was used in the definition of both the risk neutral probability measure P^0 and the risk neutral Brownian motion.

5.3 Computing the Value Functions

Late Resolution of Uncertainty

Assume that condition FV holds and let the agent's initial capital at time zero be such that $x > u_0(C_e)$. According to the result of Theorem 25 the agent's value function in this case is given by (here we use the fact that the utility function depends on the state only through the stock's terminal value)

$$V_e^\ell(x) = E\left[U_e \circ I_e\right](S(T), \hat{y}(x)H_0(T)) \tag{67}$$

where the optimal Lagrange multiplier $\hat{y}(x)$ is chosen in such a way that the agent's budget constraint (43) holds as an equality. In order to simplify the computation of the agent's value function, let us start by observing that thanks to the explicit form (65) of the stock price process and the definition of the state price density process we have

$$H_0(t) = \exp\left[-\left(\tfrac{a+r}{2}\right)\left(1 - \tfrac{\xi}{\sigma}\right) t \right] \cdot \left(\frac{S(T)}{S(0)} \right)^{-\frac{\xi}{\sigma}}.$$

Using the above relation between the stock price process and the state price density process it is now easily seen that the expectations in (67) and (43) depend only on the stock's terminal value. Observing that the distribution of the stock's terminal value on

$\mathbb{R}_+$ is explicitly given by (with the usual informal probabilistic notation)

$$P\left[S(T) \in dx\right] = \frac{dx}{(2\pi)^{1/2}} \exp\left[-\tfrac{1}{2\sigma T}\left(bT - \tfrac{\sigma^2}{2}T + \log\left(\tfrac{x}{S(0)}\right)\right)^2\right]$$

we can rewrite the equations (67) and (43) as ordinary integrals with respect to the above density. Such integrals are easily computed numerically by any modern mathematical software such as Mathematica.

To compute the associated late resolution utility based prices, we now need to solve the non linear equation $V_0(x) = V_e^\ell(x - p)$. For practical purposes, and in particular to graph the prices, we found it easier to inverse the problem. We first fix a Lagrange multiplier y and compute the late resolution value function and initial capital z through the integration of (67) and (43). Using the fact that the no-contingent claim value function is known explicitly, we then compute the corresponding initial capital x through the inversion formula

$$x = V_0^{-1}\left(V_e^\ell(z)\right),$$

and finally obtain the late resolution utility based buying price of the contingent by setting $p_b(x, e) = x - z$. A similar procedure is implemented to compute the late resolution utility based selling price.

Early Resolution of Uncertainty

Let us now turn to the more difficult early resolution case where the agent monitors the realization of the event time continuously. From previous results, it is easily deduced that the early resolution utility maximization problem of an agent who has initial capital x and is endowed with one unit of the claim $(A, \bar{R})$ is equivalent to that of an agent who has initial capital $x + u_0(\bar{R})$ and is endowed with one unit of the claim $(A - \bar{R}, 0)$. Taking this fact into account, we shall from now on restrict ourselves to the computation of the value function associated with ESCC of the form

$$h \triangleq \left(Q\left(S(T)\right), \bar{R} \equiv 0\right)$$

for some well behaved function of the stock's terminal value. Consider an agent endowed with one unit of such a claim, fix an arbitrary intermediate time t and suppose that the event is still to occur. Assuming that the wealth reached by the agent at time t is equal to $X(t)$ and using the reformulation result of Theorem 15 in conjunction with the Markovian structure of our financial market model, we may write the agent's value function at the time t as

$$q\left(t, S(t), X(t)\right) = \sup_{\theta \in \Theta_0^t} E\Bigg(e^{-\lambda(T-t)} u\left(X^\theta(T) + \alpha Q\left(S(T)\right)\right) + \int_t^T \lambda e^{-\lambda(u-t)} V_0\left(u, X^\theta(u)\right) du \,\Bigg|\, S(t), X(t) \Bigg) \tag{68}$$

where X^θ denotes the value process associated with the trading strategy θ and where the no-contingent claim value function at time u is given by (66). The dynamic programming equation associated with the above stochastic control problem is the non linear, second order partial differential equation

$$\sup_{\theta \in \mathbb{R}} \left\{ \mathcal{D}^\theta q\left(t, s, x\right) + \lambda\left[V_0(t, x) - q(t, s, x)\right] + \frac{\partial q}{\partial t}\left(t, s, x\right) \right\} = 0 \tag{69}$$

where $\mathcal{D}^\theta$ denotes the infinitesimal generator of the two dimensional diffusion process with coordinates S and X^θ. Standard verification theorems as found for example in chapter 3 of Fleming and Soner show that if there exists a smooth solution to the above partial differential equation with the appropriate boundary conditions then this solution coincides with the value function. The difficulty here is that in most cases such a classical solution cannot be shown to exist. Nevertheless, under quite general assumptions the value function can be shown to constitute the unique viscosity solution to (69) with the appropriate boundary conditions and this property gives a sufficient theoretical justification for the use of finite differences discretisation schemes (sufficient conditions for this property to hold can be found in Yong and Zhou (2000, Chapter 4)).

To solve the dynamic programming equation associated with our problem we use an explicit finite difference scheme, as suggested in Fleming and Soner (1993, Chapter 9) and which basically consists in approximating the continuous time problem by a discrete time problem for a suitably chosen Markov chain. Such a discretisation actually transforms the partial differential equation (69) into a difference equation which can be solved numerically by going backwards from the terminal time to the initial time. A point worth noting is that thanks to the concavity of the utility function, the maximization in (69) can be carried out explicitly so that one can avoid the tedious task of solving the associated optimization problem numerically at each step of the algorithm.

Let us now consider an economic agent who has initial capital x at time zero and is endowed with one unit of the claim $(A, \bar{R})$. Using our finite difference discretisation scheme with the modified pay-off function $Q = A - \bar{R}$ we obtain the expected utility index reached by this agent at time zero through the formula

$$V_e(x) = q\left(0, S(0), x + u_0(\bar{R})\right)$$

where the risk neutral price $u_0(R)$ is computed by numerical integration of the pay-off function against the risk neutral distribution of the stock's terminal value. Applying a procedure similar to the one we used to calculate the late resolution utility based prices, we can now compute the early resolution prices by comparing the above expected utility index with the explicit formula for the no-contingent claim value functions given in (66).

