Mutual fund competition in the presence of dynamic flows[☆]

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A B S T R A C T

This paper analyzes competition between mutual funds in a multiple funds version of the model of Hugonnier and Kaniel (2010). We characterize the set of equilibria for this portfolio management game and show that there exists a unique Pareto optimal equilibrium. The main result of this paper shows that the funds cannot differentiate themselves through portfolio choice in the sense that they should offer the same risk/return tradeoff in equilibrium. This result brings theoretical support to the findings of recent empirical studies on the importance of media coverage and marketing in the mutual funds industry.

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1. Introduction

In recent decades, the number of mutual funds offered to investors has grown substantially and now exceeds the number of traded assets in most exchanges (see Gruber, 1996; Massa, 1998), while an increasing number of these mutual funds are operating in the same sector. Most of the funds charge a fraction of fund fees whereby the manager receives a fixed fraction of the assets under management (see Golec, 1999; Golec & Starks, 2004), but the level of these fees varies greatly across funds (see Hortaçsu & Syverson, 2004). As a result, in any given market segment various investment vehicles are offered to the investor in the form of mutual funds which differ in their management fees and, presumably, also in their investment strategies.

The aim of this paper is to investigate if mutual funds competing on the same market can differentiate from each other through portfolio management in a world where investors can move their money in and out of mutual funds. To this end, we study a generalization of the model of Hugonnier and Kaniel (2010) with multiple mutual funds. Specifically, we consider a continuous-time economy populated by a small investor and two mutual

fund managers. The small investor implicitly faces high costs that preclude her from trading directly in the equity market. These implicit costs can be related, for example, to the fact that the opportunity costs of spending her time in stock-trading related activities are high. For example, one might think that actively trading multiple risky securities requires considerably more attention than trading in one or two mutual funds. While the investor is precluded from holding equity directly, she is allowed to dynamically allocate money between the two mutual funds and a riskless asset. We impose the natural restriction that the investor cannot short the funds and assume that both funds charge fraction-of-fund fees, albeit at different rates.

To focus on the competition between the funds, while maintaining a tractable setup, we make a few simplifying assumptions. First, agents have complete information and observe the actions of each other.[1] Second, from the perspective of the funds markets are complete. Third, the investor is assumed to have a logarithmic utility function. Fourth, the fund managers are strategic whereas the investor is not. Specifically, when the investor determines her holdings in the funds, she takes the funds' portfolios as given. On the other hand, when a fund manager selects the portfolio of his fund, he takes into account the portfolio of the other and the investors' reaction to the portfolios of the two funds.

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[1] This assumption is rather natural in the context of a continuous-time model. Indeed, if one assumes that market prices are perfectly observed and that the vector of instantaneous risk premia is known to all, then the portfolio strategy of a given fund is public knowledge as soon as its returns are continuously observed.

In order to solve for the equilibria of the game, we start by studying the investor's utility maximization problem given an arbitrary pair of fund portfolios. Since the investor has logarithmic utility, her optimal strategy depends only on the current characteristics of the funds. In this context, we show that she will invest in both funds, in only one of them or not at all depending on the relative excess returns of the funds with respect to one another. Interestingly, we show that, contrary to the monopolistic case considered in Hugonnier and Kaniel (2010), the investor may find it optimal to invest in a fund whose net-of-fees Sharpe ratio is currently negative. In other words, competition for the investor's money can lead to positive externalities between mutual funds.

In a second step, we take the investor's best response strategy as given and study the Nash game that it induces between the managers. Combining traditional optimization techniques with a change of measure argument we characterize the set of equilibria for this game and show that each of these gives rise to an equilibrium for our delegated portfolio management game. Furthermore, we show that among these equilibria there exists a unique Pareto optimal equilibrium in which the funds offer the same risk/return trade-off. This implies that the investor is indifferent between the two funds in equilibrium and hence that competition does not benefit the investor. In particular, we show that the total fraction of her wealth that the investor will delegate is independent of the funds characteristics, and that its allocation among the funds is arbitrary. This indeterminacy creates a role for marketing in the mutual fund industry and corroborates the findings of recent empirical studies showing the importance of advertising and media coverage, see Barber, Odean, and Zheng (2005), Gallaher, Kaniel, and Starks (2006), Hortaçsu and Syverson (2004), Jain and Wu (2000) and Kaniel, Starks, and Vasudevan (2007) among others.

