Supplementary Appendix to:

Capital supply uncertainty, cash holdings, and investment

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This document provides the proofs omitted from the main text and discusses the extension of the model to the case of a firm with multiple growth options. To avoid confusions the numbering of sections, equations and figures is continued from the text.

H Bargaining with outside investors

This Appendix shows that introducing bargaining is equivalent to reducing the arrival rate of outside investors. To simplify the presentation, we start by considering the case of a firm with no growth option before turning to the general case.

If the firm has mean cash flow rate μ_i and no growth option, its optimization problem can be formulated as

$$V_i(c) = \sup_{\pi \in \Theta} E_c \left[\int_0^{\tau_0} e^{-\rho s}(dD_s - (f_s + \eta S(C_{s-}^\pi | (\pi, V_i)))dN_s) + e^{-\rho \tau_0} \ell_i \right]. \tag{53}$$

where

$$S(C_{t-}^\pi | (\pi, V_i)) = V_i(C_t^\pi) - V_i(C_{t-}^\pi) - f_t = V_i(C_{t-}^\pi + f_t) - V_i(C_{t-}^\pi) - f_t$$

represents the financing surplus associated with the strategy $\pi = (D, f)$, and Θ denotes the set of dividend and financing strategies such that

$$E_c \left[\int_0^{\tau_0} e^{-\rho s}(dD_s + f_s dN_s) \right] < \infty.$$

Since the firm value appears in the objective function, the optimization problem in (53) is akin to a rational expectations problem: When bargaining over financing, outside investors have to correctly anticipate the strategy that the firm will use in the future. Accordingly, if the function V_i satisfies (53) and $\pi^* \in \Theta$ attains the supremum, then we say that (π^*, V_i) is a rational expectations equilibrium for the firm.

In order to simplify the construction of such an equilibrium, consider the equivalent probability measure P^* defined by

$$\left. \frac{dP^*}{dP} \right|_{\mathcal{F}_t} = e^{\eta \lambda t}(1-\eta)^{N_t}, \qquad t \geq 0,$$

and observe that due to Girsanov's theorem for jump processes (see Dellacherie and Meyer (1980, VII.45-50)) the Poisson process has intensity $\lambda^* = \lambda(1-\eta)$ under P^*. Let also Θ^* denote the set of strategies such that

$$E_c^* \left[\int_0^{\tau_0} e^{-\rho s}(dD_s + f_s dN_s) \right] < \infty.$$

The following proposition shows that a rational expectations equilibrium can be constructed by considering the optimization problem of an auxiliary firm for which investors arrive at rate λ^* and do not bargain over the terms of financing.

Proposition H.1 *Consider the optimization problem defined by*

$$V_i^*(c) = \sup_{\pi \in \Theta^*} E_c^* \left[\int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s) + e^{-\rho \tau_0} \ell_i \right].$$

and assume that the function V_i^* is Lipschitz continuous. If there exists a strategy $\pi^* \in \Theta^* \cap \Theta$ that attains the above supremum then (π^*, V_i^*) constitutes a rational expectations equilibrium.

Proof. Let (π^*, V_i^*) be as in the statement. Using the definition of the optimization problem together with standard dynamic programming arguments, we deduce that

$$X_t^\pi = e^{-\rho t \wedge \tau_0} V_i^*(C_{t \wedge \tau_0}^\pi) + \int_{0+}^{t \wedge \tau_0} e^{-\rho s}(dD_s - f_s dN_s)$$

is a supermartingale under P^* for any strategy π and a martingale for the optimal strategy π^*. By application of the Doob-Meyer decomposition this implies that for every π there exists a predictable process ϑ^π and a non-decreasing process A^π such that

$$
\begin{aligned}
X_t^\pi &= V_i^*(C_0^\pi) + \int_0^t (\vartheta_s^\pi dB_s - dA_s^\pi) + \int_0^t e^{-\rho s}\Delta X_s(dN_s - \lambda^* ds) \\
&= V_i^*(C_0^\pi) + \int_0^t (\vartheta_s^\pi dB_s - dA_s^\pi) + \int_0^t e^{-\rho s}(\Delta V_i^*(C_s^\pi) - f_s)(dN_s - \lambda^* ds)
\end{aligned}
$$

where $\Delta Z_t = Z_t - \lim_{s \uparrow t} Z_s$ denotes the jump in the process Z and the last equality follows from the definition of the process X^π. This in turn implies that

$$
\begin{aligned}
Y_t^\pi &= e^{-\rho t \wedge \tau_0} V_i^*(C_{t \wedge \tau_0}) + \int_0^{t \wedge \tau_0} e^{-\rho s}(dD_s - (f_s + \eta S(C_{s-}^\pi|(\pi, V_i^*)))dN_s) \\
&= X_t^\pi + \int_0^t e^{-\rho s}\eta(\Delta V_i^*(C_s^\pi) - f_s)dN_s \\
&= V^*(C_0^\pi) + \int_0^t \vartheta_s^\pi dB_s - A_t^\pi + \int_0^t e^{-\rho s}(1 - \eta)(\Delta V_i^*(C_s^\pi) - f_s)(dN_s - \lambda ds)
\end{aligned}
$$

is a supermartingale under P for any strategy π and a local martingale under P for the

optimal strategy π^*. In particular, we have

$$
\begin{aligned}
V_i^*(c) = Y_0^\pi - \Delta V_i^*(C_0^\pi) &\geq E_c\left[Y_t^\pi\right] - \Delta V_i^*(C_0^\pi) \\
&= E_c\left[e^{-\rho t \wedge \tau_0} V_i^*(C_{t\wedge\tau_0}^\pi) + \int_{0+}^{t\wedge\tau_0} e^{-\rho s}(dD_s - (f_s + \eta S(C_{s-}^\pi|(\pi, V_i^*)))dN_s)\right] \\
&\quad - \Delta V_i^*(C_0^\pi) \\
&\geq E_c\left[e^{-\rho t \wedge \tau_0} V_i^*(C_{t\wedge\tau_0}^\pi) + \int_0^{t\wedge\tau_0} e^{-\rho s}(dD_s - (f_s + \eta S(C_{s-}^\pi|(\pi, V_i^*)))dN_s)\right]
\end{aligned}
$$

for all $t \geq 0$ and every π where the last inequality follows from the fact that

$$
V_i^*(c) \geq \sup_{x \geq 0}(x + V_i^*(c - x)).
$$

Using the assumed Lipschitz continuity of the function V_i^* together with the dynamics of the cash buffer process it can be shown that

$$
\sup_{t \geq 0}|Y_t^\pi| \leq A_0 + A_1 m_\infty^* + A_2 \int_0^{\tau_0} e^{-\rho s}(dD_s + f_s dN_s) \tag{54}
$$

for some constants $(A_i)_{i=0}^2$ where

$$
m_t^* = \sup_{s \leq t}\left|\int_0^t e^{-\rho s}\sigma dB_s\right|.
$$

Since the right hand side of equation (54) is integrable for any $\pi \in \Theta$ it follows from the dominated convergence theorem and the definition of V_i^* that

$$
\begin{aligned}
V_i^*(c) &\geq \sup_{\pi \in \Theta} E_c\left[\int_0^{\tau_0} e^{-\rho s}(dD_s - (f_s + \eta S(C_{s-}^\pi|(\pi, V_i^*)))dN_s) + e^{-\rho \tau_0} V_i^*(0)\right] \tag{55} \\
&= \sup_{\pi \in \Theta} E_c\left[\int_0^{\tau_0} e^{-\rho s}(dD_s - (f_s + \eta S(C_{s-}^\pi|(\pi, V_i^*)))dN_s) + e^{-\rho \tau_0}\ell_i\right].
\end{aligned}
$$

On the other hand, if $(\tau_n)_{n=1}^\infty$ denotes a localizing sequence of stopping times for the local

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martingale Y^{π^*} then

$$
\begin{aligned}
V_i^*(c) &= Y_0^{\pi^*} - \Delta V_i^*(C_0^{\pi^*}) = E_c \left[Y_{\tau_n}^{\pi^*} + \Delta D_0^* \right] - \Delta V_i^*(C_0^{\pi^*}) - \Delta D_0^* \\
&= E_c \left[e^{-\rho \tau_0 \wedge \tau_n} V^*(C_{\tau_0 \wedge \tau_n}^{\pi^*}) + \int_0^{\tau_0 \wedge \tau_n} e^{-\rho s} (dD_s^* - (f_s + \eta S(C_{s-}^{\pi^*} | (\pi^*, V_i^*))) dN_s) \right] \\
&\quad - \Delta V_i^*(C_0^{\pi^*}) - \Delta D_0^* \\
&= E_c \left[e^{-\rho \tau_0 \wedge \tau_n} V^*(C_{\tau_0 \wedge \tau_n}^{\pi^*}) + \int_0^{\tau_0 \wedge \tau_n} e^{-\rho s} dD_s^* - (f_s + \eta S(C_{s-}^{\pi^*} | (\pi^*, V_i^*))) dN_s \right]
\end{aligned}
$$

where the last equality follows from the fact that

$$
V_i^*(c) = \Delta D_0^* + V_i^*(c - \Delta D_0^*) = \Delta D_0^* + V_i^*(C_0^{\pi^*})
$$

due to the assumed optimality of π^*. Since $\pi^* \in \Theta$ it follows from equation (54), the dominated convergence theorem and the definition of the function V_i^* that

$$
V_i^*(c) = E_c \left[\int_0^{\tau_0} e^{-\rho s} (dD_s^* - (f_s + \eta S(C_{s-}^{\pi^*} | (\pi^*, V_i^*))) dN_s) + e^{-\rho \tau_0} \ell_i \right] \tag{56}
$$

Combining equations (55) and (56) shows that (π^*, V_i^*) is a rational expectations equilibrium and completes the proof. ∎

Having dealt with the case of a firm with no growth option, we now turn to the case of a firm that possesses an option to expand operations. Let (π_1^*, V_1^*) be a rational expectations equilibrium after investment that satisfies the conditions of Proposition H.1 and denote by Π (resp. Π^*) the set of triples $\pi = (D, f, T)$ where T is a stopping time and $(D, f) \in \Theta$ (resp. Θ^*) is a dividend and financing strategy. Relying on dynamic programming arguments, we have that the optimization problem of such a firm can be formulated as

$$
\begin{aligned}
V(c) = \sup_{\pi \in \Pi} E_c \Bigg[\int_0^{\tau_0 \wedge T} e^{-\rho s} (dD_s - [f_s + \eta S(C_{s-}^{\pi} | (\pi, V, V_1^*))] dN_s) \\
+ 1_{\{\tau_0 < T\}} e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq T\}} e^{-\rho T} V_1^*(C_T^{\pi}) \Bigg]
\end{aligned} \tag{57}
$$

where

$$
S(C_{t-}^{\pi} | (\pi, V, V_1^*)) = 1_{\{t \neq T\}} V(C_{t-}^{\pi} + f_t) + 1_{\{t = T\}} V_1^*(C_{t-}^{\pi} + f_t - K) - V(C_{t-}^{\pi}) - f_t
$$

represents the financing and investment surplus. In accordance with our previous definition, (π^*, V, V_1^*) forms a rational expectations equilibrium if (V, V_1^*) satisfy (57) and $\pi^* \in \Pi$ attains the supremum. The following proposition is the direct counterpart of Proposition H.1 for the case of a firm with a growth option.

Proposition H.2 *Consider the optimization problem defined by*

$$V^*(c) = \sup_{\pi \in \Pi^*} E_c^* \left[\int_0^{\tau_0 \wedge T} e^{-\rho s}(dD_s - f_s dN_s) + 1_{\{\tau_0 < T\}} e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq T\}} e^{-\rho T} V_1^*(C_T^\pi) \right]$$

and assume that the function V^* is Lipschitz continuous. If there exists $\pi^* \in \Pi^* \cap \Pi$ that attains the above supremum then the triple (π^*, V^*, V_1^*) constitutes a rational expectations equilibrium.

Proof. The proof is similar to that of Proposition H.1 and therefore is omitted. ∎

I High-contact condition

This Appendix shows why the high-contact condition in Dumas (1991) can be applied in Section 2.1 and throughout the paper. By Lemma B.2 in the main appendix we have that

$$L_i(c; b) = E_c \left[e^{-(\rho+\lambda)\tau_{i,0}} 1_{\{\tau_{i,0} \leq \tau_{i;b}\}} \right] = \frac{G_i(b)F_i(c) - F_i(b)G_i(c)}{G_i(b)F_i(0) - F_i(b)G_i(0)},$$

$$H_i(c; b) = E_c \left[e^{-(\rho+\lambda)\tau_{i;b}} 1_{\{\tau_{i;b} \leq \tau_{i,0}\}} \right] = \frac{F_i(0)G_i(c) - G_i(0)F_i(c)}{G_i(b)F_i(0) - F_i(b)G_i(0)},$$

where the functions $F_i(c)$ and $G_i(c)$ are two linearly independent solutions to the second order ordinary differential equation

$$\lambda \phi(c) = \mathcal{L}_i \phi(c).$$

Using these properties in conjunction with equations (5) and (8) of the main text allows us to derive an explicit expression for the value $v_i(b; b)$ at the boundary. Now let

$$f_i(c; b) = \frac{\partial v_i(c; b)}{\partial c}.$$

denote the function describing the first order condition with respect to the barrier level. Combining the result of the first step with equation (8) of the main text and the definition

of the functions $F_i(c)$ and $G_i(c)$, we easily get

$$f_i(c;b) = f_i(b;b) \left[\frac{\rho H_i(c \wedge b; b)}{\lambda + \rho} + \frac{\lambda(1 - L(c \wedge b; b))}{\lambda + \rho} \right], \qquad c; b > 0.$$

Since the bracketed term on the right hand side is strictly positive for all $b > 0$ and $c > 0$ we obtain that the validity of the first order condition at $c = b$ is equivalent to its validity at all $c > 0$. To conclude the argument let

$$g_i(b) = \frac{\partial^2 v_i(b;b)}{\partial c^2} = 0$$

denote the function describing the high contact condition. A direct calculation using once again (8) and the definition of the functions $F_i(c)$ and $G_i(c)$ shows that

$$g_i(b) = f_i(b;b) \left[\frac{\lambda}{\lambda + \rho} \frac{\partial L_i(b;b)}{\partial c} - \frac{\rho}{\lambda + \rho} \frac{\partial H_i(b;b)}{\partial c} \right]. \tag{58}$$

The function $H_i(c;b)$ is increasing while the function $L_i(c;b)$ is decreasing and therefore we have that the bracketed term on the right hand side is negative or zero. Since these two functions are linearly independent by construction we have that their Wronskian determinant

$$\mathcal{W}[H_i, L_i](c;b) = L_i(c;b) \frac{\partial H_i(c;b)}{\partial c} - H_i(c;b) \frac{\partial L_i(c;b)}{\partial c}$$

is everywhere different from zero and combining this with the fact that $H_i(b;b) - 1 = L_i(b;b) = 0$ we deduce that $\frac{\partial L_i(b;b)}{\partial c} < 0$. This in turn implies that the bracketed term on the right hand side of (58) is strictly negative and it follows that the first order condition is equivalent to the high contact condition.

J   Probabilities of investment

The probabilities of investment from internal and external funds can be computed as

$$P_I(c) = f(c; (0, C_U^*)),$$
$$P_E(c) = 1 - f(c; (0, C_U^*)) - g(c; (0, C_U^*))$$

for $K < K^{**}$, and

$$P_I(c) = 1_{\{c > C_L^*\}} f(c; (C_L^*, C_H^*)),$$

$$P_E(c) = 1_{\{c \le C_L^*\}} \left(1 - h(c \wedge C_W^*; (0, C_W^*)) \right)$$
$$+ 1_{\{c > C_L^*\}} \left(1 - f(c; (C_L^*, C_H^*)) - g(c; (C_L^*, C_H^*)) h(C_W^*; (0, C_W^*)) \right)$$

otherwise. In these equations, the bounded functions f, g and h are defined by

$$f(c; (A, B)) = E_c \left[e^{-\lambda \tau_{0,B}} 1_{\{\tau_{0,B} \le \tau_{0,A}\}} \right],$$

$$g(c; (A, B)) = E_c \left[e^{-\lambda \tau_{0,A}} 1_{\{\tau_{0,A} \le \tau_{0,B}\}} \right],$$

$$h(c; (A, B)) = E_c \left[e^{-\lambda \widehat{\tau}_{0,A}(B)} \right],$$

for some $A \le B$ where $\tau_{0,b}$ denotes the first time that the uncontrolled cash buffer process with mean cash flow rate μ_0 reaches the nonnegative level b and $\widehat{\tau}_A(B)$ denotes the first time that the cash buffer process with mean cash flow rate μ_0 reaches the level $A \le B$ given that it is reflected from above at the level B.

The following proposition relies on standard methods to provide closed form expressions for the functions f, g and h and thereby allows to compute the probabilities of investment.