5.4 Comparison of Results for an Event Digital

We first analyze a security which pays a fixed amount at maturity, contingent on occurrence of the event before maturity. This is analogous to, for example, a credit derivative or an IO/PO security.

Figure 2 shows the utility based buying and selling prices of such a security as a function of the agent's initial capital for the two scenarios of resolution of uncertainty. Figure 2 graphs the early resolution buy and sell premia, that is the differences between early and late resolution buy prices and between late and early resolution sell prices. The figures confirm:

1. The ordering of the utility based prices prescribed by (40).
2. As the agent's initial capital increases all prices converge towards the risk neutral price $u_0(e)$ which attributes a zero risk premium to the event risk.
3. As the agent's initial capital tends to zero, the utility based selling prices converge to the upper hedging price and the utility based buying prices to lower hedging price.
4. The early resolution premium is positive and tends to zero as the agent's initial capital tends to zero and infinity.

When his initial capital tends to zero, the CRRA agent becomes highly risk averse and prices the claim assuming the worst case scenario, that is assuming that the event never occurs if he buys, and that it always occurs if he sells. Likewise, when his initial capital increases relative to his position in the claim, the investor becomes less risk averse relative to the event risk and is therefore is willing to buy and sell it at the risk-neutral price $u_0(e)$ which attributes a zero

Insert Figure 1 here

risk-premium to the event risk. In other words, the agent uses the historical ($P-$measure) intensity of the event time τ to discount payoffs under the risk neutral probability measure. Although this result is intuitive, it is interesting to see how fast the convergence occurs. For credit exposures (measured by the ratio of the credit derivative nominal to the investor's initial capital) less than 75%, the utility based prices already fall within a few basis points of nominal from the risk neutral price. These results provide a justification for the approach of some practitioners (e.g. JP Morgan's CreditMetrics®) who use historical estimates of default probabilities to price credit risky securities, while maintaining credit limits with their different counterparties.

With respect to the temporal resolution of uncertainty, our figure confirms that late resolution of uncertainty leads to higher bid/ask spreads. Agents always prefer early to late resolution of uncertainty since, with incomplete markets, the former allows them to modify to their advantage their portfolio strategy in response to the event. In that sense, the temporal resolution of uncertainty affects the investor's wealth process and therefore is not irrelevant. Interestingly,

Insert Figure 2 here

willingness to pay for early resolution depends on the agent's initial capital since the early and late resolution utility based prices coincide for both low and high levels of initial capital. As illustrated in Figure 2 the amount which the investor would be willing to pay for early resolution of uncertainty is therefore (almost) concave in the agent's initial capital. As pointed out by Ross (1989) the temporal of uncertainty is irrelevant for the purpose of arbitrage pricing. Interestingly his result, which was originally proved in a complete markets model, extends here to our incomplete markets setting when taking the appropriate concept of almost sure hedging prices. As the agent's initial capital increases relative to his position in the claim, the impact of event risk on his wealth becomes negligible. Thus, the opportunity to readjust his portfolio decision in response to information becomes less valuable and the difference between early and late resolution utility based prices vanishes. Quantitatively, the early resolution premium appears small but not insignificant: around 30 basis points at its maximum for a logarithmic utility investor.

5.5 Defaultable Bonds

We next investigate the utility based pricing of a defaultable bond which pays \$100 at maturity and has a recovery rate of 50% in the event of default. In such a case the event time τ represents the time at which the bond seller defaults on its obligation. Since the utility maximization problems studied in this paper do not take into account the possibility that the agent who prices the claim may default on his obligations, we focus on the utility based buying prices only.¶

Figure 3 graphs the utility based buying price of this defaultable bond as a function of the agent's initial capital for various levels of the historical default intensity. As for an event digital option, we see that the price a logarithmic agent would be willing to pay to buy the bond is rapidly increasing towards the

Insert Figure 3 here

risk neutral price.‖ As the default intensity λ increases, the risk neutral price (which is independent of wealth) decreases accordingly since the default probabilities are the same under both the historical and the risk neutral measure and so does the utility based buying price.

Figure 4 graphs the per unit utility based buying price $n \to u_b(x, n \cdot e)/n$ of the defaultable bond under consideration as a function of the position size for various levels of initial capital. For a given position size the utility based buying price per unit of contingent claim is increasing in the agent's initial capital. The figure effectively displays a decreasing inverse demand function for defaultable bonds for each level of initial capital. Because the financial market faced by the agent does not allow him to hedge the default event, his demand function is not

Insert Figure 4 here

perfectly elastic and the price at which he is willing to purchase the defaultable bond is decreasing in the size of his order. For a given level of initial capital the utility based unit buying price converges to the risk neutral price as the position size goes to zero. However, the larger the initial capital the slower the convergence. This obviously contrasts with the traditional complete markets analysis where inverse demand functions are constant at the no-arbitrage price because the investor can offset an arbitrary position in the contingent claim by an appropriate hedging position in marketed securities.

Figure 5 graphs the utility based buying price as a function of the investor's initial capital capital for two different levels of expected return on the stock. Although, changing the drift and hence the risk premium on traded assets does not change the risk neutral and arbitrage prices of the defaultable bond, Figure 5 clearly shows that such a modification has an impact on the corresponding utility based buying price. For a given level of initial

¶Duffie and Huang (1996) and Collin Dufresne and Hugonnier (2000) investigate pricing in the presence of bilateral counterparty credit risk.

‖Since, for logarithmic agents, the late resolution prices provide reasonable approximation of the early resolution utility based prices we present only graphs for late resolution prices. These are also simpler to compute.

capital, the utility based buying price is a decreasing function of the risk premium on the stock. This is due to a substitution effect: as the drift of the underlying stock price increases

Insert Figure 5 here

the defaultable bond becomes less interesting compared to the stock itself, and, as a result its utility based buying price decreases. However, Figure 5 also shows that the impact of this substitution effect decreases as the agent's initial capital increases. The rationale for this is the following: as the agent's initial capital increases, the effect of his position in the defaultable bond on his utility index becomes marginal thus making the substitution effect smaller. As we shall see in the next section, the impact of a change of drift on the utility based prices may be ambiguous when the contingent claim's payoff is state dependent.