Fraction-of-fund fees are by far the predominant compensation contract in the mutual fund industry. However, some funds have a performance component in their compensation contract. Basak, Pavlova, and Shapiro (2007), Carpenter (2000) and Grinblatt and Titman (1989) among others, have studied the optimal portfolio strategy of a manager receiving convex performance fees in a setting where the manager receives an exogenous amount of money to manage at the initial date. An analysis of the equilibrium asset pricing implications of both fulcrum fees and asymmetric performance fees is conducted in Cuoco and Kaniel (2001). In that paper both fund managers and unconstrained investors trade directly in equity markets, but investors who use the fund services make allocation decisions only at the initial date. High water mark fees, used in the hedge fund industry, are discussed in Goetzmann, Ingersoll, and Ross (2003). Note however that all these models consider the case of a single mutual fund, and hence abstract from the strategic aspects of competition among mutual funds.

Since the focus of this paper is the impact of dynamic flows on the competition between mutual funds, we take the fee structures of the funds as given. However, it is important to emphasize that we are not taking a stance on whether fraction-of-fund fees is the optimal compensation contract, but instead rely on its widespread use as the motivation for our analysis. Papers that analyze these optimal contracting issues include Carpenter, Dybvig, and Farnsworth (2010), Das and Sundaram (2002a,b), Lynch and Musto (1997), Ou-Yang (2003) and Roll (1992) among many others.

The remainder of the paper is organized as follows. In Section 2, we describe the economic setting, the financial market and the dynamics of the mutual funds. In Section 3, we introduce the players and their objective functions. Section 4 describes the game and defines the notion of equilibrium that we use in this paper. In Section 5, we solve for the best responses of the investor and the managers. In Section 6, we obtain the equilibrium and discuss the impact of imperfect competition between mutual funds. Section 7 concludes. All proofs are deferred to the Appendix.

2. The model

We consider a continuous-time economy with a finite horizon $[0, T]$. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ on which is defined a standard n-dimensional Brownian motion represented by the column vector B. The filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ is the usual augmentation of the filtration generated by the Brownian motion and we let $\mathcal{F} = \mathcal{F}_T$.

In the sequel, all processes are assumed to be adapted to $\mathbb{F}$ and all statements involving random quantities are understood to hold either almost surely or almost everywhere. We shall also make use of the following vectorial notation: a $'$ denotes transposition, $\| \cdot \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$ and $\mathbf{1}_k$ is a k-dimensional vector of ones.

2.1. Securities

There is a single perishable good (the numéraire) in units of which all quantities are expressed. The financial market consists of $n + 1$ long-lived securities. The first of these is a locally riskless asset whose price S^0 is given by

$$S_t^0 = e^{rt} \tag{1}$$

for some constant interest rate r. The remaining n assets are risky and are referred to as the stocks. The price S_t^i of a share of the i-th stock evolves according to

$$S_t^i = S_0^i + \int_0^t S_s^i \left(a_i ds + \sigma_i' dB_s \right), \tag{2}$$

for some drift a_i and some volatility vector $\sigma_i \in \mathbb{R}^n$ which are both assumed to be constant. We let $a \in \mathbb{R}^n$ denote the column vector of stock drifts, $\sigma \in \mathbb{R}^{n \times n}$ denote the square matrix obtained by stacking up the individual stock volatilities and we assume that σ is invertible.

The assumptions imposed on the coefficients of the model imply that the relative risk premium, or market price of risk, $\xi := \sigma^{-1}(a - r\mathbf{1}_n)$ is well defined. As a result, the formula

$$\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = M_t := \exp \left(-\xi' B_t - \frac{1}{2} \|\xi\|^2 t \right) \tag{3}$$

defines an equivalent *risk-neutral probability measure*. Since the volatility matrix of the stocks is invertible this risk-neutral probability measure is uniquely defined and it follows that the financial market is dynamically complete in the absence of trading constraints.

2.2. Mutual funds

We consider two mutual funds, indexed by $i \in \{1, 2\}$, both of which have access to the $n + 1$ securities described above. The management fees are assumed to be withdrawn continuously from fund i at the constant rate γ_i applied to the market value of the assets under management and we denote by $\gamma = (\gamma_1, \gamma_2)'$ the vector of instantaneous fee rates.

A trading strategy for fund i is a vector process θ_i specifying the share of the fund's assets invested in each of the stocks. Given such a trading strategy, the return on investments in fund i evolves according to

$$dF_{it} = \left(1 - \theta_{it}' \mathbf{1}_n \right) F_{it} \frac{dS_t^0}{S_t^0} + \sum_{k=1}^{n} F_{it} \theta_{it}^k \frac{dS_t^k}{S_t^k} - \gamma_i F_{it} dt$$

$$= F_{it} \left(r - \gamma_i + \psi_{it}' \xi \right) dt + F_{it} \psi_{it}' dB_t, \tag{4}$$

where $\psi_i = \sigma' \theta_i$ is the corresponding fund-volatility process. Since the volatility matrix is invertible there is a one-to-one correspondence between fund trading strategies and fund

volatilities and we will from now on identify the manager's strategy with the volatility of his fund.