Proposition J.1 *The functions f, g and h solve*

$$(rc + \mu_0)\phi'(c) + \frac{1}{2}\sigma^2 \phi''(c) - \lambda \phi(c; b) = 0, \qquad c \in (A, B),$$

subject to the boundary conditions

$$f(A; (A, B)) = g(B; (A, B)) = 0,$$

$$g(A; (A, B)) = f(B; (A, B)) = 1,$$

$$h(A; (A, B)) = 1 - h'(B; (A, B)) = 1,$$

and are explicitly given by

$$f(c, (A, B)) = \frac{G_0(A)F_0(c)}{G_0(A)F_0(B) - F_0(A)G_0(B)} + \frac{F_0(A)G_0(c)}{F_0(A)G_0(B) - G_0(A)F_0(B)},$$

$$g(c, (A, B)) = \frac{G_0(B)F_0(c)}{F_0(A)G_0(B) - G_0(A)F_0(B)} + \frac{F_0(B)G_0(c)}{G_0(A)F_0(B) - F_0(A)G_0(B)},$$

$$h(c; (A; B)) = \frac{G_0'(B)F_0(c)}{F_0(A)G_0'(B) - G_0(A)F_0'(B)} + \frac{F_0'(B)G_0(c)}{G_0(A)F_0'(B) - F_0(A)G_0'(B)},$$

where the functions F_0 and G_0 are defined in equations (35) *and* (36).

K   Gambling

K.1   Proof of the results in Section 3.1

In this Appendix we establish the results of section 3.1 regarding the optimal policies of the firm in an environment where management is allowed to engage in risky gambles. Since $V_1(c)$ is concave we have that the firm never gambles post-investment, and it follows that the critical investment cost that determines the present value of the growth option remains unchanged when the firm is allowed to gamble. Accordingly, we assume throughout this Appendix that $K < K^*$ so that the growth option has strictly positive net present value.

The method of proof that we use is similar to that of the previous appendices: We will construct a solution to the HJB equation and then rely on verification arguments to show that this solution coincides with the value of the firm. Consider the operator

$$\mathcal{L}_G = \mathcal{L}_0 + \frac{1}{2}G^2\frac{d^2}{dc^2}$$

and let $\mathcal{L}^*\phi(c) = \max\{\mathcal{L}_0\phi(c), \mathcal{L}_G\phi(c)\}$. With this notation we have that the HJB equation associated with the firm's problem is given by

$$\max\{\mathcal{L}^*V(c) + \lambda(V_1(C_1^*) - C_1^* - K + c - V(c)), V_1(c - K) - V(c), 1 - V'(c)\} = 0, \quad (59)$$

subject to the initial condition $V(0) = \ell_0$. Assume that the firm invests from internal funds at some level $b \geq K$ of cash reserves and denote by $v(c; b)$ the corresponding value. Following the same logic as in the previous appendices we expect that given the optimal threshold this

function satisfies both the value matching condition

$$v(c;b) = V_1(c - K), \qquad c \geq b,$$

and the smooth pasting condition

$$v'(b;b) = V_1'(b - K).$$

Combining these conditions with the definition of $V_1(c)$ and the fact that $v(c;b)$ must satisfy the differential equation

$$\mathcal{L}^* v(c;b) + \lambda(V_1(C_1^*) - C_1^* - K + c - v(c;b)) = 0 \tag{60}$$

in a left neighborhood of b we deduce that $\operatorname{sign} v''(b;b) = \operatorname{sign} \mathcal{I}(b - K)$ where

$$\mathcal{I}(c) = (\mu_1 - \mu_0 - rK)V_1'(c) + \frac{\sigma^2}{2}V_1''(c).$$

This shows that the second derivative at b has the same sign as $\mathcal{I}(b - K)$, and allows to easily determine when a given investment threshold will give rise to gambling. Building on this observation we now construct an auxiliary function $\Gamma(c;b)$ as follows:

CASE 1: IF $\mathcal{I}(b - K) \leq 0$ then we define $\Gamma(c;b)$ as the unique solution to

$$\mathcal{L}_0 \Gamma(c;b) + \lambda\left(V_1(C_1^*) - C_1^* - K + c - \Gamma(c;b)\right) = 0, \qquad c \in [0, b] \tag{61}$$

subject to the value matching and smooth pasting conditions

$$\Gamma(c;b) - V_1(c - K) = \Gamma'(b;b) - V_1'(b - K), \qquad c \geq b.$$

In this case it follows from Lemmas B.5 and B.6 imply that the function $\Gamma(c;b)$ is concave and satisfies $\Gamma'(c;b) \geq 1$ for all $c \geq 0$ as well as $\Gamma(c;b) \geq V_1(c - K)$ for $c \geq K$.

CASE 2: IF $\mathcal{I}(b - K) > 0$ then the function is convex in a left neighborhood of b and it follows that the strategy will involve gambling. In this case we let

$$\Gamma(c;b) = \bar{\Gamma}(c;b), \qquad c \in [b_1^*(b), b],$$

where the function on the right hand side is the unique solution to

$$\mathcal{L}_G \bar{\Gamma}(c;b) + \lambda \left(V_1(C_1^*) - C_1^* - K + c - \bar{\Gamma}(c;b) \right) = 0, \qquad c \in [0,b], \tag{62}$$
$$\bar{\Gamma}(c;b) - V_1(c-K) = \bar{\Gamma}'(b;b) - V_1'(b-K), \qquad c \in [b,\infty)$$

and the constant

$$b_1^*(b) = \sup\{0 \le a \le b : \bar{\Gamma}''(a;b) \le 0 \text{ or } \bar{\Gamma}'(a;b) \le 1\} \ge 0$$

defines the lower end-point of the last interval over which the function $\bar{\Gamma}(c;b)$ is convex with a derivative greater or equal to one. If $b_1^*(b) = 0$ then $\Gamma(c;b)$ is convex with a derivative larger than one over the whole interval $[0,b]$. Otherwise, two subcases may arise:

CASE 2.A: IF $b_1^*(b) \wedge \mathcal{I}(b-K) > 0$ and $\bar{\Gamma}'(b_1^*(b), b) = 1$ then the firm should distribute a lumpsum dividend and abandon the option of financing investment with internal funds. In this case we let

$$\Gamma(c;b) = w(c; b_2^*(b)), \qquad c \in [0, b_1^*(b)]$$

where the function $w(c; b_2^*(b))$ is defined as in Appendix E and the constant $b_2^*(b)$ is the unique solution to the linear equation

$$b_2^*(b) - b_1^*(b) + \Gamma(b_1^*(b), b) = \frac{\mu + rb_2^*(b) + \lambda(V_1(C_1^*) - C_1^* - K + b_2^*(b))}{\rho + \lambda}.$$

Note that, by definition, the right hand side coincides with the value $w(b_2^*(b); b_2^*(b))$ of the function $w(c; b_2^*(b))$ at the target level of cash holdings.

CASE 2.B: IF $b_1^*(b) \wedge \mathcal{I}(b-K) > 0$ and $\bar{\Gamma}''(b_1^*(b), b) = 0$, then the firm will stop gambling at the point where its cash reserves fall to $b_1^*(b)$, but will not abandon the option of financing investment with internal funds as the marginal value of cash remains strictly above one. In this case we let

$$\Gamma(c;b) = \hat{\Gamma}(c;b), \qquad c \in [0, b_1^*(b)],$$

where the function on the right hand side is the unique solution to

$$\mathcal{L}_0 \hat{\Gamma}(c;b) + \lambda(V_1(C_1^*) - C_1^* - K + c - \hat{\Gamma}(c;b)) = 0, \qquad c \in [0, b_1^*(b)],$$

subject to the value matching and smooth pasting conditions

$$\hat{\Gamma}(b_1^*(b);b) - \bar{\Gamma}(b_1^*(b);b) = \hat{\Gamma}'(b_1^*(b);b) - \bar{\Gamma}'(b_1^*(b);b) = 0.$$

Note that in this case the function $\Gamma(c;b)$ is twice continuously differentiable over the interval $(0,b)$ since the function $\bar{\Gamma}(c;b)$ satisfies the differential equation (62) and $\bar{\Gamma}''(b_1^*(b),b) = 0$.

Consider now the functional transformation of Lemma D.1 corresponding to the second order differential operator

$$(\mu_1 + r(c - K))\frac{d}{dc} + \frac{1}{2}\sigma^2\frac{d^2}{dc^2} - (\rho + \lambda).$$

A direct calculation implies that under this transformation $\Gamma(c;b)$ is convex while $V_1(c - K)$ is linear and, since these functions are tangent at $c = b$, we deduce that

$$\Gamma(c;b) \geq V_1(c - K), \qquad c \geq K.$$

Furthermore, the function $\Gamma(c;b)$ solves (60) with a derivative that is always greater than or equal to one and therefore satisfies (59). To obtain a solution of the HJB equation it now only remains to show that b can be chosen so that the initial condition $\Gamma(0;b) = \ell_0$ is satisfied.

Lemma K.1 *When $\Gamma(0;K) \leq \ell_0$ the value function is given by $V(c;G) = \Gamma(c;C_G)$ where the constant C_G is the unique solution to $\Gamma(0;C_G) = \ell_0$ in $[K, C_1^* + K)$.*

Proof. The existence of a solution with the required property will follow by continuity once we show that $\Gamma(0;C_1^* + K) > \ell_0$. By construction, we have that

$$\Gamma'(C_1^* + K; C_1^* + K) - 1 = 0 < \Gamma''(C_1^* + K; C_1^* + K)$$

where the inequality follows from the fact that since $K \leq K^*$ we have $\mathcal{I}(C_1^*) > 0$ by Lemma C.3. This immediately implies that $b_1^*(C_1^* + K) = C_1^* + K$, and it now follows from the

construction of the function $\Gamma(c; b)$ that we have

$$\Gamma(c; C_1^* + K) = w(c; b_2^*(C_1^* + K)).$$

Assume towards a contradiction that we have $\Gamma(0; C_1^* + K) \leq \ell_0$. Using the above expression together with the same argument as in the proof of Lemma E.1 then shows that $C_W^* \geq b_2^*(C_1^* + K)$, which contradicts Lemma E.3 and it follows that there exists a solution C_G with the required property. Given this solution, we have that $\Gamma(c; C_G)$ solves the HJB equation, and the verification arguments Lemmas F.1 and F.5 show that it coincides with the value of the firm. In particular, the solution C_G has to be unique, and the proof is complete. ■

When $\Gamma(0; K) > \ell_0$ it should be optimal to invest as soon as possible and liquidate afterwards if investment is financed from internal funds. In this case the smooth pasting condition does not hold and, as a result, we need another function to describe the value of the firm. Let

$$\mathcal{Q} \equiv \frac{(\rho + \lambda)\ell_1 - \lambda(V_1(C_1^*) - C_1^*)}{\mu_0 + rK}$$

and consider, for each $a \geq 1$, the function defined as follows:

CASE 1′: IF $a \geq \mathcal{Q}$ then we define the function $\psi(c; a)$ as the unique solution to

$$\mathcal{L}_0\psi(c; a) + \lambda\left(V_1(C_1^*) - C_1^* - K + c - \psi(c; a)\right) = 0, \qquad c \in [0, K],$$

subject to the boundary conditions

$$\psi'(K; a) - a = \psi(c; a) - V_1(c - K), \qquad c \geq K.$$

In this case we have that

$$\frac{\sigma^2}{2}\psi''(K; a) = (rK + \mu_0)(\mathcal{Q} - a) \leq 0$$

and an application of Lemma B.7 shows that the function $\psi(c; a)$ is concave with a derivative that is everywhere greater or equal to one.

CASE $2'$: If $a < \mathcal{Q}$ then we let

$$\psi(c; a) = \psi_o(c; a), \qquad c \in [\beta_1^*(a), \infty),$$

where the function on the right hand side is the unique continuously differentiable solution to the differential equation

$$\mathcal{L}_G \psi_o(c; a) + \lambda \left(V_1(C_1^*) - C_1^* - K + c - \psi_o(c; a) \right) = 0, \qquad c \in [0, K], \tag{63}$$

subject to

$$\psi_o(c; a) - V_1(c - K) = \psi_o'(K; a) - a = 0, \qquad c \in [K, \infty),$$

and the constant

$$\beta_1^*(a) = \sup\{0 \le c \le K : \psi_o''(c; a) \le 0 \text{ or } \psi_o'(c; a) \le 1\} \ge 0$$

defines the lower end-point of the last interval over which the function $\psi_o(c; a)$ is convex with a derivative greater than or equal to one. By construction, we have $\psi_o'(K; a) = a \ge 1$ and

$$\frac{1}{2}(\sigma^2 + G^2)\psi_o''(K; a) = (rK + \mu_0)(\mathcal{Q} - a) > 0$$

so that the function $\psi(c; a)$ is convex in a neighborhood of the investment cost. If $\beta_1^*(a) = 0$ then it is convex with a derivative greater or equal to one over $[0, K]$. Otherwise, two further subcases may arise:

CASE $2'$.A: If $a < \mathcal{Q}$ and $\psi_o'(\beta_1^*(a); a) = 1$ then the firm should distribute a lumpsum dividend and abandon the option of financing investment with internal funds. In this case we let

$$\psi(c; a) = w(c; \beta_2^*(a)), \qquad c \in [0, \beta_2^*(a)]$$

where the function $w(c; \beta_2^*(a))$ is defined as in Appendix E and the constant $\beta_2^*(a)$ is the unique solution to the linear equation

$$\beta_2^*(a) - \beta_1^*(a) + \psi(\beta_1^*(a); a) = \frac{\mu + r\beta_2^*(a) + \lambda(V_1(C_1^*) - C_1^* - K + \beta_2^*(a))}{\rho + \lambda}.$$

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Note that, by definition, the right hand side coincides with the value $w(\beta_2^*(a); \beta_2^*(a))$ of the function $w(c; \beta_2^*(a))$ at the target level of cash holdings.

CASE 2′.B: If $a < \mathcal{Q}$ and $\psi_o''(\beta_1^*(a); a) = 0$ then we let

$$\psi(c; a) = \psi_p(c; a), \qquad c \in [0, \beta_1^*(a)],$$

where the function on the right hand side is the unique continuously differentiable solution to the differential equation

$$\mathcal{L}_0 \psi_p(c; b) + \lambda(\ell_1(c) - K + \psi_p(c; a)) = 0, \qquad c \in [0, \beta_1^*(a)],$$

subject to the value matching and smooth pasting conditions

$$\psi_p(\beta_1^*(a); a) - \psi_o(\beta_1^*(a); a) = \psi_p'(\beta_1^*(a); a) - \psi_o(\beta_1^*(a); a) = 0.$$

Note that in this case $\psi(c; a)$ is twice continuously differentiable over $(0, K)$ since $\psi_o(c; b)$ satisfies both the differential equation (63) and $\psi_o''(\beta_1^*(a); a) = 0$.

Lemma K.2 *When $\Gamma(0; K) > \ell_0$ the value of the firm is given by $V(c; G) = \psi(c; a^*)$ where the constant a^* is the unique solution to $\psi(0; a^*) = \ell_0$ in $(V_1'(0), \infty)$ and there is no intermediate dividend distribution region.*

Proof. By construction we have that $\psi(c; V_1'(0)) = \Gamma(c; K)$. Therefore the result follows from the same arguments as in the proof of Lemma K.1. In particular, we have that for any constant $a > \mathcal{Q}$ the function $\psi(c; a)$ is concave with a derivative greater or equal to one. This implies

$$\lim_{a \to \infty} \psi(0; a) \leq \lim_{a \to \infty} (-aK + \psi(K; a)) = -\infty$$

and the existence of a solution to $\psi(0; a^*) = \ell_0$ follows by continuity and the intermediate value theorem. Given the value a^* of the derivative at the boundary it is easily shown that the function $\psi(c; a^*)$ solves the HJB equation and the verification arguments Lemmas F.1 and F.5 show that it coincides with the value of the firm. In particular, the solution a^* has to be unique.

To establish the last claim in the statement we argue as follows. Since the value function

is non decreasing in the gambling limit we have

$$V_1(0) = \psi(K; a^*) = V(K; G) \geq V(K; 0) \geq V_1(0).$$

This implies that $V(K; 0) = V_1(0)$ and hence, when $G = 0$, the value of the firm is given by $u(c; K)$. It therefore follows from the results of the previous appendices that $K \leq \underline{K}(0) < K^{**}(0)$. Now assume that the optimal strategy for $G > 0$ includes an intermediate dividend distribution region. By construction this implies that we have

$$V(c; G) = \psi(c; a^*) = W(c), \qquad c \in [0, C_W^*]$$

and relying once again on the fact that the value function is non decreasing in the gambling limit we deduce that

$$V(c; 0) \leq V(c; G) = W(c), \qquad c \in [0, C_W^*]$$

Combining this inequality with the results of the previous appendices then shows that we must have $K \geq K^{**}$ which contradicts the fact that $K \leq \underline{K}(0) < K^{**}(0)$. ∎

Lemma K.3 *For any $G \geq 0$ there exists $\underline{K}(G)$ such that $\Gamma(0, K) > \ell_0$ if and only if $K > \underline{K}(G)$.*

Proof. We need to prove that the equation $\Gamma(0; K) = \ell_0$ can have at most one solution and to this end it suffices to show that $\Gamma(0; K)$ is strictly monotone decreasing in K. Since the function $\Gamma(c; K)$ is concave for sufficiently large investment costs the same argument as in the proof of Lemma K.2 shows that we have

$$\lim_{K \to \infty} \Gamma(0; K) = -\infty$$

and it follows that it is enough to show that $\Gamma(0, K)$ is strictly monotone. Suppose to the contrary that this function is not strictly monotone so that there are investment costs $K_1 < K_2$, which can be chosen arbitrarily close to each other, such that

$$\Gamma(0; K_1) = \Gamma(0; K_2) \equiv \ell_0'.$$

The same arguments as in the proof of Lemmas K.1 and K.2 show that the function $\Gamma(c; K_i)$

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is the value of a firm with investment cost K_i and liquidation value ℓ_0', and that the associated optimal strategy does not include an intermediate dividend distribution region.