5.6 Vulnerable Options

In this last section we investigate the utility based pricing of vulnerable call and put options. In cases where the seller of the option defaults prior to its maturity we assume that the buyer of the option receives nothing and restrict ourselves to the study of the utility based buying prices.

Figures 6 graphs the utility based buying price of a call option as a function of the agent's initial capital for different moneyness of the option. The figure shows that the convergence of utility based buying price towards the risk neutral price is somewhat slower as the option becomes in the money. However, in all

Insert Figure 6 here

cases we see that for *reasonable* levels of credit exposure as measured by the ratio of the default free option to the agent's initial capital, the utility based buying price of vulnerable options is well approximated by the risk neutral price computed by taking an expectation of the payoff under the risk neutral measure without risk adjusting the historical default intensity.

The remaining figures investigate the impact of the drift on the utility based buying prices of vulnerable options. This seems particularly interesting since one of the major insights of the contingent claim pricing literature initiated by Merton and Black and Scholes is that option prices are independent of the drift of the underlying asset price process. In our case, it is still true that arbitrage pricing (as reflected by the no-arbitrage bounds) is independent of the drift of the underlying security. However, as the figures clearly show, utility based prices depend on the drift of the underlying security.

As already mentioned, the impact will not be as clear here as for defaultable bonds since two different effects may influence prices. First, the substitution effect makes the contingent claim less interesting compared to the stock as the risk premium on the stock increases. The second effect is due to the fact that

Insert Figure 7 here

the options' payoffs depend on the terminal stock price: as the drift of the underlying stock price process increases, the objective probability that the put (resp. call) will end up in the money decreases (resp. increases).

For put options the two effect go in the same direction and we thus expect their utility based prices to decrease when the drift of the underlying stock increases. Figure 7 confirms this intuition. However, for call option the two effects compensate each other and the net effect is ambiguous. Figure 8 shows

Insert Figure 8 here

that there are sets of parameters for which the call option's utility based buying prices are increasing in the drift of the stock: the 'payoff' effect dominates the substitution effect. However, for deep in the money call options, the substitution effect dominates. The effect of a change in the drift of the underlying asset on the utility based buying price of the call is therefore undetermined in general and will depend on the particular parameter choice.

6 Conclusion

We study the utility based pricing of event sensitive contingent claims, defined as securities whose payoff is contingent on the occurrence of an event that cannot be hedged with standard marketed securities.

We solve the incomplete market problem of an investor endowed with an ESCC under two scenarios of resolution of uncertainty. In general, investors with time separable utility functions are willing to pay a premium for early resolution of uncertainty, because the event cannot be hedged with marketed securities and thus forces them to self-insure.

We analyze constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA) utility functions in more detail. For CARA investors we obtain explicit early and late resolution prices which are independent of wealth. The early resolution premium is constant and in general non zero, but we discuss a sufficient condition for it to be zero. For CRRA utility we propose a simple numerical scheme to compute both early and late resolution prices.

We analyze prices of different securities such as event digitals, defaultable bonds and vulnerable options. The early resolution premium is positive but tends to zero when wealth tends to zero or infinity.Interestingly, as the exposure (relative to initial wealth) to the ESCC becomes small, all prices tend towards a common limit: the risk-neutral price, obtained by risk-adjusting all the traded sources of risk while leaving the functional form of the event intensity unchanged. As our numerical results show, most standard pricing intuition breaks down for ESCC. For example, we find that vulnerable call options prices can be either increasing or decreasing in the drift of the underlying depending on the moneyness of the option.

A Proof of Theorem 29

Theorem 29 will be established through a series of Lemmas. Before turning to the actual proof, we start by fixing some notation that will be useful later on. Recall that the zero-coupon price process D is defined by

$$D(t) \triangleq E_t\left(\frac{H_0(T)}{H_0(T)}\right) \equiv E_t^0\left(\frac{D_0(T)}{D_0(t)}\right). \tag{A.1}$$

Taking into account the positivity and boundedness of the discount factor and applying the representation theorem to the martingale $H_0 D$ we have that there is an $\mathbb{F}-$progressively measurable, square integrable process ϑ such that

$$D(t) - D(0) = \int_0^t D(s)\left(r(s)ds + \vartheta(s)^*\left[dB(s) + \xi(s)ds\right]\right). \tag{A.2}$$

Combining this expression with the definition of the forward probability measure and applying Girsanov theorem, it follows that process

$$\tilde{B}(t) \triangleq W(t) - \int_0^t \vartheta(s)ds \equiv B(t) + \int_0^t \left(\xi(s) - \vartheta(s)\right)ds \tag{A.3}$$

is a standard Brownian motion under the forward probability measure. Using this in conjunction with (54), the definition of H_0 and the boundedness of the model coefficients and applying Itô's representation theorem under the forward probability measure we obtain that there is an $\mathbb{F}-$progressively measurable and square integrable (under the forward probability measure) process φ such that

$$\begin{aligned} d\left(\frac{Z(t)}{D(t)}\right) &= d\left(D(t)^{-1}\cdot E_t^0\left(\frac{D_0(T)}{D_0(t)}\log\frac{H_0(T)}{H_0(t)}\right)\right) = d\left(\tilde{E}_t \log\frac{H_0(T)}{H_0(t)}\right) \\ &= \varphi(t)^* d\tilde{B}(t) + \left(r(t) - \tfrac{1}{2}\left\|\xi(t)\right\|^2 + \vartheta(t)^*\xi(t)\right)dt. \end{aligned} \tag{A.4}$$

After these preparations, we are now ready to begin our proof of Theorem 29. Let now Q be an arbitrary but fixed non negative, bounded process with terminal value equal to A and for each admissible trading strategy θ set

$$V_\theta(t) \triangleq \Lambda(t) V_0\left(t, X^{x,\theta}(t) + D(t)Q(t)\right) - \int_0^t V_0\left(s, X^{x,\theta}(s) + R(s)\right) d\Lambda(s)$$

where $R(\cdot)$ is the replicating value process associated with the event insensitive contingent claim $\bar{R}$ and where $V_0(\cdot)$ is defined as in (56).