In what follows we denote by Ψ_i the set of fund-volatility processes ψ_i with the property that the solution to Eq. (4) is non-negative.

3. Agents

3.1. The investor

We consider a small investor[2] who has no direct access to the risky assets, but is allowed to trade in the riskless asset and the two mutual funds.[3]

A trading strategy for the investor is a two-dimensional vector process π specifying the share of wealth invested in each of the two mutual funds. Since investors cannot short actively managed funds, we impose the constraint

$$\pi_t \in \mathbb{R}_+^2, \quad 0 \leq t \leq T.$$

Given a pair (ψ_1, ψ_2) of fund volatilities, and under the usual self-financing condition, the investor's wealth $W = W(\pi, \psi_1, \psi_2)$ evolves according to

$$dW_t = \left(1 - \pi_t' \mathbf{1}_2\right) W_t \frac{dS_t^0}{S_t^0} + \sum_{i=1}^2 W_t \pi_{it} \frac{dF_{it}}{F_{it}}$$
$$= W_t \left(r + \pi_t'(\psi_t \xi - \gamma)\right) dt + W_t \pi_t' \psi_t dB_t \qquad (5)$$

with initial condition $W_0 > 0$ where $\psi = (\psi_1, \psi_2)'$ represents the volatility matrix of the two mutual funds.

Denote by Φ_{it} the process of cumulative management fees paid by the investor to fund i, that is

$$\Phi_{it} := \int_0^t \gamma_i \pi_{is} W_s ds. \qquad (6)$$

In what follows we let $\Pi(\psi)$ denote the set of processes $\pi \in \mathbb{R}_+^2$ such that, given ψ, the solution to Eq. (5) is a positive process and

$$\sum_{i=1}^2 E_Q \left[\int_0^T e^{-rt} d\Phi_{it} \right]$$
$$= E_Q \left[\int_0^T e^{-rt} \gamma' \pi_t W_t(\pi, \psi_1, \psi_2) dt \right] < \infty. \qquad (7)$$

Since the market is dynamically complete, the above expectation represents the market value of the future management fees to an unconstrained investor and the constraint guarantees that this value is finite for both mutual funds.

The investor is assumed to have logarithmic preferences over terminal wealth. Given a fund-volatility matrix ψ, her objective is thus to select a trading strategy $\pi \in \Pi(\psi)$ so as to maximize the expected utility

$$U(\pi, \psi_1, \psi_2) = E \left[\log(W_T(\pi, \psi_1, \psi_2)) \right].$$

The facts that the fund-value processes are driven by n independent Brownian motions and that the investor is not allowed to

short the funds imply that she effectively faces an incomplete market. The assumption of log utility is therefore critical because it is the only utility function which allows for a closed-form solution of the investor's problem for all fund volatilities.

The model can also be solved under the assumption that the investor has a constant relative risk aversion utility function but in that case it has to be established ex-ante that the managers' optimal strategies are deterministic, for otherwise one cannot solve the investor's problem given an arbitrary pair of fund volatilities. The results in that case are very similar to those obtained with a logarithmic utility function and hence do not warrant the additional difficulty.

3.2. The fund managers

In exchange for his services, the manager of fund i receives the fees that are generated by the investor's trading of the fund. Specifically, if the investor follows some strategy π then the manager of fund i receives $\gamma_i \pi_{it} W_t$ per unit of time and the corresponding cumulative fee is given by the process Φ_i defined in Eq. (6).

We assume that the manager of fund i chooses the volatility of his fund in order to maximize the initial market value

$$V_i(\pi, \psi_1, \psi_2) = E_Q \left[\int_0^T e^{-rt} \gamma_i \pi_{it} W_t(\pi, \psi_1, \psi_2) dt \right] \qquad (8)$$

of the future fees generated by the investor's trading of the fund.[4] This assumption is similar to those of Boudoukh, Richardson, Stanton, and Whitelaw (2004) and Hugonnier and Kaniel (2010) and can be justified as follows. Each of the mutual funds should be viewed as being part of a different financial services firm. In this case, Eq. (8) represents the contribution of the given fund to the market value of the firm that owns it and our specification implies that the manager of fund i acts in order to maximize the value of the firm.[5] The simplifying assumption we make is that we ignore agency conflicts between the manager and the shareholders of the financial services firm that employs him.