By definition we have that the function

$$\mathcal{I}(0; K) = (\mu_1 - \mu_0 - rK)V_1'(0) + \frac{\sigma^2}{2}V_1''(0)$$

is decreasing in the investment cost. It follows that $\mathcal{I}(0; K_1) > \mathcal{I}(0; K_2)$ and, since K_1 and K_2 can be chosen arbitrarily close to each other, we may assume without loss of generality that the quantities $\mathcal{I}(0; K_1)$ and $\mathcal{I}(0; K_2)$ have the same sign (the only exceptional case occurs when the turning point is located at the point where $\mathcal{I}(0; K) = 0$ but this non-generic case can be treated by a limiting argument). We consider two cases:

Case 1: If we have $\mathcal{I}(0, K_1) \leq 0$ then $\mathcal{I}(0; K_2) \leq \mathcal{I}(0; K_1) \leq 0$ so that the function $\Gamma(c; K_i)$ is concave on $[0, K_i]$. It follows that we have $\Gamma(c; K_i) = u(c; K_i)$ in the notation of Lemma F.7 and the same arguments as in the proof of Lemma F.8 directly leads to a contradiction.

Case 2: If we have $\mathcal{I}(0; K_2) \geq 0$ then consider the functions defined by $h_i(y) = \Gamma(y + K_i; K_i)$. By construction we have that these functions satisfy

$$\frac{1}{2}\left(\sigma^2 + G^2 1_{\{y+K_i \geq b_i\}}\right) h_i''(y) + (\mu_0 + r(y + K_i))h_i'(y) - (\rho + \lambda)h_i(y) + \lambda(V_1(C_1^*) - C_1^* + y) = 0,$$

on $[-K_i, 0]$ where $b_i = b_1^*(K_i) \leq K_i$ gives the threshold above which the function $\Gamma(c; K_i)$ is convex and satisfies

$$h_i''(b_i - K_i) = \Gamma''(b_i; K_i) = 0.$$

It follows that the function $m(c) = h_1(c) - h_2(c)$ satisfies

$$\frac{1}{2}(\sigma^2 + G^2)m''(y) + (\mu_0 + r(y + K_1))m'(y) - (\rho + \lambda)m(y) + r(K_1 - K_2)h_2'(y) = 0$$

subject to $m(0) = m'(0) = 0$ on the interval $[(b_1 - K_1) \vee (b_2 - K_2), 0]$. Since $h_2'(y) > 0$ it follows from Lemma B.6 that $m(c)$ is strictly decreasing and convex on that interval and these properties imply that we have $b_2 - K_2 > b_1 - K_1$ as well as

$$h_1(y) > h_2(y), \qquad y \in [b_2 - K_2, 0].$$

Since the function $h_1(y)$ remains convex on $[b_1 - K_1, b_2 - K_2]$ and the function $h_2(y)$ is concave on that interval, we then get that the function $m(y)$ satisfies

$$\frac{1}{2}\sigma^2 m''(y) + (\mu_0 + r(y + K_1))m'(y) - (\rho + \lambda)m(y) + r(K_1 - K_2)h_2'(y) = 0$$

on the interval $I_2 = [-K_1, b_1 - K_1]$ with $m'(b_1 - K_1) \leq 0 < m(b_1 - K_1)$. Therefore it follows from Lemma B.6 that the function $m(c)$ is strictly positive on that interval. Since the function $h_2(c)$ is strictly increasing this in turn implies that

$$\Gamma(0; K_1) = h_1(-K_1) > h_2(-K_1) > h_2(-K_2) = \Gamma(0; K_2) = \Gamma(0; K_1)$$

which provides the required contradiction. ∎

Lemma K.4 *For any $G \geq 0$ there exist $\underline{K}(G) < K^{**}(G) \leq K^*$ such that the optimal policy includes an intermediate dividend distribution region if and only if $K > K^{**}(G)$. In this case the value of the firm can be constructed as in Case 2.A above and the lower end point of the intermediate dividend distribution region is $b_2^*(C_G) = C_W^*$.*

Proof. Let us start by establishing the second part of the statement. If the optimal strategy includes an intermediate dividend distribution region then it follows from Lemmas K.1 and K.2 that the value of the firm $V(c; G) = \Gamma(c; C_G)$ satisfies

$$\mathcal{L}_0 V(c; G) + \lambda(V_1(C_1^*) - C_1^* - K + c - V(c; G)) = 0, \qquad c \in [0; b_2^*(C_G)]$$

subject to the boundary conditions

$$V(0; G) - \ell_0 = V'(b_2^*(C_G); G) - 1 = V''(b_2^*(C_G); G) = 0.$$

By the uniqueness result of Lemma E.1 this implies that have $V(c; G) = W(c)$ for all $c \leq C_W^*$ and therefore $b_2(C_G) = C_W^*$ which is what had to be proved.

If there exists an intermediate dividend distribution region then we know from the first part of the proof that this region is given by $[C_W^*, C_L^*(G)]$ and, since the thresholds C_W^* and $C_L^*(G)$ are continuous in the investment cost, the claim will follow once we that there exists a unique value of the investment cost $K = K^{**}(G)$ such that $C_L^*(G; K) = C_W^*(K)$. Assume towards a contradiction that this is not the case so that there exist investment costs $K_1 < K_2$

such that $C_L^*(G; K_i) = C_W^*(K_i)$ and denote by $\bar{W}_i(c)$ the unique solution to

$$\mathcal{L}^* \bar{W}_i(c) - \lambda \bar{W}_i(c) + \lambda(V_1(C_1^*) - C_1^* - K_i + c) = 0, \qquad c \geq 0, \tag{64}$$

which coincides with the function $W(c; K_i)$ on $[0, C_W^*(K_i)]$. From the proof of Lemmas B.7 and F.2 we know that this auxiliary function is concave on the interval $[0, c_i^*] = [0, C_W^*(K_i)]$ and convex otherwise so that

$$\bar{W}_i'(c) > \bar{W}'(c_i^*) = 1, \qquad c \neq c_i^*$$

Furthermore, Lemma E.2 shows that $c_2^* < c_1^*$ and it follows from the first part that we have $\bar{W}_i(c) = V(c; G; K_i)$ on $[0, C_G(K_i)]$ and therefore $\bar{W}_i(c) \geq V_1(c - K_i)$ for all $c \geq K_i$ convexity. Now consider the function defined by

$$k(c) = \bar{W}_2'(c) - \bar{W}_1'(c)$$

A direct calculation shows that this function satisfies the differential equation

$$\mathcal{L}_0 k(c) - (\lambda - r)k(c) + \frac{1}{2}G^2 \bar{W}_2'''(c) = 0, \qquad c \geq 0,$$

as well as the inequalities $k(c_2^*) < 0$, $k(c_1^*) > 0$. Differentiating (64) and applying Lemma B.7 we deduce that

$$\bar{W}_2'''(c) \geq 0, \qquad c \geq c_2^*.$$

Therefore, it follows from Lemma B.5 that the function $k(c)$ cannot have negative local minima and this implies that there exists a unique $c_* \in (c_2^*, c_1^*)$ such that $k(c_*) = 0, k'(c_*) > 0$ and $k(c) > 0$ if and only if $c > c_*$. In other words, the function $\bar{W}_2(c) - \bar{W}_1(c)$ attains a global minimum at the point c_* and $\bar{W}_2''(c_*) > \bar{W}_1''(c_*)$. Subtracting (??) with $i = 2$ from itself with $i = 1$, evaluating the resulting differential equation at the point c^* and using the fact that the function $\bar{W}_2(c)$ is convex on $[c_2^*, \infty)$ we obtain

$$\bar{W}_2(c_*) - \bar{W}_1(c_*) > \frac{\lambda}{\rho + \lambda}(K_1 - K_2),$$

and therefore

$$\bar{W}_1(c) - \bar{W}_2(c) < \frac{\lambda}{\rho + \lambda}(K_2 - K_1), \qquad c \geq 0,$$

by definition of c_*. However, since $\bar{W}_1(c) \geq V_1(c - K_1)$ for $c \geq K_1$ and $V_1'(c) \geq 1$ for all $c \geq 0$ we finally obtain that

$$\frac{\lambda}{\rho + \lambda}(K_2 - K_1) \geq W_1(C_G(K_2)) - W_2(C_G(K_2))$$
$$= W_1(C_G(K_2)) - V_1(C_G(K_2) - K_2)$$
$$\geq V_1(C_G(K_2) - K_1) - V_1(C_G - K_2) \geq K_2 - K_1,$$

which establishes the required contradiction. ∎

Proof of Theorem 5. Assume first that $K > \underline{K}(G)$. By Lemmas K.1 and K.3 we have that the value of the firm is $\Gamma(c; C_G)$. The existence of the critical investment cost $K^{**}(G)$ follows from Lemma K.4 and the remaining results follow by setting $C_U^*(G) = C_G$ and

$$C_g^*(G) = 1_{\{U''(C_U^*)>0\}}b_1^*(C_U^*(G))$$

whenever the investment cost is such that $K \leq K^{**}(G)$, and

$$(C_L^*(G), C_H^*(G)) = (b_1^*(C_G), C_G)$$

otherwise. When the investment cost is such that $K \leq \underline{K}(G)$. the result follows directly from Lemmas K.2 and and K.3. ∎

Lemma K.5 *Let (a_1, a_2, b) be such that $b > 0$ and denote by $\Psi(c; G)$ be the unique solution to*

$$\mathcal{L}_G\Psi(c; G) + \lambda(V_1(C_1^*) - (C_1^* + K - c) - \Psi(c; G)) = 0, \qquad c \geq 0, \tag{65}$$

subject to

$$\Psi(b; G) - a_1 = \Psi'(b; G) - a_2 = 0 \tag{66}$$

Then the function $\Psi(c; G)$ converges to the linear function $a_1 + (c - b)a_2$ uniformly on compact subsets of the positive real line as $G \to \infty$.

Proof. Rewriting (65) as

$$-\frac{1}{2}\Psi''(c;G) = \frac{(rc+\mu_0)\Psi'(c;G) - \rho\Psi(c;G) + \lambda(V_1(C_1^*) - (C_1^* + K - c) - \Psi(c;G))}{\sigma^2 + G^2}$$

and using standard continuous dependence results for solutions of linear differential equations, we get that $\Psi(c;G)$ converges uniformly to the unique solution of the equation $\Psi''(c) = 0$ subject to (66), and the claim follows. ∎

Lemma K.6 *The critical investment costs $\underline{K}(G)$ and $K^{**}(G)$ are respectively decreasing and increasing in the gambling limit.*

Proof. Assume that $K \leq \underline{K}(G)$ so that we have $V(c;G) = \psi(c;a^*)$ for some $a^* > 1$. Using the fact that the value of the firm is non decreasing in the gambling limit we get

$$V_1(0) = \psi(K;a^*) = V(K;G) \geq V(K;G'), \qquad G' \leq G.$$

This implies that it is optimal to exercise at K when the gambling limit is equal to G' and it immediately follows that $\underline{K}(G) \leq \underline{K}(G')$, which is the desired result. Similarly, if the costs is such that $K > K^{**}(G)$ then the construction of the function $\Gamma(c;b)$ and the fact that the firm value is non decreasing in the gambling limit jointly imply that

$$W(c) = \Gamma(c;C_G) = V(c;G) \geq V(c;G'), \qquad c \leq C_W^*, G' \leq G$$

and it immediately follows that we have $K > K^{**}(G')$ which is the desired result. ∎

Lemma K.7 *Assume that $K > K^{**}(G)$. Then the dividend distribution threshold $C_L^*(G)$ is monotone decreasing in G and there exists a constant $\bar{G} < \infty$ such that $C_L^*(\bar{G}) = C_W^*$.*

Proof. Let $y \geq C_W^*$ be a fixed constant and denote by $\Phi(c;y;G)$ the unique solution to the second order differential equation

$$\mathcal{L}_G\Phi(c;y;G) + \lambda(V_1(C_1^*) - (C_1^* + K - c) - \Phi(c;y,G)) = 0, \qquad c \in [y,\infty), \tag{67}$$

subject to the boundary conditions

$$\Phi'(y;y;G) - 1 = \Phi(y;y;G) - (y - C_W^* + W(C_W^*)) = 0. \tag{68}$$

The same argument as in the proof of Lemma D.2 (based on Lemma D.1) implies that this function satisfies the inequality

$$\Phi(c; y; G) \geq c - C_W^* + W(C_W^*), \qquad c \geq 0. \tag{69}$$

Indeed, under the transformation of Lemma D.1 the function $\Phi(c; y, G)$ becomes a line that is tangent to the transformation of the function $c - C_W^* + W(C_W^*)$ at the point y and the inequality follows by noting that the latter transformed function is concave. Combining (69) with the differential equation shows that $\Phi''(y; y; G) \geq 0$ and it thus follows from an application of Lemma B.7 that the function $\Phi(c; y; G)$ is convex on the interval $[y, \infty)$.

Fix now two arbitrary gambling limits $G_1 < G_2$. Under the transformation of Lemma D.1 associated to the operator $\mathcal{L}_{G_1} - \lambda$, we have that the function $\Phi(c; y; G_1)$ becomes a line that is tangent to the transformation of the function $\Phi(c; y; G_2)$. Using this property in conjunction with (67), (68) and the above convexity then gives

$$\mathcal{L}_{G_1}\Phi''(c; y; G_2) + \lambda(V_1(C_1^*) - (C_1^* + K - c) - \Phi(c; y; G_2)) = \frac{1}{2}(G_1^2 - G_2^2)\Phi''(c; y; G_2) \leq 0.$$

This implies that the function $\Phi(c; y; G_2)$ is concave under the transformation of Lemma D.1, associated to the operator $\mathcal{L}_{G_1} - \lambda$ and it now follows from the same argument as in the proof of Lemma D.2 that

$$\Phi(c; y; G_2) \leq \Phi(c; y; G_1), \qquad c \geq y.$$

This shows that $\Phi(c; y; G)$ is monotone decreasing in G and the same argument, using the fact that the transformation of the function $\Phi(c; y; G)$ parametrized by the constant y is a tangent line moving along the graph of the concave transformation of the function $c - C_W^* + W(C_W^*)$, shows that $\Phi(c; y; G)$ is also monotone decreasing in y.

After these preparations, we now turn to the proof of the statement. By continuity, we have that the threshold $C_L^*(G)$ is the smallest value of y for which the graph of $\Phi(c; y; G)$ touches that of $V_1(c - K)$ from above, and the required monotonicity follows from the monotonicity properties that we established in the first part. In particular, we have that $C_L^*(G) = C_W^*$ if and only if the graph of $\Phi(c; C_W^*; G)$ touches that of $V_1(c - K)$ from above. Since by assumption $K > K^{**}(G) > K^{**}(0)$ we have that $C_L^*(G) > C_W^*$, and it follows that $\Phi(c; C_W^*; G)$ is strictly larger than $V_1(c - K)$ for small values of the gambling limit. On the

other hand, the results of Lemmas E.3 and K.5 imply that the graph of $\Phi(c; C_W^*; G)$ crosses that of $V_1(c - K)$ for sufficiently large values of the gambling limit and the required result now follows from the intermediate value theorem. ∎

Lemma K.8 *The investment threshold, which can be either $C_U^*(G)$ or $C_H^*(G)$ depending on the investment cost, is monotone increasing in G.*

Proof. Let $G_1 < G_2$ be given gambling limits, denote by $C_{G,i}$ the corresponding investment threshold, and suppose that $C_{G,1} > C_{G,2}$. Using the value matching condition together with the fact that the value of the firm is increasing in the gambling limit and satisfies $V(c; G_i) > V_1(c - K)$ for all $c < C_{G,i}$ we immediately deduce that

$$V(C_{G,2}; G_2) = V_1(C_{G,2} - K) < V(C_{G,2}; G_1) \leq V(C_{G,2}; G_2),$$

which is a contradiction. ∎

Lemma K.9 *The function $\bar{G}(K)$ is monotone increasing in K.*

Proof. By definition we have that $\bar{G}(K)$ is the inverse of $K^{**}(G)$ and the result thus follows from Lemma K.6. ∎

Proof of Proposition 6. The proof follows directly from Lemmas K.6, K.7, K.8, and K.9. ∎

We complete this section with a characterization of the firm value in the limit where gambling becomes unconstrained. For each candidate investment threshold $b > 0$, let

$$\Gamma_\infty(c; b) = \Psi(c; b), \qquad c \leq b_\infty^*(b),$$

$$\Gamma_\infty(c; b) = A(c; b) \equiv V_1(b - K) + (c - b)^+ - V_1'(b - K)(c - b)^-, \qquad c \geq b_\infty^*(b),$$

where the nonnegative constant $b_\infty^*(b) \leq b$ is the unique solution to

$$(\rho + \lambda)A(b_\infty^*(b); b) = (\mu_0 + rb_\infty^*(b))V_1'(b - K) + \lambda(V_1(C_1^*) - (C_1^* + K - b_\infty^*(b))),$$

and the function $\Psi(c; b)$ is the unique solution to the differential equation (61) subject to the value matching and smooth pasting conditions

$$\Psi(b_\infty^*(b); b) - A(b_\infty^*(b); b) = \Psi'(b_\infty^*(b); b) - V_1'(b - K).$$

The following result constitutes the direct counterpart of Lemma K.1 for the case where size of the firm's gambling position is unconstrained.