Lemma A.1 *For each admissible trading strategy $\theta \in \Pi_0$, the process V_θ is uniformly bounded from below by an integrable random variable.*

Proof. Combining Jensen's inequality with (A.4), the convexity of the function $\exp(\cdot)$ and the definition of the forward probability measure we get that

$$D(t) \cdot \exp\left(-\frac{Z(t)}{D(t)}\right) \leq D(t) \cdot \tilde{E}_t \exp\left(-\log \frac{H_0(T)}{H_0(t)}\right) = 1$$

holds almost surely for every $t \in [0, T]$. Plugging this estimate back into the definition of V_θ and using the boundedness of the processes $(\lambda, \Lambda, D, R, Q)$ it is now easily seen that there are non negative constants C_i such that

$$\begin{aligned} -V_\theta(t) &\leq C_1 \cdot \exp\left(-\gamma \frac{X^{x,\theta}(t)}{D(t)}\right) + \int_0^t C_2 \cdot \exp\left(-\gamma \frac{X^{x,\theta}(s)}{D(s)}\right) ds \\ &\leq C_1 \cdot \exp\left(-\gamma \inf_{t \in [0,T]} \frac{X^{x,\theta}(t)}{D(t)}\right) + C_3 \cdot \exp\left(-\gamma \inf_{t \in [0,T]} \frac{X^{x,\theta}(s)}{D(s)}\right) \cdot T. \end{aligned}$$

By definition of the class Π_0 of admissible trading strategies the negative random variable on the right hand side of the above expression is integrable under the objective probability measure and our proof is complete. ■

Let now (Σ, μ) denote respectively the volatility and the drift of the unknown process Q under the objective probability measure. Using the definition of the no-contingent claim value function in (56) and applying Itô's lemma we obtain (after simplification) that for an arbitrary trading strategy

$$\begin{aligned} dW_\theta(t) &\triangleq dV_0\left(t, X^{x,\theta}(t) + D(t)Q(t)\right) \\ &= W_\theta(t)\left(\vartheta(t) - \gamma\widehat{\theta}(t) - \varphi(t) - \gamma\Sigma(t)\right)^* dB(t) \\ &+ \frac{1}{2} W_\theta(t) \left\|\gamma\widehat{\theta}(t) + \varphi(t) + \gamma\Sigma(t) - \xi(t)\right\|^2 dt \\ &- \gamma W_\theta(t)\left(\mu(t) + \Sigma(t)^*\left(\vartheta(t) - \xi(t)\right)\right) dt \end{aligned}$$

where we have set $\widehat{\theta} := (\sigma^*\theta - X^{x,\theta}\vartheta)/D$. Plugging this back into the definition of the process V_θ and applying Itô's lemma once again we finally obtain that for an arbitrary trading strategy the dynamics of the process V_θ are given by

$$\begin{aligned} dV_\theta(t) &= d\left(\Lambda(t)W_\theta(t)\right) + \lambda(t)\Lambda(t)W_\theta(t)\exp\left(\gamma\cdot[Q(t)-R(t)/D(t)]\right)dt \\ &= \Lambda(t)W_\theta(t)\left(\vartheta(t)-\gamma\widehat{\theta}(t)-\varphi(t)-\gamma\Sigma(t)\right)^* dB(t) \\ &+ \frac{1}{2}\Lambda(t)W_\theta(t)\left\|\gamma\widehat{\theta}(t)+\varphi(t)+\gamma\Sigma(t)-\xi(t)\right\|^2 dt \\ &- \gamma\Lambda(t)W_\theta(t)\left(\mu(t)+\Sigma(t)^*\left(\vartheta(t)-\xi(t)\right)\right)dt \\ &- \lambda(t)\Lambda(t)W_\theta(t)\left(1-e^{\gamma\cdot[Q(t)-R(t)/D(t)]}\right)dt. \end{aligned}$$

Using the above expression in conjunction with Lemma B.1, the fact that a local martingale which is uniformly bounded from below is a supermartingale and the negativity of the process W_θ, it is now easily seen that if the coefficients of the unknown process are such that

$$\mu(t) = \Sigma(t)^*\left(\xi(t)-\vartheta(t)\right) - \frac{\lambda(t)}{\gamma}\left(1-e^{\gamma\cdot[Q(t)-R(t)/D(t)]}\right)$$

then the process V_θ is a supermartingale for every admissible trading strategy and a martingale for the optimal trading strategy defined implicitly by

$$\widehat{\theta}_*(t) \triangleq -\Sigma(t) + (1/\gamma)\cdot\left(\xi(t)-\varphi(t)\right). \tag{A.5}$$

In order to complete the first part of the proof of Theorem 29 we therefore have to establish that

(i) there exists a non negative, bounded process with terminal value equal to A almost surely and whose coefficients verify the above restriction,

(ii) that the candidate optimal trading strategy defined implicitly by (A.5) indeed belongs to the set Π_0 of admissible trading strategies.

Before establishing these we start by observing that thanks to (A.2) and the definition of the forward probability measure, item (i) above is equivalent to the existence of a non negative, bounded process with terminal value equal to A almost surely and whose dynamics are given by

$$-dQ(t) = \frac{\lambda(t)}{\gamma}\left(1-e^{\gamma\cdot[Q(t)-R(t)/D(t)]}\right)dt - \Sigma(t)^* d\tilde{B}(t) \tag{A.6}$$

where $\tilde{B}$ is an $n-$dimensional standard Brownian motion under the forward measure. Such equations are known in the literature as Backward Stochastic Differential Equations and have been extensively studied in the past years (see the monograph edited by El Karoui and Mazliak (1997) for references).

In the following statement we say that the pair (Σ, Q) of adapted processes is a maximal solution to the backward equation if its trajectory dominates that of any other solution.