4. The game

We assume that the investor is a small agent who does not realize that her decisions have an impact on the managers' strategies.[6] On the other hand, we assume that the managers act independently and non-cooperatively and take into account the reaction function of the investor in choosing their strategy. The 3-player game is thus a Nash game between the two managers, and a Stackelberg game between each manager and the investor. Notice that, even if they have access to the same market, the Nash players are not symmetrical, because they have potentially different management fees.

The sequence of events is as follows. At the initial time, the managers announce simultaneously their portfolio strategies through the fund-volatility processes ψ_1 and ψ_2. The investor then reacts by dynamically allocating her wealth between the two funds and the riskless asset. Accordingly we have the following definition of an equilibrium for our delegated portfolio management game.

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[2] While it would be interesting to include heterogenous investors into the model, we focus on the case of a single investor for tractability. Indeed, the inclusion of heterogenous investors would make our model similar to a general equilibrium model with incomplete markets, a class of models which are known to be very difficult to solve, even numerically.

[3] The inability of the investor to trade stocks directly should be viewed as a reduced form representing the fact that it is more costly for her to trade stocks efficiently than it is for the funds.

[4] It is important to note that maximizing the market value of the fees is different from assuming that the manager is risk neutral. In particular, if the manager is risk averse and can trade in the market without constraint then his value function depends positively on the market value of the future fees and it follows that the optimal fund volatility maximizes Eq. (8). This striking equivalence only holds under the assumption that markets are dynamically complete and this is the sense in which this assumption is crucial to our results.

[5] If both funds belongs to the same financial services firm then the managers will cooperate in order to maximize the total value of the fees paid by the investor.

[6] An equivalent assumption is that the investor does not have to commit in advance to her allocation strategy.

Definition 1. An equilibrium is a fund-volatility matrix $\psi^* = (\psi_1^*, \psi_2^*)'$ with $\psi_i^* \in \Psi_i$ and an investor trading strategy $\pi^* \in \Pi(\psi^*)$ such that

1. Given the fund volatility ψ^*, the trading strategy π^* is optimal for the investor in the sense that

$$U\left(\pi^*, \psi_1^*, \psi_2^*\right) \geq U\left(\pi, \psi_1^*, \psi_2^*\right)$$

for all trading strategies $\pi \in \Pi(\psi^*)$.

2. The fund volatility ψ_i^* is optimal for manager i in the sense that

$$V_i\left(\pi^*, \psi_i^*, \psi_j^*\right) \geq V_i\left(\hat{\pi}(\psi_i, \psi_j^*), \psi_i, \psi_j^*\right)$$

for all $\psi_i \in \Psi_i$ where the trading strategy $\hat{\pi}(\psi_i, \psi_j^*) \in \Pi(\psi_i, \psi_j^*)$ is the investor's reaction to the pair (ψ_i, ψ_j^*) of fund volatilities.

In order to solve for the equilibrium of the game we will proceed in two steps. First we will solve the investor's problem given a pair of admissible fund volatilities in order to obtain her reaction function. Second, we will solve the Nash game obtained by plugging this reaction function into the objective functions of the managers. Since the coefficients of the market are deterministic, we expect that there exist equilibria of the game in open-loop strategies. This intuition will be confirmed in Section 6.

5. Optimal strategies

5.1. The investor's reaction function

We first characterize the investor's best response to a given fund-volatility matrix ψ. To simplify the interpretation of the optimal strategy in terms of the relative performances of funds, we start by introducing some notation. We denote by

$$\lambda_{it} = \frac{\psi_{it}'\xi - \gamma_i}{\|\psi_{it}\|^2}$$

the net-of-fees Sharpe ratio of fund $i \in \{1, 2\}$, that is the instantaneous net excess return on fund i per unit of risk. If the funds are not collinear, that is if $\det[\psi_t \psi_t'] \neq 0$, then we denote by

$$\Lambda_t = \left[\psi_t \psi_t'\right]^{-1} (\psi_t \xi - \gamma) = \frac{1}{\det[\psi_t \psi_t']} \begin{pmatrix} A_{1t}\|\psi_{2t}\|^2 \\ A_{2t}\|\psi_{1t}\|^2 \end{pmatrix}$$

the vector representing the relative risk premia of the funds with respect to each other. In the above equation, the scalar process

$$A_{it} = A_i(\psi_t) = \left(\psi_{it}'\xi - \gamma_i\right) - \frac{\psi_{it}'\psi_{jt}}{\|\psi_{jt}\|^2} \left(\psi_{jt}'\xi - \gamma_j\right)$$

represents the risk-adjusted instantaneous net excess return of fund i with respect to fund $j \neq i$, i.e. its "alpha" with respect to fund j.

The following proposition characterizes the solution to the investor's optimization problem.