Lemma K.10 *When $G \to \infty$ the value of the firm is given by $V(c; \infty) = \Gamma_\infty(c; C_{G,\infty})$ where the constant C_G^∞ is the unique solution to $\Gamma_\infty(0; C_G^\infty) = \ell_0$. In particular, the value of the firm is concave on the positive real line with a flat piece on the interval $[b_\infty^*(C_G^\infty), C_G^\infty]$.*

Proof. The proof follows from Lemmas K.1 and K.5. ∎

K.2 Proof of the results in Section 3.2

Denote by $\tilde{V}(c; \Lambda; b)$ the value of a firm that searches for new investors, optimally gambles when opportunities arise, and invests as soon as $c \geq b$.

Lemma K.11 *The function $\tilde{V}(c; \Lambda; b)$ belongs to $C^2((0, b)) \cap C(\mathbb{R}_+)$, and there exist a pair of thresholds $C_{g,1}^*(b) \leq b \leq C_{g,2}^*(b)$ such that it satisfies*

$$0 = \mathcal{L}_0 \tilde{V}(c; \Lambda; b) + 1_{\{c \in [C_{g,1}^*(b), C_{g,2}^*(b)]\}} \Lambda(\Theta(c; b) - \tilde{V}(c; \Lambda; b))$$
$$+ \lambda(V_1(C_1^*) - C_1^* - K + c - \tilde{V}(c; \Lambda; b))$$

for all $c \in (0, b)$ with the function

$$\Theta(c; b) = \tilde{V}(C_{g,1}^*(b); \Lambda; b) + \frac{c - C_{g,1}^*(b)}{C_{g,2}^*(b) - C_{g,1}^*(b)}(\tilde{V}(C_{g,2}^*(b); \Lambda; b) - \tilde{V}(C_{g,1}^*(b); \Lambda; b)).$$

Proof. The same arguments as in the proof of Proposition 3 in Hugonnier, Malamud and Morellec (2014) imply that $\tilde{V}(c; \Lambda; b)$ belongs to $C^2((0, b)) \cap C(\mathbb{R}_+)$ and satisfies

$$\mathcal{L}_0 \tilde{V}(c; \Lambda; b) + \mathcal{O}(c; b) = 0, \qquad 0 \leq c \leq b \tag{70}$$

with the nonnegative function

$$\mathcal{O}(c; b) = \max_{x: E[x] \leq 0} E[\tilde{V}(c + x; \Lambda, b) - \tilde{V}(c; \Lambda, b)] = \mathcal{C}[\tilde{V}](c; \Lambda; b) - \tilde{V}(c; \Lambda; b).$$

Therefore, the proof will be complete once we show that

$$\mathcal{O}(c; b) = 1_{\{c \in [C_{g,1}^*(b), C_{g,2}^*(b)]\}}(\Theta(c; b) - \tilde{V}(c; \Lambda; b)), \qquad c \leq b$$

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for some thresholds $C_{g,1}^*(b) \leq b \leq C_{g,2}^*(b)$, and to obtain this result it is sufficient to show that there can be at most one interval over which $\tilde{V}(c; \Lambda; b)$ is convex.

Suppose first $\tilde{V}''(b; \Lambda; b) \leq 0$ so that the function is concave in a left neighborhood of b. In this case we claim that the function is actually concave on $[0, b]$ so that gambling in never optimal. Indeed, suppose that this is not the case, and let c_1 denote the first point below b where the second derivative is equal to zero. Then

$$\phi(c) = \Lambda\big(\mathcal{C}[\tilde{V}](c; \Lambda; b) - \tilde{V}(c; \Lambda; b)\big)$$

is decreasing on the interval $[c_1, b \wedge C_{g,2}(c_1)]$ and it thus follows from Lemma B.7 that we must have $\tilde{V}''(c_1; \Lambda; b) < 0$. This contradicts the definition of the point c_1 and therefore establishes the required concavity.

Suppose now that either $\tilde{V}''(b; \Lambda; b) > 0$ or that the function $\tilde{V}(c; \Lambda; b)$ has a convex kink at b due to non smooth pasting with post investment firm value. In this case the last gambling interval to the left of b necessarily contains b and we claim that there can be only one such interval. Indeed, by definition of the optimal gambling strategy we must have that $\tilde{V}''(L_{g,1}) \leq 0$ where $L_{g,1}$ denotes the left end point of the last gambling interval, and combining this inequality with the same argument as above based on Lemma B.7 shows that the function $\tilde{V}(c; \Lambda; b)$ is concave on the interval $[0, L_{g,1}]$. ∎

Lemma K.12 *We have $\tilde{V}(c; \Lambda; b) \leq \lim_{G \to \infty} V(c; G) = V(c; \infty)$ for all $c \geq 0$ and $b \geq K$.*

Proof. By Lemma K.10 we have that $V(c; \infty)$ is increasing and weakly concave. Combining this with Jensen's inequality immediately shows that

$$\max_{x:E[x]\leq 0} E[V(c + x; \infty) - V(c; \infty)] = 0, \qquad c \geq 0,$$

and, because $V'(c; \infty) \geq 1$, the desired inequality will follow from standard verification arguments provided that we can show that $(\mathcal{L}_0 + \mathcal{F})V(c; \infty) \leq 0$ for all $c \geq 0$. In the no-gambling region, this inequality holds as an equality. In the gambling region, we have $(\mathcal{L}_G + \mathcal{F})V(c; G) = 0$ for all $G \geq 0$ and therefore

$$\rho V(c; G) - (rc + \mu_0)V'(c; G) - \mathcal{F}V(c; G) = \frac{1}{2}(\sigma^2 + G^2)V''(c; G) \geq 0$$

where the inequality follows from the fact that continuous gambling only occurs at points where the value of the firm is convex. Letting $G \to \infty$ on both sides and using the fact that

the function $V(c; \infty)$ is linear in the gambling region by Lemma K.10, we get that

$$(\mathcal{L}_0 + \mathcal{F})V(c; \infty) = (rc + \mu_0)V'(c; \infty) - \rho V(c; \infty) + \mathcal{F}V(c; \infty) \leq 0$$

and the proof is complete. ∎

Lemma K.13 *Either* $\tilde{V}'(C_{U,\ell}^*(\Lambda); \Lambda; C_{U,\ell}^*(\Lambda)) = V_1'(C_{U,\ell}^*(\Lambda) - K)$, *for some* $C_{U,\ell}^*(\Lambda) \in [K, C_U^*(\infty)]$ *or the optimal investment trigger is explicitly given by* $C_U^*(\Lambda) = K$.

Proof. If we have $\tilde{V}'(K; \Lambda; K) \geq V_1'(K)$ then the same arguments as in the case without gambling (see Lemma F.6) implies that the firm value is given by $\tilde{V}(c; \Lambda; K)$. Suppose now that we have $\tilde{V}'(K; \Lambda; K) < V_1'(K)$. Since

$$\tilde{V}(C_U^*(\infty); \Lambda; C_U^*(\infty)) = V(C_U^*(\infty); \infty) = V_1(C_U^*(\infty) - K)$$

it immediately follows from Lemma K.12 that we have

$$\tilde{V}'(C_U^*(\infty); \Lambda; C_U^*(\infty)) \geq V'(C_U^*(\infty); \infty) = V_1'(C_U^*(\infty) - K),$$

and the desired result now follows by continuity and the intermediate value theorem. ∎

Proof of Proposition 7. The first part follows from the fact that when $U''(C_U^*) \leq 0$ the value of the firm $U(c)$ is concave and, therefore, satisfies

$$\max_{x:E[x]\leq 0} E[U(c + x) - U(c)] = 0.$$

Let us now turn to the second part, denote by $\tilde{v}(c; \Lambda) \equiv \tilde{V}(c; \Lambda; C_{U,\ell}^*(\Lambda))$ the candidate value function and let $C_{g,i}^*(\Lambda) \equiv C_{g,i}^*(C_{U,\ell}^*(\Lambda))$. Our first observation is that

$$\tilde{v}'(c; \Lambda) \geq 1, \qquad c \geq 0,$$

as soon as Λ is sufficiently large. Indeed, because $\tilde{V}(c; \Lambda, b)$ is convex over at most one interval we have that the derivative can fall below one only in the gambling region. However, Lemma K.14 implies that in this region $\tilde{v}(c; \Lambda)$ and its derivative converge to $\Theta(c; C_{U,\ell}^*(\Lambda))$ and its derivative as $\Lambda \to \infty$, and the latter is larger than one because

$$\Theta'(c; C_{U,\ell}^*(\Lambda)) = V_1'(C_{g,2}^*(\Lambda) - K) \geq 1.$$

Our claim regarding the derivative of the function $\tilde{v}(c; \Lambda)$ for large Λ then follows by continuity. The same arguments as in the proof of Lemma F.1 imply that

$$V_1(c - K) \leq \tilde{v}(c; \Lambda), \qquad c \geq K,$$

and it remains to show that

$$\mathcal{H}(c) \equiv (\mathcal{L}_0 + \mathcal{F})\tilde{v}(c; \Lambda) + \Lambda \max_{x : E[x] \leq 0} E[\tilde{v}(c + x; \Lambda) - \tilde{v}(c; \Lambda)] \leq 0, \qquad c \geq 0. \tag{71}$$

By construction we have that this inequality holds as an equality on the interval $[0, C^*_{U,\ell}(\Lambda)]$, and is equivalent to $(\mathcal{L}_0 + \mathcal{F})V_1(c - K) \leq 0$ on the interval $[C^*_{g,2}(\Lambda), \infty)$. Because the latter has been established in the proof of Lemma F.4 it now only remains to establish the result on the interval $[C^*_{U,\ell}(\Lambda), C^*_{g,2}(\Lambda)]$. Combining the value matching and smooth pasting conditions with the fact that equation (71) holds as an equality for all $c < C^*_{U,\ell}(\Lambda)$ gives

$$\mathcal{H}(C^*_{U,\ell}(\Lambda)_+) = \frac{\sigma^2}{2} \left(V_1''(C^*_{U,\ell}(\Lambda) - K) - \tilde{v}''(C^*_{U,\ell}(\Lambda); \Lambda) \right) < 0$$

where the last inequality follows the strict concavity of the post-investment firm value and the fact that we must have $\tilde{v}''(C^*_{U,\ell}(\Lambda); \Lambda) > 0$ if gambling is optimal. As shown by Lemma K.14 below we have that

$$\lim_{\Lambda \to \infty} (C^*_{g,2}(\Lambda) - C^*_{U,\ell}(\Lambda)) = 0$$

and it thus follows by continuity that (71) holds on $[C^*_{U,\ell}(\Lambda), C^*_{2,g}(\Lambda)]$ for Λ sufficiently large. Combining the above inequalities shows that

$$0 = \max\{1 - \tilde{v}'(c; \Lambda), V_1(c - K) - \tilde{v}(c; \Lambda),$$
$$(\mathcal{L}_0 + \mathcal{F})\tilde{v}(c; \Lambda) + \Lambda \max_{x : E[x] \leq 0} E[\tilde{v}(c + x; \Lambda) - \tilde{v}(c; \Lambda)]\}$$

for all $c \geq 0$ and the desired result now follows from verification arguments similar to those of the previous sections. ∎

Lemma K.14 *We have*

$$0 = \lim_{\Lambda \to \infty} (C^*_{g,2}(\Lambda) - C^*_{U,\ell}(\Lambda)) \tag{72}$$

and

$$0 = \lim_{\Lambda \to \infty} 1_{\{c \in [C_{g,1}^*(\Lambda), C_{g,2}^*(\Lambda)]\}} \left(\tilde{v}(c; \Lambda) - \Theta(c; C_{U,\ell}^*(\Lambda)) \right) \tag{73}$$

$$= \lim_{\Lambda \to \infty} 1_{\{c \in [C_{g,1}^*(\Lambda), C_{g,2}^*(\Lambda)]\}} \left(\tilde{v}'(c; \Lambda) - \Theta'(c; C_{U,\ell}^*(\Lambda)) \right) \tag{74}$$

uniformly for all $c \geq 0$.

Proof. Let us start with the second part of the statement. The function $\tilde{v}(c; \Lambda)$ is bounded from above by the first best firm value, and using this property together with standard arguments (see, e.g., Hugonnier, Malamud, and Morellec (2014)) we deduce that both $\tilde{v}'(c; \Lambda)$ and $\tilde{v}''(c; \Lambda)$ are uniformly bounded. Therefore, it follows from (70) that

$$\max_{x: E[x] \leq 0} E[\tilde{v}(c + x; \Lambda) - \tilde{v}(c; \Lambda)] = 1_{\{c \in [C_{g,1}^*(\Lambda), C_{g,2}^*(\Lambda)]\}} \left(\Theta(c; C_{U,\ell}^*(\Lambda)) - \tilde{v}(c; \Lambda) \right)$$

is bounded and converges uniformly to zero for all $c \geq 0$, that is equation (73) holds. Suppose now towards a contradiction that equation (74) does not hold. Then, passing if necessary to a subsequence, we get that there exists $\{\Lambda_k\}_{k \geq 1}$ such that $\Lambda_k \to \infty$, the thresholds $C_{g,i}^*(\Lambda_k)$ both converge to finite limits, and

$$\lim_{k \to \infty} 1_{\{c \in [C_{g,1}^*(\Lambda_k), C_{g,2}^*(\Lambda_k)]\}} \left(\tilde{v}'(c; \Lambda_k) - \Theta'(c; C_{U,\ell}^*(\Lambda_k)) \right) > 0.$$

Assume without loss of generality that the limit interval $I = [C_{g,1}^*(\infty), C_{g,2}^*(\infty)]$ is non degenerate, let $c^* \in I$ and pick $\varepsilon > 0$ such that

$$A = [c^* - \varepsilon, c^* + \varepsilon] \subset [C_{g,1}^*(\Lambda_k), C_{g,2}^*(\Lambda_k)], \qquad k \geq 1.$$

Since $\tilde{v}'(c; \Lambda)$ and $\tilde{v}''(c; \Lambda)$ are uniformly bounded, the Arzela-Ascoli theorem implies that the family

$$\left\{ \Theta'(u; C_{U,\ell}^*(\Lambda_k)) - \tilde{v}'(u; \Lambda_k) \right\}_{u \in A, k \geq 1}$$

is compact. Therefore, there exists a further subsequence $\Lambda_{k_n} \to \infty$ such that the derivative of the function defined by

$$\varphi_{k_n}(u) = \Theta(u; C_{U,\ell}^*(\Lambda_{k_n})) - \tilde{v}(u; \Lambda_{k_n})$$

converges to a function such that $F(u) \neq 0$ for all $u \in A$. Because convergence is uniform

this implies that we have

$$\lim_{n\to\infty} \left(\varphi_{k_n}(u_2) - \varphi_{k_n}(u_1) \right) = \lim_{n\to\infty} \int_{u_1}^{u_2} \varphi'_{k_n}(x)dx = \int_{u_1}^{u_2} F(x)dx \neq 0$$

for any $u_1 \leq u_2 \in A$. This contradicts the fact that $\varphi_k(u) \to 0$ for all $u \in A$ and completes the proof of (74). To establish (72) we argue as follows. Because the function

$$\varphi(u; \Lambda) = \Theta(u; C^*_{U,\ell}(\Lambda)) - \tilde{v}(u; \Lambda)$$

converges uniformly to zero on the interval $[C^*_{1,g}(\Lambda), C^*_{U,\ell}(\Lambda)]$ we have that

$$\lim_{\Lambda \to \infty} \left(\Theta(C^*_{U,\ell}(\Lambda); C^*_{U,\ell}(\Lambda)) - \tilde{v}(C^*_{U,\ell}(\Lambda); \Lambda) \right) = \lim_{\Lambda \to \infty} \varphi(C^*_{U,\ell}(\Lambda); \Lambda) = 0,$$

and the required result follows by noting that the function $\Theta(c; C^*_{U,\ell}(\Lambda))$ is tangent to the function $V_1(c - K)$ at the point $C^*_{2,g}(\Lambda)$. ∎

Lemma K.15 *The function $\tilde{V}(c; \Lambda)$ is monotone increasing in Λ and converges as $\Lambda \to \infty$ to a piecewise C^2 and continuous function $\tilde{V}(c; \infty)$ that is concave and satisfies*

$$0 \geq \max\{1 - \tilde{V}'(c; \infty), V_1(c - K) - \tilde{V}(c; \infty),$$
$$(\mathcal{L}_0 + \mathcal{F})\tilde{V}(c; \infty) + \hat{\Lambda} \max_{x:E[x]\leq 0} E[\tilde{V}(c + x; \infty) - \tilde{V}(c; \infty)]\}$$

for all $c \geq 0$ and $\Lambda \geq 0$.