Lemma A.2 *Let* $(A, \bar{R})$ *be non negative, bounded random variables and* R *be defined by* (36)*. Then the backward stochastic differential equation* (A.6) *admits a maximal solution* (Σ, Q) *whose trajectory is non negative and bounded.*

Proof. Using Lemma 5 in conjunction with (A.2) and applying Itô's lemma, we have that there is an $\mathbb{F}-$progressively measurable process ρ such that

$$d\left(R(t)/D(t)\right) = -\rho(t)^* \left(dB(t) + [\xi(t) - \vartheta(t)]\, dt\right) = -\rho(t)^* d\tilde{B}(t) \tag{A.7}$$

where the last equality follows from the definition of the process $\tilde{B}$ in (A.3). Combining this with the fact that the terminal value of D is equal to one, it is now easily seen that the result of the lemma will follow once we have established that the backward stochastic differential equation

$$-d\bar{Q}(t) = \frac{\lambda(t)}{\gamma}\left(1 - e^{\gamma \bar{Q}(t)}\right) dt - \bar{\Sigma}(t)^* d\tilde{B}(t) \tag{A.8}$$

with terminal condition equal to $A - \bar{R}$ admits a maximal bounded solution whose trajectory dominates that of the non positive process $-R/D$. To this end, we start by observing that because of the boundedness of λ we have

$$|f\,(t, \omega, x)| \triangleq \left|\tfrac{\lambda(t)}{\gamma}\left(1 - e^{\gamma x}\right)\right| \leq C \cdot |1 - e^{\gamma x}| \triangleq \ell(x)$$

for some non negative constant C. Straightforward computations using the definition of the non negative function ℓ then show that the pair of (ordinary) backward differential equations given by

$$\begin{aligned} A_1(t) &= a_1 - \int_t^T \ell\left(A_1(s)\right) ds \\ A_2(t) &= a_2 + \int_t^T \ell\left(A_2(s)\right) ds \end{aligned}$$

admit bounded global solutions for every $a_1 \leq 0 \leq a_2$ and the existence of a maximal bounded solution to the backward stochastic differential equation (A.8) now follows from Lepeltier and San Martin (1997, Theorem 1).

In order to complete the proof, we are only left to show that the trajectory of this maximal solution dominates that of the process $-R/D$ but this easily follows from the negativity of the latter process, the fact that its terminal condition is dominated by that of $\bar{Q}$ and the comparison theorem for backward stochastic differential equations (see Lepeltier and San Martin (1997, Corollary 2) or El Karoui and Mazliak (1997, Theorem 2.5)). ■

Let now $\widehat{\theta}_*$ be defined by (A.5) and denote by θ_* the corresponding candidate optimal trading strategy. Using (A.3) in conjunction with the dynamics of the zero-coupon bond price process given by (A.2) and applying Itô's lemma we get

$$d\left(\frac{X^{x,\theta_*}(t)}{D(t)}\right) \triangleq dX_*(t) = \widehat{\theta}(t)^* d\tilde{B}(t) = \left(\tfrac{1}{\gamma}\xi(t) - \tfrac{1}{\gamma}\varphi(t) - \Sigma(t)\right)^* d\tilde{B}(t).$$

Comparing the previous expression with the dynamics of the processes Q, Z/D and $\log H_0$ it follows that the candidate optimal wealth process is given by

$$\begin{aligned}\frac{X^{x,\theta_*}(t)}{D(t)} &= (Q(0) - Q(t)) - \tfrac{1}{\gamma}Z(t) + \left(x + \tfrac{1}{\gamma}Z(0)\right)\frac{1}{D(0)} \\ &- \frac{1}{\gamma}\left(\log H_0(t) + \int_0^t \lambda(s)\left\{1 - e^{\gamma\cdot[Q(s)-R(s)/D(s)]}\right\} ds\right).\end{aligned}$$

Using the boundedness of the three dimensional process $(D, R/D, Q)$ in conjunction with the definition of the process Z/D and of the forward probability measure, it is now easily seen that there are non negative constants K_i such that we have

$$\begin{aligned}-\left(K_1 + \inf_{t\in[0,T]}\frac{X^{x,\theta_*}(t)}{D(t)}\right) &\leq \sup_{t\in[0,T]}\frac{1}{\gamma}\left(\log H_0(t) + \frac{Z(t)}{D(t)}\right) \\ &\leq \sup_{t\in[0,T]} E_t^0\left(\frac{K_2 D_0(T)}{\gamma}\log^+ H_0(T)\right) \triangleq -X(\omega).\end{aligned}$$

The process D being bounded, it follows that in order to complete the proof of (ii) it is sufficient to show that the random variable X belongs to the space L and is such that $\exp(-\gamma X)$ is integrable under the objective probability measure. To establish the first property we observe that there are non negative constants C_i such that

$$\begin{aligned}E^0|X|^p &\leq C_1 \cdot E^0 \sup_{t\in[0,T]}\left(E_t^0 \log^+ H_0(T)\right)^p \leq C_1 \cdot E^0\left(\log^+ H_0(T)\right)^p \\ &\leq C_1 \cdot E^0\,|\log H_0(T)|^p \leq C_2 + C_3 \cdot E^0\left|\int_0^T \xi(t)^* dW(t)\right|^p\end{aligned}$$

where the first inequality follows from non negativity of the interest rate process, the second from Doob's maximal inequality and the fourth from the definition of the state price density H_0 in conjunction with the boundedness of the processes r and ξ. Using Burkholder-Davis-Gundy inequality we then get that

$$E^0 \left| \int_0^T \xi(t)^* dW(t) \right|^p \leq C_4 \cdot E^0 \left| \int_0^T \|\xi(t)\|^2 \, ds \right|^{\frac{p}{2}}$$

holds for some non negative constant C_4 and conclude that the random variable X belongs to the space L. To establish the second property we observe that there are non negative constant C_5 and C_6 such that

$$\begin{aligned} \exp(-\gamma X) C_5 &\leq \sup_{t\in[0,T]} \exp\left(E_t^0 D_0(T) \log^+ H_0(T)\right) \\ &\leq \sup_{t\in[0,T]} E_t^0 \exp\left(D_0(T) \log^+ H_0(T)\right) \\ &\leq \sup_{t\in[0,T]} E_t^0 \exp\left(\log^+ H_0(T)\right) \leq \sup_{t\in[0,T]} \left(1 + E_t^0 H_0(T)\right) \\ &\leq \sup_{t\in[0,T]} \left(1 + E_t^0 Z_0(T)\right) \leq \sup_{t\in[0,T]} \left(1 + C_6 \cdot Z_0(t)\right) \end{aligned}$$

where the second inequality follows Jensen's inequality, the third and fifth from the non negativity of the interest rate and the sixth from the definition of the process Z_0 and the boundedness of ξ. Using the latter property once again we obtain that the random variable on the right hand side is integrable under the objective probability measure and conclude that θ_* is indeed optimal for the agent's utility maximization problem.