Proposition 2. For a fund-volatility matrix $\psi = (\psi_1, \psi_2)'$ with $\psi_i \in \Psi_i$ the investor's optimal strategy $\hat{\pi}_t = \hat{\pi}_t(\psi_1, \psi_2)$ is given by

1. If $\psi_t \in \mathcal{C}_1 = \{x : A_1(x) > 0, A_2(x) > 0\}$ then $\hat{\pi}_t = \Lambda_t$.
2. If $\psi_t \in \mathcal{C}_2 = \{x : A_1(x) > 0, A_2(x) \leq 0\}$ then $\hat{\pi}_t = (\lambda_{1t}^+, 0)'$.
3. If $\psi_t \in \mathcal{C}_3 = \{x : (A_1(x), A_2(x)) \in \mathbb{R}_-^2 \setminus (0, 0)\}$ then $\hat{\pi}_t = (\lambda_{1t}^+, \lambda_{2t}^+)'$.
4. If $\psi_t \in \mathcal{C}_4 = \{x : A_1(x) \leq 0, A_2(x) > 0\}$ then $\hat{\pi}_t = (0, \lambda_{2t}^+)'$.

Finally, if $\psi_t \in \mathcal{C}_0 = \{x : A_1(x) = A_2(x) = 0\}$ so that $\gamma_2\psi_{1t} = \gamma_1\psi_{2t}$, then the investor is indifferent between the funds and her optimal strategy is given by $\hat{\pi}_{1t} = \varepsilon\lambda_{1t}^+$ and $\hat{\pi}_{2t} = (1-\varepsilon)\lambda_{2t}^+$ for any $\varepsilon \in [0, 1]$.
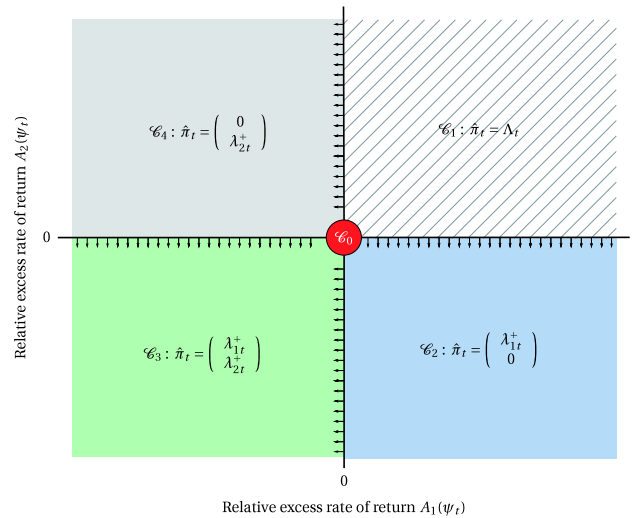

Fig. 1. Investor's optimal strategy as a function of the relative excess rates of return on the funds.

The result of Proposition 2 is illustrated by Fig. 1 which shows the investor's best response in the (A_1, A_2) space. At any time, four cases may arise depending on the fund-volatility matrix ψ which is announced by the managers. The first case happens when the investor allocates all her wealth to the bond and none to the mutual funds. This solution occurs when the funds are dominated by the riskless asset in the sense that the net-of-fees Sharpe ratio λ_{it} is negative for both funds. This can happen in any of the three regions $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_3$, at the intersection with the set of fund volatilities for which both λ_{1t} and λ_{2t} are negative.

The second case happens when the investor invests in the riskless asset and in only one of the two funds, say fund 1. This solution occurs when fund 1 dominates fund 2 in the sense that either $A_{1t} > 0$ and $A_{2t} \leq 0$, which corresponds to region $\mathcal{C}_2$, or $A_{1t} \leq 0, A_{2t} < 0$ and $\lambda_{2t} \leq 0$, which corresponds to the intersection of region $\mathcal{C}_3$ with the set of fund volatilities for which λ_{2t} is negative.[7] In that case the proportion of her wealth that the investor allocates to each fund is equal to the positive part of the fund's net-of-fees Sharpe ratio.

The third case happens when the investor invests in both funds. This occurs when none of the funds dominates the other, that is when both have strictly positive risk-adjusted net excess return relative to the other. This corresponds to region $\mathcal{C}_1$ where the proportion of her wealth that the investor allocates to each of the two funds is given by the vector Λ_t of relative risk premia. Interestingly, in this case the investor may allocate a positive proportion of her wealth to a fund which offers a negative net-of-fees Sharpe ratio as long as both funds have strictly positive risk-adjusted net excess return relative to the other. An examination of the optimal investment strategy shows that the fees charged by fund i have a direct impact on the amount invested in the rival fund j. This indirect impact of the fees can be positive or negative depending on the correlation between the funds returns.