Proof. Monotonicity in Λ is clear and convergence follows by the same arguments as above. Now pick a sequence $(\Lambda^n)_{n=1}^\infty$ such that the thresholds $C^*_{g,1}(\Lambda^n)$ and $C^*_{g,2}(\Lambda^n)$ converge to finite limits. Since all derivatives stay bounded, it has to be that the term

$$1_{\{c\in[C^*_{g,1}(\Lambda^n),C^*_{g,2}(\Lambda^n)]\}}\Lambda^n(\Theta(c; C^*_U(\Lambda^n)) - \tilde{V}(c; \Lambda^n; C^*_U(\Lambda^n)))$$

also stays bounded, and it follows that

$$\lim_{n\to\infty} 1_{\{c\in[C^*_{g,1}(\Lambda^n),C^*_{g,2}(\Lambda^n)]\}}(\Theta(c; C^*_U(\Lambda^n)) - \tilde{V}(c; \Lambda^n; C^*_U(\Lambda^n))) = 0.$$

Thus, the function $\tilde{V}(c; \infty)$ is concave outside $[C^*_{g,1}(\infty), C^*_{g,2}(\infty)]$ and linear inside that interval. Since it is either continuously differentiable or has a concave kink at point $c = K$ we have it is globally concave and the proof is complete. ∎

Lemma K.16 *We have $\tilde{V}(c; \infty) = V(c; \infty)$ for all $c \geq 0$.*

Proof. By Lemma K.12 we have that $\tilde{V}(c; \Lambda) \leq V(c; \infty)$ for all $\Lambda \geq 0$ and taking the limit shows that $\tilde{V}(c; \infty) \leq V(c; \infty)$. On the other hand, the same argument as in the proof of Lemma K.12 shows that the concave function $\tilde{V}(c; \infty)$ satisfies (59) with an inequality for any $G \geq 0$. As a result, standard verification arguments imply that we have $V(c; G) \leq \tilde{V}(c; \infty)$ for any value of G and the result now follows by letting $G \to \infty$. ∎

Proof of Proposition 8. By Lemma K.16 we have that the value functions implied by the two specifications coincide in the limit and remaining claims in the statement follow from the definition of the thresholds. ∎

L   Finitely many growth options

A key and novel feature of the optimal policy for a firm with a growth option is that it may include an intermediate payout region where shareholders optimally abandon the option of investing with internal funds. This feature is unexpected, but we contend that it is in fact universal in models with fixed costs and capital supply frictions.

To make this point, we consider in this section a firm with assets in place and $N \geq 1$ growth options that arrive sequentially over time. The initial mean cash flow rate of the firm is μ_0 and the exercise of the i'th growth option allows to increase the mean cash flow rate from μ_{i-1} to $\mu_i > \mu_{i-1}$ by paying a constant cost K_i. To prevent the simultaneous exercise of multiple growth options, we assume that the firm can hold at most one growth option at a time and that, after exercising each growth option, the firm enters a waiting phase in which the next growth option arrives at an exponentially distributed random time with intensity λ_o. As in the benchmark model, management seeks to maximize shareholders' wealth and has full flexibility over the investment, payout, and financing policies of the firm. The sequential arrival and exercise of the growth options is illustrated in Figure 8.

Insert Figure 8 Here

To solve this extension of the model, we use the fact that there are finitely many investment opportunities and proceed backwards in time starting from the last period where the firm has exhausted its growth potential. Let $V_{o,i}(c)$ denote the value of the firm as a function of its cash holdings in the period where it holds the i'th growth option, and $V_{n,i}(c)$

denote the value of the firm in the waiting period following the exercise of this growth option. After the exercise of the last growth option, the value of the firm is $V_{n,N}(c) = V_N(c)$, where the later is the value of a firm with a mean cash flow rate μ_N and no growth option that was derived in section 2.1.

Similarly, in the period prior to the exercise of the last growth option, the value of the firm is given by $V_{o,N}(c) = V(c)$ where the later is the value of a firm with a single growth option that was derived in section 2.2. To proceed further in this backward recursion, we now have to solve the problem of the firm in the waiting period between the exercise of a growth option and the arrival of the next one.

L.1 Optimal policy in the waiting period

In the waiting period following the exercise of the i'th growth option, the firm may retain earnings to avoid inefficient closure and to exercise not only the next growth option but potentially each of the $N - i$ growth options that it stands to receive. Following the logic of the previous section, we therefore conjecture that the optimal strategy in the waiting period can be characterized in terms of an optimal target level and up to $N - i$ intermediate payout intervals, whose upper ends correspond to the points where the firm decides to temporarily stop hoarding cash to finance a future investment opportunity.

In order to describe the class of all such strategies, let $s = (a, b, x)$ where $x \geq 0$ is a constant that represents the target level for the cash holdings of the firm when raising outside funds and $a, b \in \mathbb{R}^n_+$ for some $n \in [0, N - i]$ are vectors with

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n \leq x$$

that specify the earnings retention intervals and the intermediate dividend distribution intervals associated with the strategy. Specifically, for every given s as above, the set

$$\mathcal{R}(s) = (0, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_{n-1}, a_n) \cup (b_n, x) = \bigcup_{k=0}^{n} \mathcal{R}_k(s)$$

gives the region over which the firm retains earnings and searches for new investors while its complement in $[0, x]$, that is

$$\mathcal{D}(s) = \bigcup_{k=1}^{n} \mathcal{D}_k(s) = \bigcup_{k=1}^{n} [a_k, b_k],$$

gives the collection of intermediate dividend distribution intervals (i.e. the "bands"). When its cash holdings are above the target x, the firm makes a lump sum payment $c - x$. When its cash holdings are in $\mathcal{R}_k(s)$, the firm retains earnings, distributes dividends to remain in the same interval, and searches for new investors in order to adjust its cash holdings to the target level x. If its cash holdings fall to the lower endpoint of the interval before outside funds can be secured, then the firm is liquidated if $k = 0$ and otherwise stops hoarding cash towards the exercise of one of its future growth options. In the latter case, the firm makes a lump sum payment to shareholders given by

$$b_k - a_k = |\mathcal{D}_k(s)| = \inf(\mathcal{R}_k(s)) - \sup(\mathcal{R}_{k-1}(s)) \geq 0,$$

in order to bring its cash buffer down to the next earnings retention interval and then follows an entirely similar payout, financing, and liquidation strategy. Figure 9 provides an illustration of the value of a firm as a function of its cash holdings in the waiting period.

Insert Figure 9 Here

Let $v_{n,i}(c, s)$ denote firm value under such a strategy. Standard arguments show that in in the retention region $\mathcal{R}(s)$, this function is twice continuously differentiable and satisfies

$$\rho v_{n,i}(c; s) = (rc + \mu_i)v'_{n,i}(c; s) + \frac{\sigma^2}{2}v''_{n,i}(c; s)$$
$$+ \lambda^* \left[v_{n,i}(x; s) - x + c - v_{n,i}(c; s)\right] + \lambda_o \left[V_{o,i+1}(c) - v_{n,i}(c; s)\right],$$

subject to the value matching conditions

$$v_{n,i}(0, s) = \ell_i$$

at the point where the firm is liquidated. This differential equation is similar to equation (2) in the main text with the exception of the last term on the second line which accounts for the change in the value of the firm that occurs upon the arrival of the next investment opportunity.

In each of the dividend distribution intervals, the value of the firm is defined by imposing the value matching condition

$$v_{n,i}(c; s) = c - a_k + v_{n,i}(a_k; s), \qquad c \in \mathcal{D}_k(s),$$

and the fact that, once in the interval $\mathcal{R}_k(s)$, the firm distributes dividends to maintain its cash holdings at or below the right endpoint of the interval implies that

$$\lim_{c \uparrow a_k} v'_{n,i}(c; s) = \lim_{c \uparrow x} v'_{n,i}(c; s) = 1.$$

The above equations provide a complete characterization of the value associated with a given band strategy and can be solved using techniques similar to those of section 2.2 of the paper. To determine the optimal strategy, we further impose the smooth pasting conditions

$$\lim_{c \downarrow b^*_{i,k}} v'_{n,i}(c; s^*_i) = 1, \tag{75}$$

at each of the points where the firm stops hoarding cash towards the exercise of one of its future growth opportunities, and the high contact conditions

$$\lim_{c \uparrow a^*_{i,k}} v''_{n,i}(c; s^*_i) = \lim_{c \uparrow x^*_i} v''_{n,i}(c; s^*_i) = 0. \tag{76}$$

at the target level x^*_i and each of the intermediate target levels $a^*_{i,k}$. Later in the Appendix, we show that there always exist a unique s^*_i such that these conditions are satisfied and a detailed analysis of the Bellman equation associated with the problem of the firm in the waiting period allows us to prove that the corresponding strategy is optimal. The following proposition summarizes our findings.

Theorem L.1 *The value of the firm in the waiting period following the exercise of the i'th growth option is given by*

$$V_{n,i}(c) = v_{n,i}(c; s^*_i)$$

and satisfies

$$V_{n,i}(x^*_i) = \frac{1}{\rho + \lambda_o}\left(rx^*_i + \mu_i + \lambda_o V_{o,i+1}(x^*_i)\right)$$

where the triple s^*_i that determines the optimal earnings retention and dividend distribution intervals is the unique solution to (75) *and* (76).

Theorem L.1 shows that in the waiting period that follows the exercise of the i'th growth option, the optimal strategy may include up to $N - i$ intermediate dividend distribution

intervals (bands). But this upper bound is rough as many of these intervals may actually collapse. While it does not seem possible to determine ex-ante the number of intermediate payout intervals, we provide later in the Appendix an explicit algorithm that allows to analytically construct these intervals for each given target level. We also prove the following result.

Proposition L.2 *Suppose that the exercise of the i'th growth option changes the tangibility of assets from φ_{i-1} to φ_i and capital supply from λ_{i-1} to λ_i. Then, the dividend distribution region $\mathcal{D}$ in the waiting period following the exercise of the i'th growth option is increasing with respect to φ_i in the inclusion order, and the target cash level x_i^* is monotone decreasing with respect φ_i and λ_i.*

L.2   Optimal strategy for a firm with a growth option

Having constructed the value of the firm in the waiting period, we now consider the optimal policy of a firm that already holds a growth option. As a first step towards the solution to this problem, the next result provides a sufficient condition for the growth option to have a positive net present value.

Proposition L.3 *A sufficient condition for the i'th growth option to have positive net present value is that*

$$V_{n,i}(x_i^*) - x_i^* - K_i \geq V_{i-1}(C_{i-1}^*) - C_{i-1}^*,$$

where the function $V_{i-1}(c)$ and the constant C_{i-1}^* denote the value and optimal target level of a firm with mean cash flow rate μ_{i-1} and no growth option.

The intuition for this result is clear. Indeed, the left hand side of the inequality gives the maximal value that the firm can attain by exercising the growth option. The right hand side gives the maximal value that it can achieve by abandoning the growth option. In the later case, the firm not only abandons the next growth option but also all subsequent ones.

To simplify the presentation of our results, we assume below that

$$K_i \leq K_i^* = \min\left\{V_{n,i}(x_i^*) - x_i^* + C_{i-1}^* - V_{i-1}(C_{i-1}^*), \frac{\mu_i - \mu_{i-1}}{r}\right\} \tag{77}$$

for all i. In the single option case, this condition is necessary for a positive net present value but this is not so with multiple options because in that case the net present value of

an individual option can no longer be determined on a stand-alone basis. In the present context, this assumption allows us to guarantee that the regions over which the firm invests with internal funds are half-lines instead of unions of disjoint intervals. This assumption can be relaxed at the cost of significantly more involved notation.

When the firm holds a growth option, cash holdings serve two purposes: Reducing the risk of inefficient closure and financing investment. Following the logic of the single option case, we therefore conjecture that the optimal strategy can be described in terms of thresholds $C_{i,W}^* \leq C_{i,L}^* \leq C_{i,H}^*$ with $C_{i,H}^* \geq K_i$. When the investment cost is low, it should never be optimal for the firm to abandon the option of investing with internal funds. We therefore expect the firm to follow a barrier strategy as in section 2.2.1 of the paper with $0 = C_{i,W}^* = C_{i,L}^*$. On the contrary, when the investment cost is high, we expect the firm to follow a strategy similar to that of section 2.2.2 of the main text, with an intermediate dividend distribution interval at the point where the firm optimally abandons the option to invest with internal funds.

To verify our conjecture, we start by constructing the value $v_{o,i}(c; b)$ of a firm that follows a strategy as above with thresholds $b = (b_1, b_2, b_3)$ where $b_1 \leq b_2 \leq b_3$ and $b_3 \geq K$. Standard arguments imply that in the region $(0, b_1) \cup (b_2, b_3)$ over which the firm retains earnings and searches for investors, $v_{o,i}(c; b)$ is twice continuously differentiable and satisfies the differential equation

$$\rho v_{o,i}(c; b) = (rc + \mu_{i-1}) v'_{o,i}(c; b) + \frac{\sigma^2}{2} v''_{o,i}(c; b) + \lambda^* \left[V_{n,i}(x_i^*) - x_i^* - K_i + c - v_{o,i}(c; b) \right]$$

subject to the value matching conditions

$$v_{o,i}(0; b) = \ell_{i-1}.$$
$$v_{o,i}(c; b) = V_{n,i}(c - K_i), \qquad c \geq b_3.$$

This differential equation is similar to that of section 2.2.2 of the main text with the exception of the last term which reflect the fact that upon finding new investors the firm raises funds to invest and simultaneously adjust its cash holdings to the target level x_i^* that is optimal in the waiting period following the exercise of the growth option.

If the thresholds under consideration are such that $[b_1, b_2] = \{0\}$, then the above equations are sufficient to determine the value of the firm and can be solved in closed form using a

modification of equation (14) of the main text. Otherwise, the value of the firm satisfies

$$v_{o,i}(c; b) = c - b_1 + v_{o,i}(b_1; b), \qquad c \in [b_1, b_2],$$

and the fact that in the lowest retention region $(0, b_1)$ the firm distributes dividends to maintain its cash holdings at or below b_1 implies that we have

$$\lim_{c \uparrow b_1} v'_{o,i}(c; b) = 1.$$

In this case, the value of the firm under the given strategy can be derived in closed-form using a modification of equations (22) and (23) of the main text.

To determine the optimal strategy, we distinguish two cases depending on the level of the investment cost. If the investment cost is sufficiently low, then we set $b_i^* = (0, 0, C_{i,H}^*)$ and determine the optimal investment trigger by imposing the smooth pasting condition

$$\lim_{c \uparrow C_{i,H}^*} v_{o,i}(c; b_i^*) = V'_{n,i}(C_{i,H}^* - K_i). \tag{78}$$

If the investment cost is high then, the optimal investment trigger is still determined by the above smooth pasting condition but this equation now needs to be solved in conjunction with the smooth pasting and high contact conditions

$$\lim_{c \uparrow C_{i,W}^*} v''_{o,i}(c; b_i^*) = \lim_{c \downarrow C_{i,L}^*} v'_{o,i}(c; b_i^*) - 1 = 0 \tag{79}$$

that determine the intermediate payout interval.

In the Appendix, we show that in either case the above equations admit a unique solution and a detailed analysis of the Bellman equation associated with the problem of the firm allows us to confirm our conjecture regarding the optimality of the corresponding strategies.

Theorem L.4 *Assume that condition (77) holds. Then there exists a constant $K_i^{**} \leq K_i^*$ such that the value of a firm holding a growth option is*

$$V_{o,i}(c) = v_{o,i}(c; b_i^*)$$

where the thresholds b_i^* are given by the unique solutions to (78) and (79) when $K_i \geq K_i^{**}$ and by the unique solution to (78) such that $C_{i,W}^* = C_{i,L}^* = 0$ otherwise.

Theorem L.4 shows that the results derived in the one growth option case naturally extend to a model in which the firm has multiple growth options. Later in the Appendix, we show that these results also hold if we incorporate search and issuance costs in the model. In this case however, firms only raise funds when the cash buffer is below some threshold C_F^*, where the financing surplus equals 0. We conclude this section with the following proposition, which provides analytic comparative static results on the optimal investment trigger.

Proposition L.5 *Suppose that the exercise of the i'th growth option changes the mean cash flow rate from μ_{i-1} to μ_i, asset tangibility from φ_{i-1} to φ_i, and capital supply from λ_{i-1} to λ_i. Then, the investment trigger $C_{i,H}^*$ is monotone increasing in capital supply λ_{i-1}, current drift μ_{i-1} and current asset tangibility φ_{i-1}, and is decreasing in φ_i.*

L.3 Proofs

In this Appendix, we consider the extension of the model to finitely many growth options outlined in the main text with the additional feature that upon raising outside funds the firm incurs not only the bargaining cost $\eta \mathcal{S}_f V(c)$ but also a fixed cost κ. In such a model, the firm will look for outside funds only when the financing surplus $\mathcal{S}_f V(c)$ exceeds the fixed cost and we will show below that this occurs precisely when the firm's cash reserve are below a constant trigger level.