In order to complete the proof of Theorem 29 we are now only left to show that as the agent's absolute risk aversion goes to zero the utility based buying price $D(0)Q(0)$ converges to the risk neutral price

$$\begin{aligned} u_0(e) &= E^0 \left(D_0(T) A 1_{\{\tau>T\}} + D_0(T)\bar{R} 1_{\{\tau\leq T\}}\right) \\ &= D(0)\tilde{E}\left(\Lambda(T)A + (1-\Lambda(T))\bar{R}\right). \end{aligned}$$

Equivalently, we need to show that as γ goes to zero, the initial value $\bar{Q}(0) \equiv \bar{Q}_\gamma(0)$ of the maximal solution to the backward stochastic differential equation (A.8) converges to the initial value of the unique solution to the backward stochastic differential equation with dynamics

$$-dY(t) = -\lambda(t)Y(t)dt - H(t)^* d\tilde{B}(t) \tag{A.9}$$

and terminal value $A - \bar{R}$. Observing that $-x \leq (1 - e^{\gamma x})/\gamma := g(\gamma, x)$ and applying the comparison theorem for backward stochastic differential equations, we obtain that

$\bar{Q}_\gamma(t) \geq Y(t)$ holds almost everywhere for all non negative γ. Taking into account the fact that these two processes have the same terminal value and using their respective dynamics (A.8)-(A.9), we then get that

$$\begin{aligned} 0 \leq \bar{Q}_\gamma(0) - Y(0) &= \tilde{E} \int_0^T \lambda(t) \left(Y(t) + g\left(\gamma, \bar{Q}_\gamma(t)\right)\right) dt \\ &\leq \tilde{E} \int_0^T \lambda(t) \left(Y(t) + g\left(\gamma, Y(t)\right)\right) dt \end{aligned}$$

where the inequality follows from the decrease of $g(\gamma, \cdot)$. Letting the agent's absolute risk aversion go to zero on both sides of the previous expression and using the fact that

$$\lim_{\gamma \to 0} -g(\gamma, Y(t)) = Y(t)$$

in conjunction with the boundedness of (λ, Y) (recall that the random variables $(A, \bar{R})$ are assume to be bounded) we conclude that $\lim_{\gamma \to 0} \bar{Q}_\gamma(0) = Y(0)$ and our proof is complete.

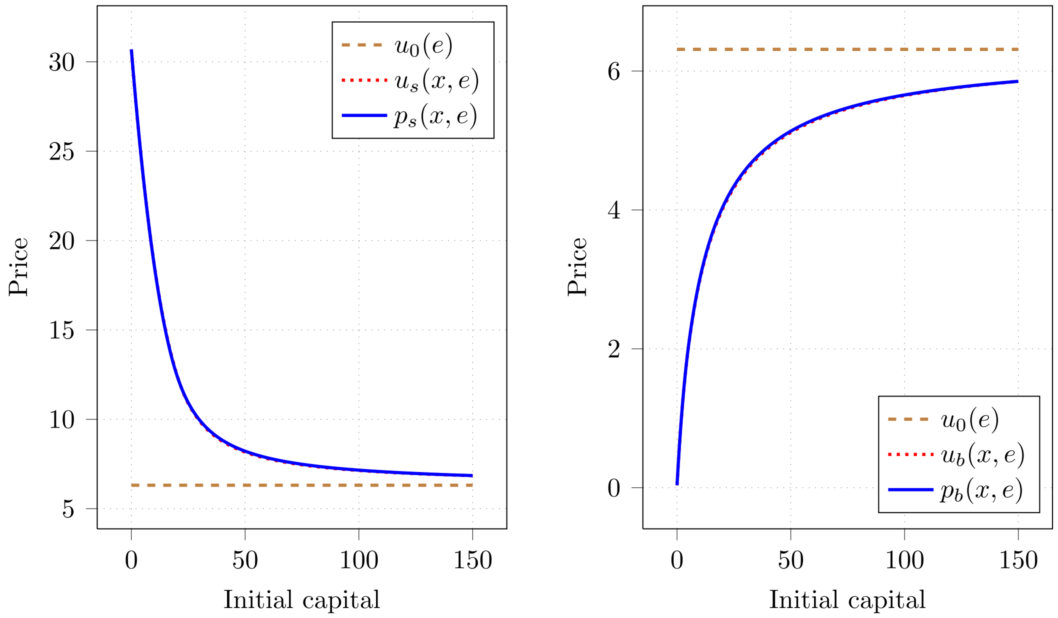

Figure 1: Utility based buying prices (right panel) and selling prices (left panel) for an event digital as functions of the initial capital when the parameters of the model are given by $(A, \bar{R}, T, \lambda, \gamma, a, r, \sigma) = (0, 30, 1, 0.25, 1, 0.1, 0.05, 0.2)$

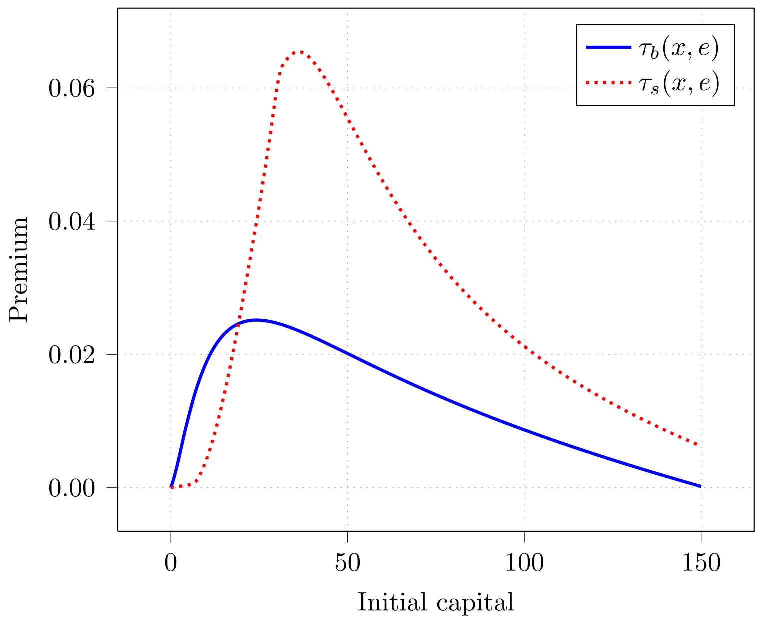

Figure 2: Early resolution premia for an event digital as functions of the initial capital when the parameters of the model are given by $(A, \bar{R}, T, \lambda, \gamma, a, r, \sigma) = (0, 30, 1, 0.25, 1, 0.1, 0.05, 0.2)$