The fourth and last case happens when the investor is indifferent between the two mutual funds. This occurs when both funds have vanishing risk-adjusted net excess return relative to the other and corresponds to the region $\mathcal{C}_0$. In that case, the funds are equivalent because they both offer the same investment

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[7] Note that in the region $\text{int}(\mathcal{C}_3)$, where both A_{1t} and A_{2t} are strictly negative, the individual Sharpe ratios λ_{1t} and λ_{2t} cannot be strictly positive at the same time.

opportunity in net terms.[8] As a result, the optimal volatility of the investor's wealth

$$\psi_t' \hat{\pi}_t = \lambda_{1t} \psi_{1t} = \lambda_{2t} \psi_{2t}$$

is uniquely defined but, as shown by the proposition, there are infinitely many trading strategies which attain the investor's maximal level of expected utility.

5.2. The managers' best responses

In this section we turn to the characterization of the managers' best responses. Specifically, we now determine the optimal course of action of manager i given the volatility of fund j and the reaction function of the investor to the fund volatilities announced by the two managers.

Fix $i \in \{1, 2\}$ and assume that the manager of fund $j \neq i$ announces a fund volatility ψ_j. If the manager of fund i announces a fund volatility ψ_i then the investor will respond by playing the trading strategy $\hat{\pi}(\psi_i, \psi_j)$ prescribed by Proposition 2 and this determines the present value

$$R_i(\psi_i, \psi_j) = V_i \left(\hat{\pi}(\psi_i, \psi_j), \psi_i, \psi_j \right)$$

$$= E_Q \left[\int_0^T e^{-rt} \gamma_i \hat{\pi}_{it}(\psi_i, \psi_j) \hat{W}_t(\psi_i, \psi_j) dt \right]$$

of the future fees to be received by the manager of fund i. In the above equation, the process $\hat{W} = \hat{W}(\psi_i, \psi_j)$ is the wealth process generated by the investor's reaction function, that is the solution to

$$d\hat{W}_t = \hat{W}_t \left(r - \gamma' \hat{\pi}_t(\psi_i, \psi_j) \right) dt + \hat{W}_t \hat{\pi}_t(\psi_i, \psi_j)' \psi_t dZ_t, \quad (9)$$

where the process Z is an n-dimensional standard Brownian motion under the risk-neutral probability measure defined by Eq. (3).

As a result of the above discussion, the best response of manager i to the fund volatility ψ_j is the solution to the stochastic control problem

$$\hat{R}_i(\psi_j) = \sup_{p \in \Psi_i} R_i(p, \psi_j)$$

subject to the dynamics in Eq. (9). In order to simplify the study of this problem we start by showing that the manager's best response cannot be such that, over a time interval of positive length, both funds offer strictly positive risk-adjusted net excess return relative to the other.

Proposition 3. *Fix $\psi_j \in \Psi_j$ and let $\psi_i \in \Psi_i$ be a fund volatility such that for some $u \in [0, T)$ and $\varepsilon \in (0, T - u]$ the matrix ψ_t belongs to $\mathcal{C}_1$ for all $t \in [u, u + \varepsilon]$. Then ψ_i cannot be the best response of manager i.*

As a result of the above proposition we have that the manager's best response is equivalent to the maximization of $R_i(\cdot, \psi_j)$ over the set

$$\Psi_i^*(\psi_j) = \left\{ p \in \Psi_i : (p_t, \psi_{jt}) \notin \mathcal{C}_1, \forall t \in [0, T] \right\}. \quad (10)$$

In order to simplify this problem further, we now introduce an equivalent change of measure. Using Lemma 9 in the Appendix, we have that for each $\psi_i \in \Psi^*(\psi_j)$ the non-negative process

$$\hat{M}_t = \hat{M}_t(\psi_i, \psi_j) = e^{\int_0^t (r + \gamma' \hat{\pi}_s(\psi_i, \psi_j)) ds} \frac{\hat{W}_t(\psi_i, \psi_j)}{W_0}$$

is a martingale with initial value one under the risk-neutral probability measure and it follows that the formula

$$\frac{d\hat{Q}(\psi_i)}{dQ} = \hat{M}_T(\psi_i, \psi_j)$$

defines an equivalent probability measure. Using this family of equivalent probability measures, we have that the problem of manager i can be written as

$$\hat{J}_i(\psi_j) = \frac{\hat{R}_i(\psi_j)}{W_0} = \sup_{p \in \Psi_i} J_i(p, \psi_j)$$

$$= \sup_{p \in \Psi_i} E_{\hat{Q}(p)} \left[\int_0^T \gamma_i x_t(p, \psi_j) \hat{\pi}_{it}(p, \psi_j) dt \right]$$

where the new state variable x is defined by

$$x_t = x_t(\psi_i, \psi_j) = \exp \left(- \int_0^t \gamma' \hat{\pi}_s(\psi_i, \psi_j) ds \right).$$

Now, if we assume that the manager of fund j restricts his choice to the set Ψ_i^d of deterministic fund volatilities, then the above problem is no longer stochastic since the investor's reaction function $\hat{\pi}_{it}$ only depends on time and the market coefficients. As a result, it follows that if ψ_j is deterministic then the best response of manager i is also deterministic.