Remark 1 (Search costs) Since financing opportunities arrive at the jumps times of a Poisson point process, the presence of a fixed cost of financing κ is equivalent to that of a search cost $\kappa_s = \kappa/\lambda$ that the firm incurs continuously over time when searching for investors.

To solve the firm's optimization problem in the presence of multiple growth options we start by formulating two auxiliary problems, whose solutions will serve as building blocks in the construction of the value of the firm and the optimal strategy.

Problem 1. Let V_o be a nonnegative function, denote by τ_o a random time distributed according to an exponential distribution with parameter $\lambda_o > 0$, and consider

$$V_{n,i}(c; V_o) = \sup_{(f,D) \in \Theta} E_c \left[\int_0^{\tau_o \wedge \tau_0} e^{-\rho t} (dD_t - a(f_t)dN_t) + 1_{\{\tau_0 < \tau_o\}} e^{-\rho \tau_0} \ell_i + 1_{\{\tau_o < \tau_0\}} e^{-\rho \tau_o} V_o(C_{\tau_o}) \right],$$

subject to

$$dC_t = (rC_{t-} + \mu_i)dt + \sigma dB_t - dD_t + f_t dN_t, \tag{80}$$

where the stopping time τ_0 stands for the firm's liquidation time, the set Θ is defined as in Appendix B.1 of the main text and we have set $a(x) = x + \kappa 1_{\{x>0\}}$.

The goal of Problem 1 is to determine the optimal financing and payout strategy in the waiting period between growth options, given that the next option arrive at the random time τ_o at which point the value of the firm will be some given function $V_o(C_{\tau_o})$. Following the same logic as in previous appendices, we have that the HJB equation associated with Problem 1 is

$$\max\{\mathcal{L}_i V_{n,i}(c) + \mathcal{F}V_{n,i}(c) + \lambda_o(V_o(c) - V_{n,i}(c)), 1 - V'_{n,i}(c), \ell_i(c) - V_{n,i}(c)\} = 0,$$

where the operator $\mathcal{F}$ now takes the form

$$\mathcal{F}V(c) = \lambda \max_{f \geq 0}\left(V(c+f) - a(f) - V(c)\right).$$

to take into account the presence of the fixed cost. Using the methods developed above for the case without the investment option, it is possible to show that the optimal policy for Problem 1 is of barrier type as soon as the function $V_o(c)$ is concave. However, in the many options case studied here the function $V_o(c)$ will coincide with the value function $V_{o,i}(c)$ of a firm with a growth option and we know from the analysis of the single option case in the main body of the paper that this function generally fails to be concave. This non-concavity significantly alters the optimal policy of the firm and leads to the multiple dividend distribution intervals reported in Theorem L.1.

To deal with the optimization problem of the firm in the phase where it already holds a growth option, we now introduce a second auxiliary problem:

Problem 2. Given a nonnegative and piecewise C^2 function V_n, consider the optimal dividend, financing, and investment problem defined by

$$V_{o,i}(c; V_n) = \sup_{\pi \in \Pi} E_c\left[\int_0^{\tau \wedge \tau_0} e^{-\rho t}\left(dD_t - a(f_t)dN_t\right) + 1_{\{\tau \geq \tau_0\}}e^{-\rho \tau_0}\ell_i + 1_{\{\tau < \tau_0\}}e^{-\rho \tau}V_n(C_\tau)\right].$$

subject to (80) where Π denotes the set of $\pi = (\tau, D, f)$ such that $(D, f) \in \Theta$ is an admissible financing and payout strategy, and τ is a stopping time.

The goal of Problem 2 is to determine the optimal financing, payout, and investment strategy of a firm holding a growth option, given that upon investing the value of the firm will be $V_n(c)$. The corresponding HJB equation takes the form

$$\max\{\mathcal{L}_i V_{o,i}(c) + \mathcal{F} V_{o,i}(c), 1 - V'_{o,i}(c), \ell_i(c) - V_{o,i}(c), V_n(c) - V_{o,i}(c)\} = 0.$$

Having solved Problems 1 and 2 for arbitrary functions V_o and V_n, we will construct the value function of a firm with multiple growth options recursively. Using the result of Proposition 1 in the main text, we know the value $V_N(c) = V_{n,N}(c)$ of a firm that has exhausted its growth potential. Taking this function as given, we solve Problem 2 with $V_o(c) = V_N(c - K_N)$ to determine the value $V_{o,N}(c)$ of the firm in the period where it holds its last growth option. Then, we solve Problem 1 with the function $V_n(c) = V_{o,N}(c)$ to determine the value $V_{n,N-1}(c)$ of the firm in the period where it awaits the arrival of the last growth option, and continuing this process allow to compute the value of the firm in all phases.

L.4 Solution to Problem 1

We start by solving Problem 1 for a fixed function $V_o(c)$. Since the algorithm for constructing the value function of this problem is quite involved, we briefly describe the main idea. As in the case without growth options, we use a shooting-based construction that starts from a target cash level and shoots backward towards the value matching condition at zero.

- We start with a conjectured target cash level b and require that the value function be C^2 at that point. This requirement together with the ODE (81) below for the conjectured value function uniquely pins down the value at that cash level. Then, we define the conjectured value function $\psi(c; b)$ as the unique solution to the corresponding ODE

- If this solution satisfies $\psi'(c; b) > 1$ for all $c \geq 0$, we are done. Otherwise, let $\xi_1(b)$ be the first level of cash from the right at which $\psi'(c; b) = 1$. This defines the upper bound of the first interior dividend distribution region and we define ψ to be linear in c below that cash level, until it hits the lower bound. At the lower bound, the value function has to satisfy the C^2-condition, which, as we show below, means that ψ has to hit the graph of an explicitly given function $\phi(c; b)$.

- At that point, the algorithm restarts: we again define ψ to be the solution to the ODE until the derivative ψ' hits 1 again, in which case we define it to be linear until it hits the graph of ϕ again, etc.

Once the function $\psi(c; b)$ has been constructed for any target cash level b, the optimal target level x_i^* is determined to match the value matching $\psi(0; x_i^*) = \ell_i$ at the origin.

By construction, the function $\psi(c; b)$ defined above always satisfies $\psi'(c; b) \geq 1$, however, in order to complete the verification argument, we need to check the supermartingale condition inside all the dividend distribution intervals. This is technically non-trivial. Throughout this process, we will always make the following technical assumption:

Assumption L.6 *The function $V_o(c)$ is such that $\max_c V_o(c) > \ell_i$, $V_o'(c) \geq 1$ for all $c \geq 0$ and $V_o'(c) = 1$ for sufficiently large c.*

This assumption will always be fulfilled in the problems under consideration because, as we show below, it is always optimal for the firm to distribute dividends when its cash buffer is sufficiently large. Denote by $Y(c) = Y(c; b)$ the unique twice continuously differentiable solution to

$$\mathcal{L}_i Y(c) + \lambda_o(V_o(c) - Y(c)) + \lambda(Y(b) - (b - c) - Y(c) - \kappa)^+ = Y'(b) - 1 = Y''(b) = 0 \quad (81)$$

for $c \leq b$ and satisfying

$$Y(c; b) = Y(b; b) + (c - b)$$

otherwise. The fact that such a function exists follows from the results in Appendix E and we note that the smoothness of the function $Y(c; b)$ and (81) jointly imply that

$$Y(b; b) = \frac{rb + \mu + \lambda_o V_o(b)}{\rho + \lambda_o}.$$

Having computed the function $Y(c; b)$ we let

$$\theta(b) = \arg\max_{x \geq b} \left\{ \frac{rx + \mu + \lambda_o V_o(x)}{\rho + \lambda_o} - x = \frac{rb + \mu + \lambda_o V_o(b)}{\rho + \lambda_o} - b \right\},$$

and define

$$\phi(x; b) = \max\left\{ \frac{rx + \mu + \lambda_o V_o(x)}{\rho + \lambda_o}, \frac{rx + \mu + \lambda_o V_o(x) + \lambda(Y(b; b) - (\theta(b) - x) - \kappa)}{\rho + \lambda_o + \lambda} \right\}. \quad (82)$$

Note that we have $Y(b; b) = Y(\theta(b); \theta(b))$ by construction and that since $\phi'(x; b) < 1$ for sufficiently large x we know that the function $\phi(x; b) - x$ is monotone decreasing for sufficiently

large x. Then, we define the function

$$\psi(c;b) = \psi(c;b;V_o(\cdot)) = Y(c;\theta(b))$$

for $c \geq \zeta_1(b)$ where

$$\zeta_1(b) = \arg\max_{c \in [0,\theta(b))} \{Y(c;\theta(b)) = \phi(c;b)\}.$$

Here, and everywhere in the sequel, we use the convention that the maximum of an empty set is zero, i.e., $\max\{\emptyset\} = 0$. We then have the following result.

Lemma L.7 *The function $\psi(c;b)$ is concave on $[\zeta_1(b), \theta(b)]$ and satisfies $\psi'(c;b) \geq 1$ in that interval.*

Proof. Since $Y'(\theta(b); \theta(b)) = 1$, it suffices to prove concavity. Let $\tilde{Y}(c;\theta(b)) = Y(c;\theta(b)) - c$. Differentiating (81) and evaluating the result at the point $c = \theta(b)$, we get

$$\frac{1}{2}\sigma^2 \tilde{Y}'''(\theta(b); \theta(b)) = \rho + \lambda_o - r - \lambda_o V_o'(\theta(b)) \geq 0 \tag{83}$$

because $\phi'(\theta(b); b) - 1 \leq 0$ by the definition of $\theta(b)$. Since $Y''(\theta(b); \theta(b)) = 0$, this in turn implies that $\tilde{Y}(c;\theta(b))$ is concave in a small neighbourhood of $\theta(b)$. Now assume that (83) is strict (the general case follows by a small modification of the arguments), suppose that the function is not concave on $[\zeta_1(b), \theta(b)]$ and let

$$c_* = \arg\max\{c \leq \theta(b) : \tilde{Y}''(c;\theta(b)) = 0\}$$

Since the function $\tilde{Y}(c;\theta(b)$ is concave on $[c_*, \theta(b)]$, and $\tilde{Y}(c;\theta(b) \leq \phi(c;b)$ on $[\zeta_1(b); \theta(b)]$ by definition of $\zeta_1(b)$, we get that $\tilde{Y}'(c^*; \theta(b) \geq 0$ and therefore

$$0 = \frac{1}{2}\sigma^2 \tilde{Y}''(c_*; \theta(b)) = (\rho + \lambda)(\tilde{Y}(c_*; \theta(b)) - \phi(c_*; b) + c_*) - (rc_* + \mu)\tilde{Y}'(c_*; \theta(b)) < 0,$$

which is a contradiction. ∎

In order to proceed further in the construction, let $\kappa_1(b)$ denote the first point below $\zeta_1(b)$ where the functions $Y(c;\theta(b))$ and $\phi(c;b)$ coincide, that is

$$\kappa_1(b) = \arg\max_{c \in [0,\zeta_1(b))} \{Y(c;\theta(b)) = \phi(c;b)\},$$

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and consider the following algorithm: If

$$\delta_1(b) = \min_{c\in[\kappa_1(b),\zeta_1(b)]} Y'(c;\theta(b)) > 1$$

then we continue the function $\psi(c;b)$ further for lower values of c as the solution to the above ODE. If on the contrary $\delta_1(b) \le 1$ then we let

$$\xi_1(b) = \arg\max_{c\in[\kappa_1(b),\zeta_1(b)]} \{c : Y'(c;\theta(b)) = 1\}$$

$$\theta_1(b) = \arg\max_{c\in[0,\xi_1(b))} \{c : \phi(c;b) = c - \xi_1(b) + Y(\xi_1(b);\theta(b))\}$$

and continue the function for lower values of c by letting

$$\psi(c;b) = c - \xi_1(b) + Y(\xi_1(b);\theta(b))$$

for $c \in [\theta_1(b), \zeta_1(b)]$ and $\psi(c;b) = H(c;\theta_1(b))$ for $c \in [\zeta_2(b), \theta_1(b)]$ where $H(c) = H(c;\theta_1(b))$ is the unique twice continuously differentiable solution to

$$0 = \mathcal{L}_i H(c) + \lambda_o(V_o(c) - H(c)) + \lambda(Y(b;b) - (\theta(b) - c) - H(c) - \kappa)^+ \tag{84}$$
$$= H'(\theta_1(b);\theta_1(b)) - 1 = H''(\theta_1(b);\theta_1(b))$$

and we have set

$$\zeta_2(b) = \arg\max_{c\in[0,\theta_1(b))} \{H(c;\theta_1(b)) = \phi(c;b)\}.$$

Continuing this process, we arrive at a function $\psi(c;b)$ defined on $[0,\theta(b)]$ that is linear on the finite union of intervals given by $\bigcup_{j=1}^k [\theta_j(b), \xi_j(b)]$ for some $k \in \mathbb{N}$ and satisfies the differential equation (84) on the complement of these intervals.

Lemma L.8 *The function $\psi(c;b)$ satisfies both $\psi'(c;b) \ge 1$ and*

$$\mathcal{L}_i\psi(c;b) + \lambda\left(Y(\theta(b);\theta(b)) - (\theta(b) - c) - \psi(c;b) - \kappa\right)^+ + \lambda_o(V_o(c) - \psi(c;b)) \le 0 \tag{85}$$

for all $c \ge 0$.

Proof. First, we need to show that $\psi'(c;b) \ge 1$. We will only consider the first step in

the above construction of $\psi(c; b)$ (i.e., up to the boundary $\theta_1(b)$). Since the construction of $\psi(c; b)$ follows the same steps for $c \in [0, \theta_1(b)]$, the general claim follows by induction.

By Lemma L.7, the function $\psi(c; b)$ is concave for $c \geq \zeta_1(b)$. Thus, if $Y'(c; \theta(b))$ hits 1 before $\psi(c; b)$ hits the graph of $\phi(c; b)$, we have that the desired inequality $\psi'(c; b) \geq 1$ holds for all c above the first interior candidate dividend distribution region. Suppose now that $Y(c; \theta(b))$ hits the graph of $\phi(c; b)$ again before $Y'(c; \theta(b))$ hits 1 and let

$$\tilde{Y}(c; \theta(b)) = Y(c; \theta(b)) - c$$

as before. If $\tilde{Y}''(\theta_1(b); \theta(b)) \leq 0$ then the same argument as in the proof of Lemma L.7 implies that $\psi(c; b)$ stays concave as long as $\psi(c; b) \leq \phi(c; b)$. Assume now that $\tilde{Y}''(\theta_1(b); \theta(b)) > 0$ so that the function $\tilde{Y}'(c; \theta(b))$ is increasing in a small neighbourhood of $\theta_1(b)$ and suppose that the required assertion is not true. In this case, $\tilde{Y}'(c; \theta(b))$ is increasing in a right neighbourhood of

$$c_* = \arg\max\{c \leq \theta_1(b) : \tilde{Y}''(c; \theta(b)) = 0\}.$$

In conjunction with the differential equation this implies that

$$\frac{1}{2}\sigma^2 \tilde{Y}''(c_*; \theta(b)) = (\rho + \lambda)(\tilde{Y}(c_*; \theta(b)) - \phi(c_*; b) + c_*) < 0$$

which is a contradiction. Continuing by induction, we get that $\psi'(c; b) \geq 1$ for all $c \geq 0$ and it only remains to prove the supermartingale property. In the regions where the derivative $\psi'(c; b) \neq 1$ we have that $\psi(c; b)$ solves (85) by construction. Therefore, the supermartingale property only has to be shown in the regions where the function is linear, but this follows by direct calculation because in those regions we have $\psi(c; b) \geq \phi(c; b)$. ∎

Theorem L.9 *There exists a unique constant x_i^* such that $\psi(0; x_i^*) = \ell_i$ and $\psi(c; x_i^*) = V_{n,i}(c; V_o)$ gives the value function of Problem 1. Furthermore, the region in which the firm optimally searches for outside investors is given by $\{c \leq \underline{C}\}$ for some $\underline{C} \geq 0$.*

Proof. Let $b^* = \arg\max_{c \geq 0}\{\phi(c; b) - c\}$. For this choice we clearly have that $\psi(c; b^*) = \phi(b^*; b^*)$ for all $c \geq 0$ and it follows from Assumption L.6 that $\psi(0; b^*) > \ell_i$. On the other hand, we have

$$\lim_{b \to \infty}\{\psi(0; b) - b\} \leq \lim_{b \to \infty}\{\phi(b; b) - b\} = -\infty$$

because $V_o'(b) = 1$ for sufficiently large b by Assumption L.6 and the existence claim follows by continuity. Uniqueness follows from the verification argument. Indeed, by the same verification argument as in the case with no investment options, $V_{n,i}(c; V_o)$ is the value function of the firm. Therefore, the target cash level is unique. To complete the proof it remains to establish that the search region

$$\{c \geq 0 : \ V_{n,i}(x_i^*; V_o) - x_i^* + c - \kappa \geq V_{n,i}(c; V_o)\}$$

is an interval but this immediately follows from the fact that $V_{n,i}'(c; V_o) = \psi'(c; x_i^*) \geq 1$. ∎

An immediate consequence of our algorithm is

Lemma L.10 *The number of dividend distribution intervals does not exceed the number of local minima of $\phi(c; x_i^*) - c$ plus one.*

Proof of Theorem L.1. The proof of Theorem L.1 follows directly from Theorem L.9, Lemma L.10, the characterization of $V_{o,i}(c)$ provided below and Lemma L.20. ∎

Proof of Proposition L.2. The proof is based on the observation that $\psi(c; b)$ and $\psi'(c; b)$ are, respectively, decreasing and increasing in b. The proof of this claim is analogous to that of Lemma B.4 of the paper and is omitted. It immediately follows that x_i^* is decreasing in φ_i. Similarly, the claim about the dividend distribution region follows because $\psi'(c; b)$ is monotone increasing in b and therefore the region $\{c \geq 0 : \psi'(c; x_i^*) = 1\}$ is expanding as the target level x_i^* decreases. The proof of monotonicity in λ_i follows by similar arguments, based on Lemma B.6 of the paper. ∎

Lemma L.11 *We have*

$$V_{n,i}(x_i^*; V_o) \geq \frac{\mu_{i-1} + r(x_i^* + K_i)}{\rho}$$

Proof. By construction we have that $V_{n,i}(c; V_o)$ is twice continuously differentiable and satisfies $0 = 1 - V_{n,i}'(x_i^*; V_o) = V_{n,i}''(x_i^*; V_o)$. Combining this with (84) shows that

$$V_{n,i}(x_i^*; V_o) = \frac{r x_i^* + \mu_i + \lambda_o V_o(x_i^*)}{\rho}.$$

and the desired claim follows since, by assumption, $\mu_i - \mu_{i-1} \geq rK_i$ and $V_o(c) \geq 0$. ∎

The following observation allows to determine the critical value κ_{max} of the fixed cost above which the firm optimally decides to never raise outside funds and concludes our discussion of the solution to Problem 1.