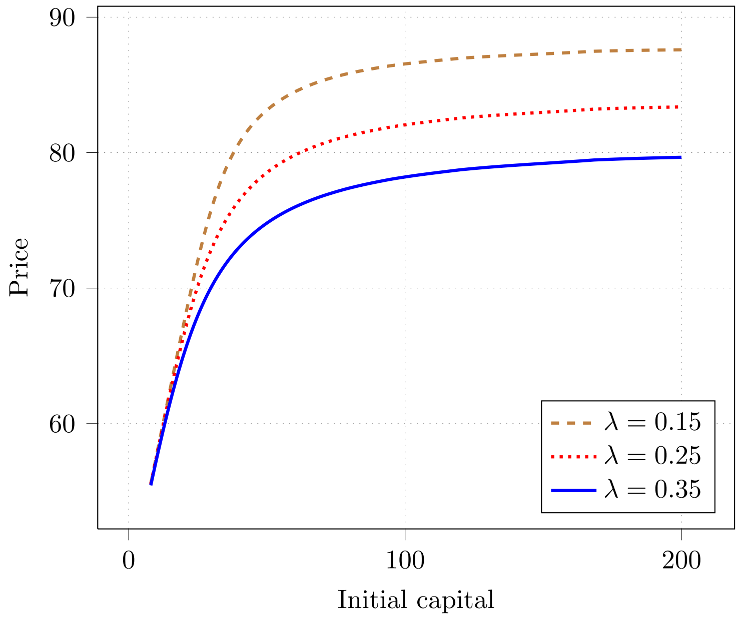

Figure 3: Utility based buying prices for an event digital as functions of the agent's initial capital for various level of the event intensity when when the other parameters of the model are given by $(A, \bar{R}, T, \gamma, a, r, \sigma) = (0, 30, 1, 0.25, 0.1, 0.05, 0.2)$

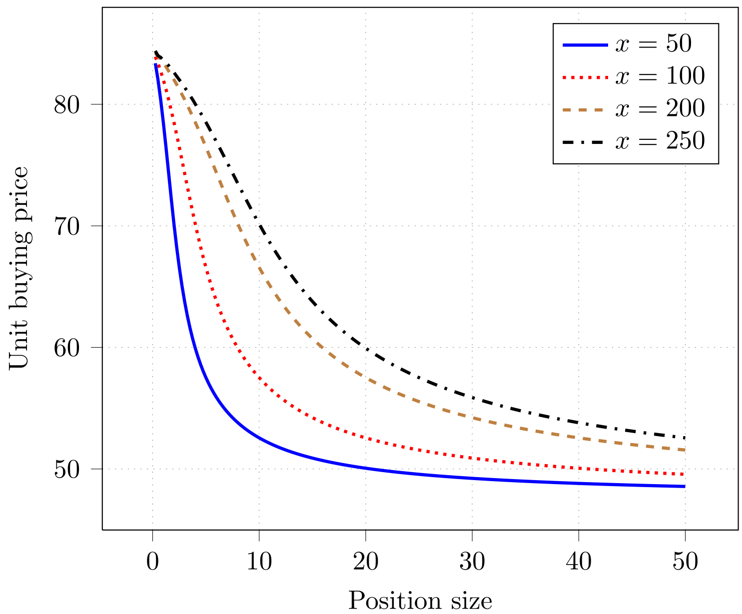

Figure 4: Utility based unit buying prices for a defaultable bond as functions of the number of bonds in the portfolio when the parameters of the model are given by $(A, \bar{R}, T, \gamma, a, r, \sigma) = (100, 50, 1, 0.25, 1, 0.1, 0.05, 0.2)$.

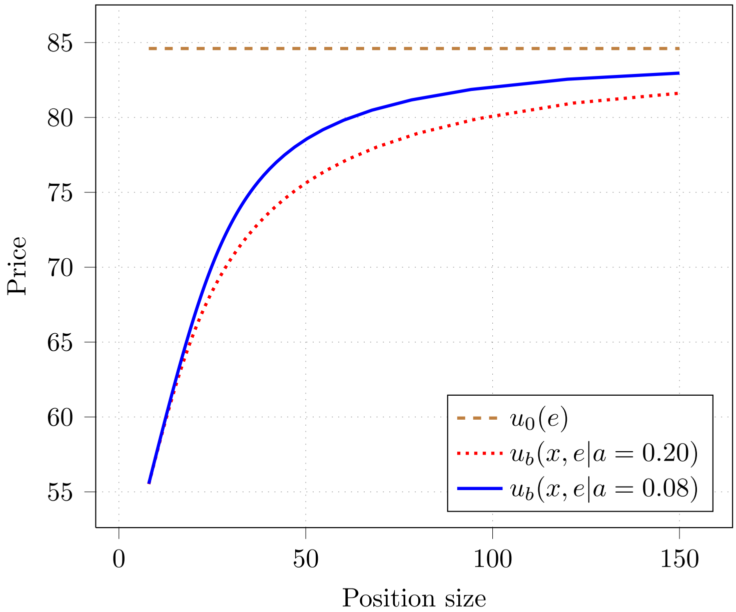

Figure 5: Utility based unit buying prices for a defaultable bond as functions of the agent's initial capital for various values of the asset return when the other parameters of the model are given by $(A, \bar{R}, T, \gamma, r, \sigma) = (100, 50, 1, 0.25, 1, 0.05, 0.2)$.