Fix an arbitrary, but deterministic, fund volatility $\psi_j \in \Psi_j^d$. In order to proceed towards a solution to manager i's optimization problem we now show that his best response cannot be such that his fund is excluded by the investor over some time interval of strictly positive length.

Proposition 4. *Fix $\psi_j \in \Psi_j^d$ and let $\psi_i \in \Psi_i^d$ be such that for some $u \in [0, T)$ and $\varepsilon \in (0, T - u]$ the investor's best response satisfies $\hat{\pi}_{it}(\psi_i, \psi_j) = 0$ for all $t \in [u, u + \varepsilon]$. Then ψ_i cannot be manager i's best response.*

Using the results of the above propositions, we deduce that the best response of manager i must belong to the set

$$\mathcal{B}_i(\psi_j) = \left\{ p \in \Psi_i^d : p_t' \xi > \gamma_i \text{ and } A_{jt}(p, \psi_{jt}) \leq 0, \forall t \in [0, T] \right\}. \quad (11)$$

Combining the definition of this set with Proposition 2 and footnote 7 we have that on the set $\mathcal{B}_i(\psi_j)$ the investor's reaction function is given by $\hat{\pi}_{it} = \lambda_{it}$ and $\hat{\pi}_{jt} = 0$. As a result, the objective of manager i can be equivalently written as

$$J_i(p, \psi_j) = \int_0^T \gamma_i \lambda_{it} e^{- \int_0^t \gamma_i \lambda_{is} ds} dt = 1 - e^{- \int_0^T \gamma_i \lambda_{it} dt}$$

and it follows that his best response can be obtained by solving

$$\sup_{p \in \mathcal{B}_i(\psi_j)} \lambda_{it} = \sup_{p \in \mathcal{B}_i(\psi_j)} \frac{p_t' \xi - \gamma_i}{\|p\|^2}$$

for each $t \in [0, T]$. The solution to this nonlinear constrained optimization program is characterized by the following proposition.

Proposition 5. *The best response $\hat{\psi}_i = \hat{\psi}_i(\psi_j) \in \mathcal{B}_i(\psi_j)$ of manager $i \neq j$ to a given $\psi_j \in \Psi_j^d$ is characterized by the following conditions:*

1. *If $\psi_{jt}' \xi < 2\gamma_j$ then $\hat{\psi}_{it} = 2\gamma_i \xi / \|\xi\|^2$.*
2. *If $\psi_{jt}' \xi \geq 2\gamma_j$ and ψ_{jt} is collinear to ξ then $\hat{\psi}_{it} = \gamma_i(\psi_{jt}/\gamma_j)$.*
3. *If $\psi_{jt}' \xi \geq 2\gamma_j$ and ψ_{jt} is not collinear to ξ then $(\hat{\psi}_{it}, \psi_{jt}) \notin \mathcal{C}_0$.*

6. Equilibrium

We now aggregate the results of the previous section to characterize the Pareto dominant equilibrium for the 3-player game.

Applying Propositions 3 and 4 successively to both managers we deduce that if an equilibrium exists it must be such that the fund volatilities belongs to

$$\mathcal{B} = \left\{ (\psi_1, \psi_2) \in \Psi_1^d \times \Psi_2^d : \psi_1 \in \mathcal{B}_1(\psi_2) \text{ and } \psi_2 \in \mathcal{B}_2(\psi_1) \right\}$$

where the sets $\mathcal{B}_1(\psi_2)$ and $\mathcal{B}_2(\psi_1)$ are defined as in Eq. (11). Using the definition of (λ_1, λ_2) in conjunction with footnote 7 and straightforward algebra it can be shown that $\mathcal{B}$ reduces to the set of fund volatilities such that (ψ_{1t}, ψ_{2t}) lies in $\mathcal{C}_0$ at all times. As a result, an equilibrium can only be such that the investor is indifferent between the funds in the sense that

$$\frac{\psi_{1t}^*}{\gamma_1} = \frac{\psi_{2t}^*}{\gamma_2}, \quad 0 \le t \le T,$$