Proposition L.12

$$\kappa_{\text{max}} = \left(V_{n,i}(x_i^*; V_o) - V_{n,i}(0; V_o) - x_i^*\right)\big|_{\kappa=0}.$$

L.5 Solution to Problem 2

Having constructed the solution to Problem 1, we now present a general algorithm for solving Problem 2. Proceeding as in the previous cases, we start by picking a candidate option exercise threshold b that we will later vary to obtain value matching at the origin. In order to construct the associated value, we start by defining an auxiliary function $Y_{o,i}(c; b)$ that is set to coincide with $V_{n,i}(c - K_i)$ for $c \geq b$ and is constructed as follows on the interval $[0, b]$:

1. If $b - K_i \geq x_i^*$ then we let

$$\zeta_0 = \max\left\{c < x_i^* + K_i : Y_{o,i}(c; b) > \frac{rc + \mu_i}{\rho}\right\}$$

and define the auxiliary function by setting

$$Y_{o,i}(c; b) = V_{n,i}(c - K_i) = V_{n,i}(x_i^*) + (c - K_i - x_i^*)$$

for $c \in [\zeta_0, b]$ and $Y_{o,i}(c; b) = H(c)$ for $c \in [0, \zeta_0]$ where the function $H(c)$ is the unique twice continuously differentiable solution to

$$\mathcal{L}_{i-1}H(c) + \lambda(V_{n,i}(x_i^*) - (c - K_i - x_i^*) - H(c) - \kappa)^+ = 0 \tag{86}$$

subject to the value matching and smooth-pasting conditions

$$0 = H(\zeta_0) - V_{n,i}(\zeta_0 - K_i) = 1 - H'(\zeta_0)$$

at the point ζ_0 (recall the convention that the supremum of an empty set is zero).

2. If $b - K_i < x_i^*$ then we let $Y_{o,i}(c; b) = H(c)$ for all $c \in [0, b]$ where the function $H(c)$ is the unique twice continuously differentiable solution to the differential equation (86)

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subject to the value matching and smooth-pasting conditions

$$0 = H(b) - V_{n,i}(b - K_i) = H'(b) - V'_{n,i}(b - K_i)$$

at the point b.

Given the auxiliary function $Y_{o,i}(c; b)$ we let

$$\zeta_1(b) = \max\{c \in [0, b] : Y'_{o,i}(c; b) = 1\}$$

denote the first point at which the derivative reaches one (and zero if such a point does not exist), and define the function

$$\phi(x) = \max\left\{\frac{rx + \mu}{\rho}; \frac{rx + \mu + \lambda(V_{n,i}(x_i^*) - (x_i^* + K_i - c) - \kappa)}{\rho + \lambda}\right\}.$$

in complete analogy with (82). The same arguments as the study of Problem 1 imply that we necessarily have $Y_{o,i}(\zeta_1(b); b) > \phi(\zeta_1(b))$. Consequently,

$$\theta_1(b) = \max\{c \in (0, \zeta_1(b)] : \phi(c) = Y_{o,i}(\zeta_1(b)) + c - \zeta_1(b)\}$$

is well-defined. Finally, we define the value function $w_{o,i}(c; b)$ associated with the given candidate investment trigger by setting

$$w_{o,i}(c; b) = 1_{\{c \geq \theta_1(b)\}}\left[Y_{o,i}(c \vee \zeta_1(b); b) + (c - \zeta_1(b))^+\right] + 1_{\{c \leq \theta_1(b)\}}H(c)$$

where the function $H(c)$ is the unique twice continuously differentiable solution to (86) subject to the value matching and smooth-pasting conditions

$$0 = H(\theta_1(b)) - Y_{o,i}(\theta_1(b)) = H'(\theta_1(b)) - 1.$$

Note that due to the definition of $\phi(x)$ we have that $w_{o,i}(c; b)$ is twice continuously differentiable at the point $\theta_1(b)$ where it satisfies the high contact condition $w''_{o,i}(\theta_1(b); b) = 0$.

Proposition L.13 *The function $w_{o,i}(c; b)$ satisfies the HJB equation*

$$\max\{\mathcal{L}_{i-1}w_{o,i}(c; b) + \lambda\mathcal{F}w_{o,i}(c; b), V_{n,i}(c - K_i) - w_{o,i}(c; b), 1 - w'_{o,i}(c; b)\} = 0.$$

Proof. By construction we have that $w'_{o,i}(c; b) \geq 1$ for $c \geq \theta_1(b)$ and the same arguments

as in the study of Problem 1 imply that $w_{o,i}(c;b)$ is concave on the interval $[0, \theta_1(b)]$ and therefore satisfies $w'_{o,i}(c;b) \geq 1$ on this interval. This immediately implies that there exists a threshold $\underline{C}_{i,o}(b) \geq 0$ such that we have

$$\mathcal{F}w_{o,i}(c;b) = (V_{n,i}(x_i^*) - (x_i^* + K_i - c) - w_{o,i}(c;b) - \kappa)^+$$
$$= 1_{\{c \leq \underline{C}_{i,o}(b)\}}(V_{n,i}(x_i^*) - (x_i^* + K_i - c) - w_{o,i}(c;b) - \kappa).$$

On the other hand, the inequality

$$\mathcal{L}_{i-1}w_{o,i}(c;b) - \rho w_{o,i}(c;b) + \mathcal{F}w_{o,i}(c;b) \leq 0$$

holds as an identity for $c \in [0, \theta_1(b)] \cup [\zeta_1(b), b]$ and as an inequality in $[\theta_1(b), \zeta_1(b)]$ because $w_{o,i}(c;b) \geq \phi(c)$ in this interval. To prove the inequality for $c \geq b$, we will need to show that the search thresholds satisfy $\underline{C}_{o,i}(b) \leq \underline{C}_{n,i} + K_i$ and to this end it clearly suffices to show that $w'_{o,i}(c;b) \leq V'_{n,i}(c - K_i)$ for all $c \geq K_i$. Since the two other possible cases are completely analogous we only consider the case where the trigger satisfies $b \in (\underline{C}_{n,i} + K_i, x_i^* + K_i)$. Let

$$k(c) = w_{o,i}(c;b) - V_{n,i}(c - K_i)$$

and suppose for the moment that we have both $V'_{n,i}(c) > 1$ and $w'_{o,i}(c;b) > 1$ for all $c \geq 0$. Then, for $c \geq \max\{\underline{C}_{o,i}(b), \underline{C}_{n,i} + K_i\}$, we have

$$\mathcal{L}_{i-1}k(c) + (rK_i - (\mu_i - \mu_{i-1}))V'_{n,i}(c - K_i) = 0 \tag{87}$$

and it follows from Lemma B.6 that the function $k(c)$ is decreasing. This immediately implies that we have $\underline{C}_{o,i}(b) \leq \underline{C}_{n,i} + K_i$ and it follows that the function $k(c)$ satisfies

$$0 = \mathcal{L}_{i-1}k(c) + (rK_i - \mu_i + \mu_{i-1})V'_{n,i}(c - K_i) - (V_{n,i}(x_i^*) - (x_i^* + K_i - c) - V_{n,i}(c) - \kappa)$$

on the interval $[\underline{C}_{o,i}(b), \underline{C}_{n,i} + K_i]$ and (87) on the interval $[0, \underline{C}_{o,i}(b)]$. If there are intervals where either of the derivatives is equal to one then we only need to show that

$$V'_{n,i}(c - K_i) = 1 \implies w'_{o,i}(c;b) = 1.$$

Let $(\theta_1(x_i^*), \xi_1(x_i^*)), \ldots, (\theta_k(x_i^*), \xi_k(x_i^*))$ with $\theta_1 > \ldots > \theta_k$ give the dividend distribution intervals for the function $V_{n,i}(c)$. We only consider the first interval. The general case follows

by induction. By definition, we have that $V'_{n,i}(\zeta_1(b)) = 1$. Therefore, the same argument as above implies that we have $w'_{o,i}(\zeta_1(b) + K_i; b) \leq 1$ and therefore $w'_{o,i}(\zeta_1(b) + K_i; b) = 1$ since $w'_{o,i}(c; b) \geq 1$ by construction. Now, consider the function

$$\phi_{n,i}(c) = \frac{rc + \mu_i + \lambda_o(V_{o,i}(c) - V_{n,i}(c)) + \lambda(V_{n,i}(x_i^*) - x_i^* + c - \kappa)^+}{\rho + \lambda}$$

By the construction of the interval $[\theta_1(b), \zeta_1(b)]$ we have

$$V_{n,i}(c - K_i) \geq \phi_{n,i}(c - K_i), \qquad c \in [\theta_1(b) + K_i, \zeta_1(b) + K_i]$$

Furthermore, the same argument as above based on Lemma B.6 of the paper shows that $w_{o,i}(c; b) \geq V_{n,i}(c - K_i)$ for all $c \in [\zeta_1(b), \theta(b)]$. Let

$$\phi_{o,i}(c) = \frac{rc + \mu_{i-1} + \lambda(V_{n,i}(x_i^*) - x_i^* + c - \kappa)^+}{\rho + \lambda}$$

Using the fact that $V_{o,i}(c) - V_{n,i}(c) > 0$ for all $c > 0$ and $\mu_i - \mu_{i-1} > rK_i$ by assumption we deduce that the inequality

$$\phi_{n,i}(c - K_i) > \phi_{o,i}(c) \tag{88}$$

holds if either $c > \max\{\underline{C}_{n,i} + K_i, \underline{C}_{o,i}\}$ or $\underline{C}_{n,i} + K_i > \underline{C}_{o,i}$. Since, as shown above, the latter inequality necessarily holds whenever $\zeta_1(b) + K_i \leq \max\{\underline{C}_{n,i} + K_i, \underline{C}_{o,i}\}$ we have that (88) holds in a neighbourhood of $\zeta_1(b)$. By continuity this in turn implies that

$$w_{o,i}(c; b) > V_{n,i}(c - K_i) \geq \phi_{n,i}(c - K_i) > \phi_{o,i}(c) \tag{89}$$

in a neighbourhood of $\zeta_1(b)$ and it follows that $w_{o,i}(c; b)$ is linear with slope equal to one in that neighbourhood. By definition, $w_{o,i}(c; b)$ remains linear with slope one until it hits the graph of $\phi_{o,i}(c)$ at some level $\xi_1(b)$ of the cash buffer and it only remains to show that $\xi_1(b) < \theta_1(b)$. Suppose to the contrary. By definition, we cannot have $\underline{C}_{o,i} \in [\zeta_1(b), \xi_1(b)]$. Therefore $\phi_{n,i}(c - K_i) > \phi_{o,i}(c)$ holds also for $c > \xi_1(b)$ and hence (89) holds true over the interval $[\zeta_1(b), \xi_1(b)]$. But this is impossible because $w_{o,i}(\xi_1(b); b) = \phi_{o,i}(\xi_1(b))$ by definition of $\xi_1(b)$. Thus, we conclude that $w'_{o,i}(c; b) \leq V'_{n,i}(c - K_i)$ and therefore $w_{o,i}(c; b) \geq V_{n,i}(c - K_i)$ for all $c \geq K_i$. ∎

Lemma L.14 *The function $w_{o,i}(c; b)$ is monotone increasing in b*

Proof. Let $b_1 > b_2$. Since, by the above $w'_{o,i}(c; b) \leq V'_{n,i}(c - K_i)$, we have

$$w'_{o,i}(b_2; b_1) \leq V'_{n,i}(b_2 - K_i) = w'_{o,i}(b_2; b_2).$$

Thus defining $k(c) = w_{o,i}(c; b_1) - w_{o,i}(c; b_2)$, we get that $k(b_2) > 0, k'(b_2) \leq 0$ and the required assertion now follows from Lemma B.6. ∎

In order to state the next results consider the twice continuously differentiable function defined by $W_i(c; b) = \bar{W}_i(c \wedge b; b) + (c - b)^+$ where the function $\bar{W}_i(c; b)$ is the unique twice continuously differentiable solution to

$$0 = \mathcal{L}_{i-1} \bar{W}_i(c; b) + \lambda(V_{n,i}(x_i^*) - x_i^* - K_i + c - \bar{W}_i(c; b) - \kappa)^+$$

subject to the boundary conditions $1 - \bar{W}'_i(b; b) = \bar{W}''_i(b; b) = 0$.

Lemma L.15 *The function $W_i(c; b)$ is monotone increasing with respect to $b \geq 0$ there exists a unique threshold $C^*_{i,W}$ such that $W_i(0; C^*_{i,W}) = \ell_i$. Furthermore, for any $b \geq 0$ there exists a threshold $\underline{C}_{i,W}(b)$ such that*

$$\{c \leq b : V_{n,i}(x_i^*) - x_i^* - K_i + c - W_i(c; b) - \kappa > 0\} = [0, \underline{C}_{i,W}(b)]$$

and the function $\bar{W}_i(c; C^*_{i,W})$ satisfies $\bar{W}'_i(c; C^*_{i,W}) \geq 1$ for all $c \geq 0$.

Proof. The proof is similar to that of Lemma E.1 of the paper and therefore is omitted. ∎

To simplify the notation in what follows we let $\bar{W}_i(c) = \bar{W}_i(c; C^*_{i,W})$ and $W_i(c) = W_i(c; C^*_{i,W})$ unless there a risk a confusion.

Lemma L.16 *If $K_i < K_i^*$ then*

$$W_i(C^*_{i,W}) < V_{n,i}(x_i^*) - x_i^* - K_i + C^*_{i,W}$$

and either $W_i(c) < V_{n,i}(c - K_i)$ for all $c \geq K_i$ or there exists a unique crossing point $\tilde{C}_i$ with the property that $W_i(c) < V_{n,i}(c - K_i)$ if and only if $c > \tilde{C}_i$.

Proof. The proof is analogous to that of Lemmas E.3 and E.4 of the paper. ∎

Proof of Proposition L.3. The proof follows directly from Lemma L.16. ∎

The following lemma follows by the same arguments as Lemma F.8 of the paper. For simplicity, we assume that $\kappa < \kappa_{\max}$ with $\kappa_{\max}$ defined as in Proposition L.12.

Lemma L.17 *Let*

$$\tilde{\ell}_i = V_{n,i}(x_i^*) - x_i^* - K_i - \kappa.$$

and for any $z > 0$ define the function $g_i(c; z)$ to be the unique twice continuously differentiable solution to

$$\mathcal{L}_{i-1} g_i(c; z) + \lambda 1_{\{c \leq z\}}(\tilde{\ell}_i + c - g_i(c; z)) = 0$$

with the boundary conditions

$$0 = \ell_i - g_i(0; z) = z + \tilde{\ell}_i - g_i(z; z)$$

Then we have $\lim_{z \to 0} g_i'(z; z) = \infty$.

Lemma L.18 *Suppose that $K < K_i^*$ and $\min_{c \geq K_i}(\bar{W}_i(c) - V_{n,i}(c - K_i)) \leq 0$. Then, there exists a threshold $z_* > 0$ such that*

(a) *$g_i(c; z_*) \geq V_{n,i}(c - K_i)$ for all $c \geq K_i$,*

(b) *There exists a unique point $C_{U,i}^* \geq K_i$ such that $g_i(C_{U,i}^*; z_*) = V_{n,i}(C_{U,i}^* - K_i)$.*

If $C_{U,i}^* = K_i$ then the optimal policy is to exercise at $c = K_i$ and liquidate. Otherwise, the optimal policy is to exercise the growth option at $C_{U,i}^*$.