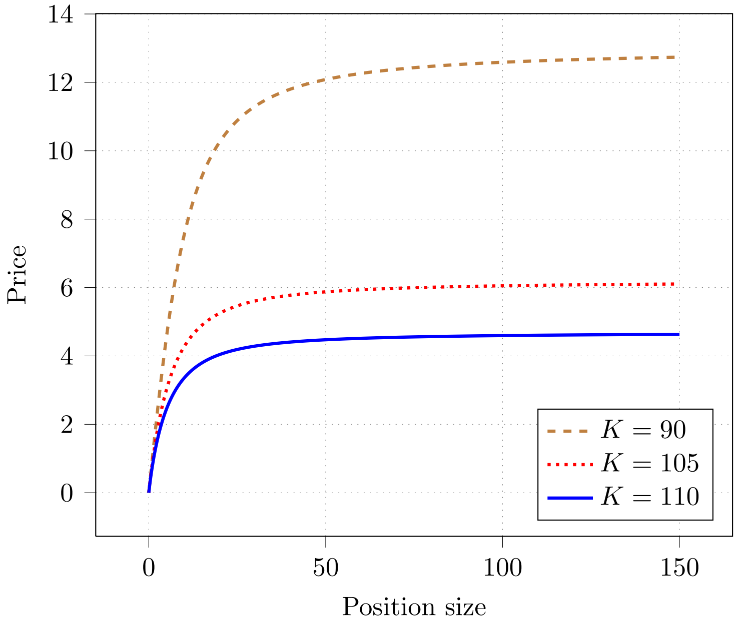

Figure 6: Utility based unit buying prices for a vulnerable call option as functions of the agent's initial capital for different exercise prices when the other parameters of the model are given by $(S(0), \bar{R}, T, \gamma, a, r, \sigma) = (100, 0, 1, 0.25, 1, 0.1, 0.05, 0.2)$.

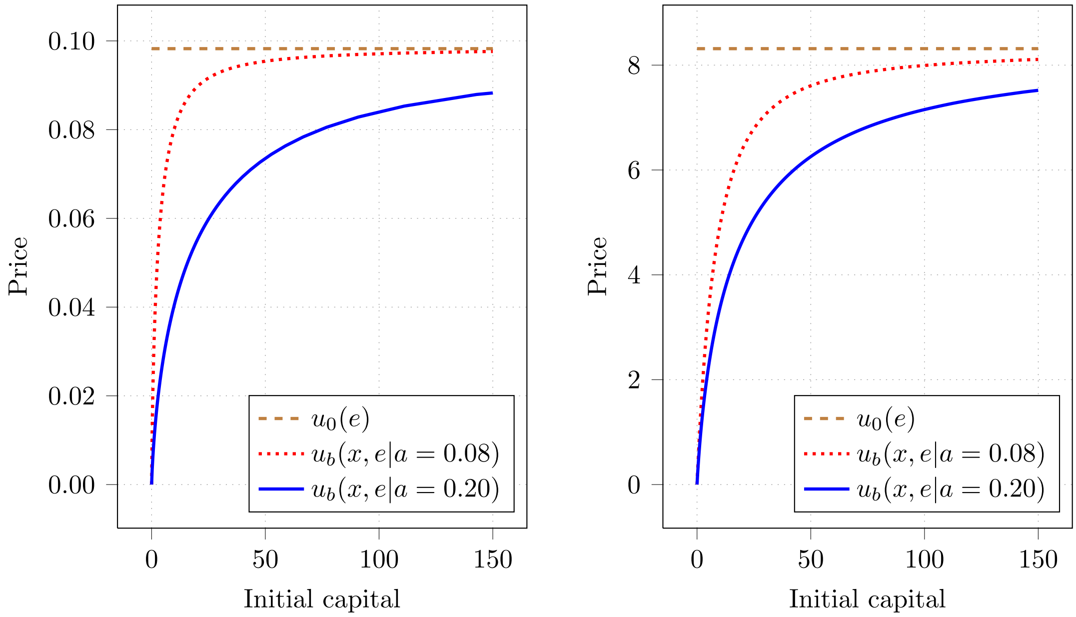

Figure 7: Utility based buying prices for an OTM put (left panel) and an ITM put (right panel) as functions of the agent's initial capital for various values of the underlying asset return when the other parameters of the model are given by $(S(0), \bar{R}, T, \gamma, r, \sigma) = (100, 0, 1, 0.25, 1, 0.05, 0.2)$.

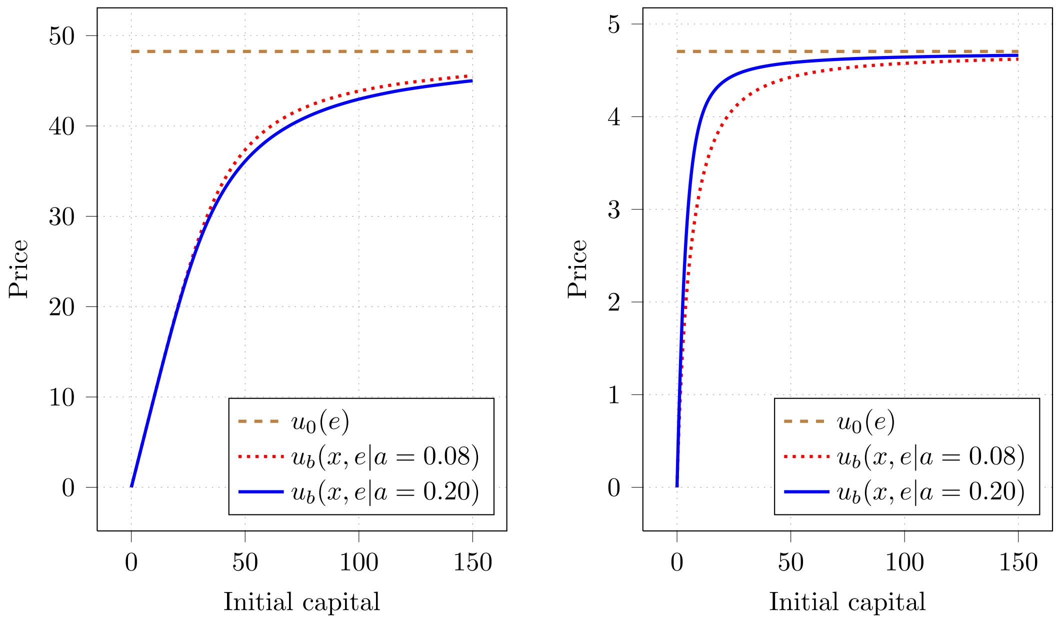

Figure 8: Utility based buying prices for an ITM call (left panel) and an OTM call (right panel) as functions of the agent's initial capital for various values of the underlying asset return when the other parameters of the model are given by $(S(0), \bar{R}, T, \gamma, r, \sigma) = (100, 0, 1, 0.25, 1, 0.05, 0.2)$.