Proof. Let us subtract from both $g_i(c; z)$ and $V_{n,i}(c)$ the function

$$\Phi_i(c) = \frac{\lambda}{\rho + \lambda}\left(V_{n,i}(x_i^*) + c - x_i^* - K_i - \kappa + \frac{\mu_{i-1} + rc}{\rho + \lambda - r}\right)$$

and denote the new functions by $\tilde{g}_i(c; z)$ and $\tilde{V}_{n,i}(c)$. Then, let us apply to these functions the transformation of Lemma D.1 of the paper with meeting intensity $\lambda = 0$, i.e. consider the equation $\mathcal{L}_{i-1} f = 0$, and apply the transformation using the solutions $F_{i,0}(c)$ and $G_{i,0}(c)$ to this equation. Denote by $\hat{g}_i(c; z)$ and $\hat{V}_{n,i}(c)$ the resulting functions. Lemma D.1 immediately yields that $\hat{g}_i(c; z)$ is concave on the interval $[0, F_i(z)/G_i(z)]$ and linear afterwards, whereas $\hat{V}_{n,i}(c)$ is globally concave. Furthermore, it follows from Lemma L.17

that on $[F_i(z)/G_i(z), +\infty)$ the slope of the function $\hat{g}_i(c; z)$ converges to infinity as z decreases to zero and this implies that $g_i(c; z) > V_{n,i}(c - K_i)$ for any sufficiently small values of z and all $c \geq K_i$.

The next important observation is that $\bar{W}_i(c) = g_i(c; \underline{C}_{i,W})$ for all $c \geq 0$. Indeed, both functions satisfy the same ODE with the same boundary conditions at the origin and the point $\underline{C}_{i,W}$ so the claim follows by the uniqueness of the solution to a second order equation. Let us now show that $g_i(c; z) > \bar{W}_i(c)$ for $z < \underline{C}_{i,W}$ and all $c > 0$. Indeed, by Lemma B.6 of the paper we have that the function

$$k(c) = g_i(c; z) - \bar{W}_i(c)$$

is either monotone increasing or monotone decreasing and the claim for $c \in [0, z]$ follows because $g_i(z; z) > \bar{W}_i(z)$ for sufficiently small values of z by Lemma L.17. The claim for $c \geq z$ follows directly from the result of Lemma B.6 of the paper.

Since $\min_{c \geq K_i}(\bar{W}_i(c) - V_{n,i}(c - K_i)) \leq 0$ by assumption, the existence of a threshold z_* satisfying the conditions of the statement follows by continuity. It is also clear that if $C^*_{U,i} > K_i$, then $g_i(c; z_*)$ satisfies the smooth pasting condition at the point $C^*_{U,i}$ and therefore

$$g_i(c; z_*) = w_{o,i}(c; C^*_{U,i}), \qquad c \leq C^*_{U,i}.$$

In particular, this implies that the function $g_i(c; z_*)$ touches the graph of the function $V_{n,i}(c)$ at a single point and the proof is complete. ∎

Lemma L.19 *Suppose that $K < K^*_i$ and $\min_{c \geq K_i}(\bar{W}_i(c) - V_{n,i}(c - K_i)) > 0$. Then, there are thresholds $C^*_{i,H} > C^*_{i,L} > C^*_{i,W}$ such that*

$$w_{o,i}(c; C^*_{i,H}) = W_i(c)$$

for all $c \leq C^*_{i,L}$.

Proof. Denote by $\delta_i(c; b)$ the unique twice continuously differentiable solution to the equation

$$\mathcal{L}_{i-1}\delta_i(c; b) + \lambda(V_{n,i}(x^*_i) - x^*_i - K_i + c - \kappa - \delta_i(c; b))^+ = 0$$

with the boundary conditions

$$\delta_i(b; b) - W_i(b) = \delta_i'(b; b) - W_i'(b).$$

The same argument as in the proof of Lemma F.4 implies that $\delta_i'(c; b)$ can have at most a single turning point. If $\delta_i'(c; b)$ is non-increasing, then there will be at most a single $\underline{C}_{\delta,i}$ at which $\tilde{\ell}_i - \delta_i(c; b)$ changes sign. Otherwise, there will be at most two such points. By construction we have

$$\delta_i(c; C_{i,W}^*) = \bar{W}_i(c) > V_{n,i}(c - K_i)$$

for all $c \geq K_i$ and it follows from Lemma L.16

$$\delta_i(\tilde{C}_i; \tilde{C}_i) = V_{n,i}(\tilde{C}_i - K_i).$$

Therefore, by continuity, there exists a threshold $b = C_{i,L}^*$ such that the function $\delta_i(c; C_{i,L}^*)$ touches the graph of the function $V_i(c - K_i)$ from above at some point $C_{i,H}^* > C_{i,L}^*$ and the same argument as in the proof of Lemma F.4 of the paper implies that $\delta_i'(c; C_{i,L}^*) \geq 1$. ∎

Proof of Theorem L.4. By Proposition L.13, we only need to show that there exists a solution to the equation $w_{o,i}(0; b) = \ell_{i-1}$ but this follows directly from Lemmas L.18 and L.19. Verification follows by the same arguments as above. ∎

Lemma L.20 *The function $V_{n,i}(c)$ has at most $N - n$ intervals of convexity.*

Proof. We prove the claim by induction. The case $i = N$ follows from Lemma B.4 of the paper. Suppose now that the claim is proved for $V_{n,i}(c)$, and let us prove it for $V_{n,i-1}(c)$. First, we note that by construction the function $V_{o,i}(c)$ can have at most one interval of convexity in the no-investment region and combining this with the induction hypothesis implies that the function $V_{o,i}(c)$ has altogether at most $N - i + 1$ intervals of convexity. Furthermore, it follows from Lemma B.7 of the paper that the second derivative of $V_{o,i}(c)$ inside a cash retention interval can change sign at most once. It follows immediately from the construction of the function $V_{n,i-1}(c)$ that it cannot have more intervals of convexity that the function $V_{o,i}(c)$ and this completes the induction step. ∎

Proof of Proposition L.5. We only prove monotonicity in λ_{i-1}. The other claims are

established similarly. Let $\lambda_1 < \lambda_2$ and define

$$k(c; b) = w_{o,i}(0; b; \lambda_2) - w_{o,i}(0; b; \lambda_1).$$

Then we have $k(b) = k'(b) = 0$ as well as $k''(b) < 0$ and it follows from Lemma B.6 of the paper that $k(c)$ cannot have negative local minima. Thus, $k(c) < 0$ and it follows that $w_{o,i}(0; b; \lambda)$ is monotone decreasing in λ. Monotonicity in φ_i is a direct consequence of Proposition L.21 below. ∎

Proposition L.21 *Suppose that an increase in a parameter α increases $V_{n,i}(c)$ and simultaneously decreases $V'_{n,i}(c)$. Then, the threshold for investment from internal funds is decreasing in α.*

Proof of Proposition L.21. For simplicity, we only consider the case without issuance costs. Let $\alpha_1 > \alpha_2$. By continuity, it suffices to consider the cases where both parameter values α_1, α_2 correspond to either the $C^*_{U,i}$ or the $C^*_{H,i}$ regimes. For simplicity, we omit the index i for the various threshold and simply denote them by C^*_U, C^*_H, C^*_L and C^*_W to denote them.

Consider first the case of a barrier policy and suppose that the desired monotonicity does not hold so that there exist $\alpha_1 > \alpha_2$ with $C^*_U(\alpha_1) = C^*_U(\alpha_2) = C^*_U$. Let $A_j = V_{n,i}(C^*_U - K_i(\alpha_1); \alpha_j)$ and consider the function defined by

$$R_j(c) = V_{o,i}(c; \alpha_j) - A_j. \tag{90}$$

By assumption, we have that

$$0 = R_1(C^*_U) = R_2(C^*_U) = R'_1(C^*_U) \leq R'_2(C^*_U) \tag{91}$$

and $A_1 > A_2$. Furthermore the function $R_j(c)$ satisfies

$$\mathcal{L}_{i-1} R_j(c) - \rho A_j + \lambda(V_{n,i}(x^*_i(\alpha_j); \alpha_j) - x^*_i(\alpha_j) - K_i(\alpha_j) - A_j + c - R_j(c)) = 0. \tag{92}$$

on the interval $[0, C^*_U]$ it follows that the function $k = R_1 - R_2$ satisfies

$$0 = \mathcal{L}_{i-1} k(c) + \lambda k(c) + \rho(A_2 - A_1) + \lambda Z = k(C^*_U) \geq k'(C^*_U)$$

where the constant Z is defined by

$$Z = V_{n,i}(x_i^*(\alpha_1); \alpha_1) - x_i^*(\alpha_1) - K_i(\alpha_1) - A_1 - (V_{n,i}(x_i^*(\alpha_2); \alpha_2) - x_i^*(\alpha_2) - K_i(\alpha_2) - A_2)$$

and the notation $K_i(\alpha_j)$ indicates the possible dependence of the investment cost on α. We claim that $Z \leq 0$. Indeed, since $1 \leq V'_{n,i}(c - K_i(\alpha_1); \alpha_1) \leq V'_{n,i}(c - K_i(\alpha_1); \alpha_2)$ by assumption we get

$$y_i(\alpha_1) = x_i^*(\alpha_1) + K_i(\alpha_1) \leq x_i^*(\alpha_2) + K_i(\alpha_1) = y_i(\alpha_1, \alpha_2)$$

and it follows that

$$Z = \int_{C_U^*}^{y_i(\alpha_1)} \left(V'_{n,i}(c - K_i(\alpha_1); \alpha_1) - 1 \right) dc - \int_{C_U^*}^{y_i(\alpha_1,\alpha_2)} \left(V'_{n,i}(c - K_i(\alpha_1); \alpha_2) - 1 \right) dc \quad (93)$$

$$\leq \int_{C_U^*}^{y_i(\alpha_1)} \left(V'_{n,i}(c - K_i(\alpha_1); \alpha_1) - V'_{n,i}(c - K_i(\alpha_1); \alpha_2) \right) dc \leq 0.$$

By Lemma B.6 this in turn implies that the function $k(c)$ is monotone decreasing and it follows that we have $0 < k(0) = A_2 - A_1 < 0$ which is a contradiction.

Suppose now that both parameters correspond to a band strategy and $C_H^*(\alpha_1) = C_H^*(\alpha_2) = C_H^*$ for some $\alpha_1 > \alpha_2$. On the interval, $[\max\{C_L^*(\alpha_1), C_L^*(\alpha_2)\}, C_H^*]$, the functions $R_j(c), j = 1, 2$ defined in (90) satisfy (91)-(92) and therefore the same argument as above implies that the function $k = R_1 - R_2$ is monotone decreasing on $[\max\{C_L^*(\alpha_1), C_L^*(\alpha_2)\}, C_H^*]$. Consequently, $R'_1 \leq R'_2$ and hence R'_1 hits the value of 1 earlier (from the right) than R'_2. That is, $C_L^*(\alpha_1) > C_L^*(\alpha_2)$ and hence $k(c)$ is decreasing on $[C_L^*(\alpha_1), C_H^*]$. Since

$$1 = R'_1(c) \leq R'_2(c), \qquad c \in I = [C_W^*(\alpha_1), C_L^*(\alpha_1)]$$

it follows that the function $k(c)$ is also decreasing on I. Since $k(C_H^*) = 0$, we have that $R_1(c) \geq R_2(c)$ on $[C_W^*(\alpha_1), C_H^*]$. We will now show that $C_W^*(\alpha_1) \leq C_W^*(\alpha_2)$. Indeed, the algorithm for the construction of the value function $w_{o,i}(c; b)$ implies that the threshold C_W^* is the first point below $C_L^*(\alpha_j)$ where $R_j(c)$ hits the graph of the function defined by

$$\phi_j(c) = \frac{(r + \lambda)c + \mu_{i-1} + Z_j}{\rho + \lambda}$$

with the constant

$$Z_j = -\rho A_j + \lambda(V_1(C_1^*(\alpha_j); \alpha_j) - C_1^*(\alpha_j) - K_i(\alpha_j) - A_j).$$

By (93) and the inequality $A_1 > A_2$, we have $Z_2 > Z_1$. Therefore, since $R_2(c) \leq R_1(c)$ on the interval $[C_L^*(\alpha_1), C_H^*]$ we have that the function $R_2(c)$ hits the graph of the function $\phi_2(c)$ at a cash level that is higher than the cash level at which the function $R_1(c)$ hits the graph of the function $\phi_1(c)$ and it follows that we have both

$$R_2(C_W^*(\alpha_1)) < R_1(C_W^*(\alpha_1)) \quad \text{and} \quad R_2'(C_W^*(\alpha_1)) > R_1'(C_W^*(\alpha_1)).$$

The same argument as in the first part of the proof now implies that $k(0) > 0$ which provides the required contradiction because $k(0) = A_2 - A_1 < 0$ by assumption. ∎
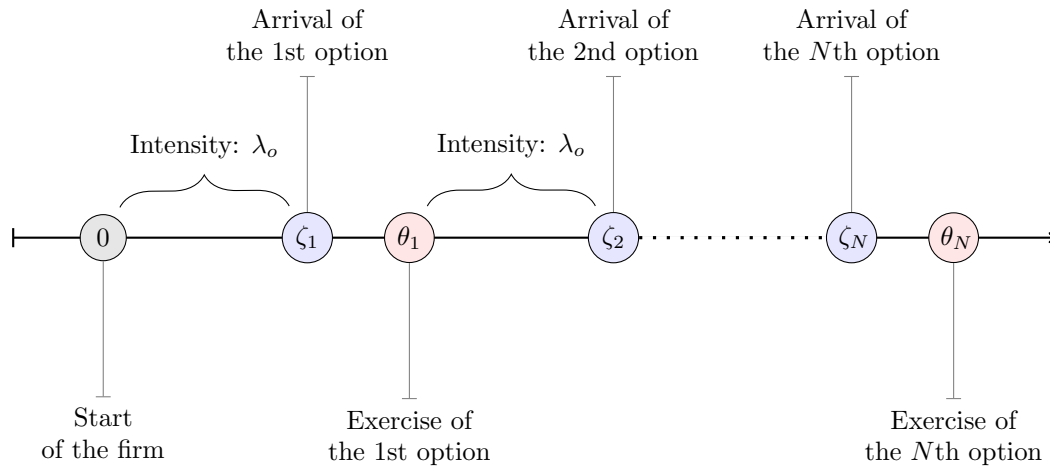
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Figure 8. The model with multiple growth options

This figure illustrates the model with multiple options: The firm initially has mean cash flow rate μ_0 and does not any growth option. At the exponentially distributed time ζ_1 the firm receives its first growth option and exercises optimally at the stopping time θ_1. The second growth option then arrives after the exponentially distributed time $\zeta_2 - \theta_1$ has elapsed and is optimally exercised at the stopping time θ_2. This goes on until the optimal exercise of the last growth option at the stopping time θ_N. After that time the mean cash flow rate of the firm remains constant.
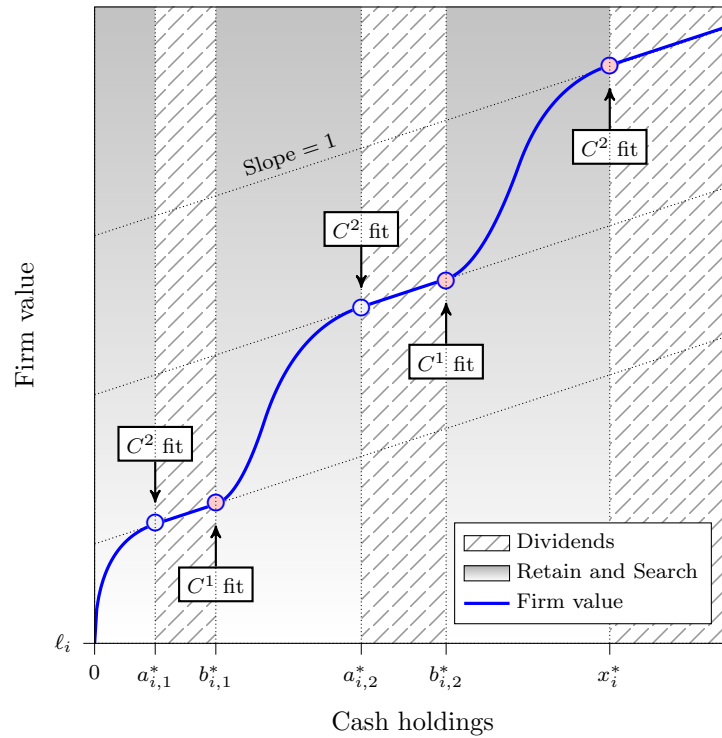
Figure 9. Value of the firm in the waiting period between growth options

This figure represents the value of a firm as a function of its cash holdings in the waiting period between the exercise of the i'th growth option and the arrival of the next one. In this picture the optimal strategy includes two intermediate dividend distribution intervals and three earnings retention intervals whose location are specified by the vectors (a_i^*, b_i^*) and the target x_i^*.