where we assume that, in case the investor is indifferent between the funds, each manager hopes to obtain the entire share of the total amount invested in the funds. Now, the fixed points of the managers' reaction functions yield an infinite number of equilibrium paths given by:

$$\psi_{1t}^* = \frac{\gamma_1}{\gamma_2} c_t \xi, \tag{12}$$

$$\psi_{2t}^* = c_t \xi, \tag{13}$$

$$\pi_{1t}^* = \varepsilon \frac{\gamma_2}{\gamma_1} \left(\frac{c_t \|\xi\|^2 - \gamma_2}{\|c_t \xi\|^2} \right), \tag{14}$$

$$\pi_{2t}^* = (1 - \varepsilon) \left(\frac{c_t \|\xi\|^2 - \gamma_2}{\|c_t \xi\|^2} \right), \tag{15}$$

where $\varepsilon \in [0, 1]$ and $c_t \ge c^* = 2\gamma_2 \|\xi\|^{-2}$. Plugging these expressions into the players' objective functions, we obtain that the market values of the fees and the investor's expected utility are given by

$$\frac{\hat{R}_2(\varepsilon, c)}{(1 - \varepsilon)W_0} = \frac{\hat{R}_1(\varepsilon, c)}{\varepsilon W_0}$$
$$= 1 - \exp\left(-\int_0^T \gamma_2 \frac{c_t \|\xi\|^2 - \gamma_2}{2\|c_t \xi\|^2} dt \right), \tag{16}$$

$$U(\varepsilon, c) = \log\left(e^{rT} W_0 \right) + \int_0^T \frac{1}{2} \left(\frac{c_t \|\xi\|^2 - \gamma_2}{\|c_t \xi\|} \right)^2 dt. \tag{17}$$

Differentiating these expressions with respect to c_t shows that the investor's expected utility is strictly increasing while the present value of the fees to the managers are strictly decreasing. Since the managers act in order to maximize the present value of their fees, they will choose the smallest possible multiplier and it follows that the Pareto dominant equilibrium is obtained by setting $c_t = c^*$ in Eqs. (12)–(15). We summarize the above discussion by the following theorem.

Theorem 6. *There exists a unique dominant equilibrium for the Nash game between the managers. In this equilibrium, the fund volatilities are given by:*

$$\psi_{1t}^\flat = \frac{2\gamma_1}{\|\xi\|^2} \xi, \tag{18}$$

$$\psi_{2t}^\flat = \frac{2\gamma_2}{\|\xi\|^2} \xi. \tag{19}$$

The corresponding trading strategy of the investor is given by

$$\pi_{1t}^\flat = \varepsilon \frac{\|\xi\|^2}{4\gamma_1}, \tag{20}$$

$$\pi_{2t}^\flat = (1 - \varepsilon) \frac{\|\xi\|^2}{4\gamma_2}, \tag{21}$$

where ε is an arbitrary constant in $[0, 1]$. Finally, the investor's equilibrium expected utility is given by

$$U^\flat = \log\left(e^{rT} W_0 \right) + \frac{\|\xi\|^2}{8} T, \tag{22}$$

and the initial market value of the fees satisfy

$$\frac{\hat{R}_1^\flat}{\varepsilon} = \frac{\hat{R}_2^\flat}{1 - \varepsilon} = W_0 \left(1 - e^{-\frac{\|\xi\|^2}{4}T} \right) \tag{23}$$

where W_0 is the investor's strictly positive initial wealth.

Eqs. (18)–(19) show that in equilibrium the equity components of the funds are proportional to the fee rate. Thus, our model predicts that funds with higher management fees will invest more in equities and hence will have more volatile returns. On the other hand, the investor's holdings in a given fund are inversely proportional to its fee rate and, as a result, her effective equity portfolio weight

$$\left(\sigma' \right)^{-1} \left(\pi_{1t}^\flat \psi_{1t}^\flat + \pi_{2t}^\flat \psi_{2t}^\flat \right) = \frac{1}{2} \left(\sigma' \right)^{-1} \xi$$

is independent of the fee rates. Furthermore, multiplying Eqs. (20) and (21) by the corresponding fee rates and summing the results shows that the fraction of the investor's wealth that is being paid as management fees is also independent of the fee rates. Combining these properties yields that, in equilibrium, the investor's wealth process, the cumulative fees processes and the players' value functions are independent of the fee rate. As in Hugonnier and Kaniel (2010), the managers' ability to select the volatility of their fund and the investor's ability to adjust her holdings compensate each other so that, as shown by Eqs. (22)–(23), the players' welfare is independent of the fee rates in equilibrium.

A salient feature of the above result is that the volatilities of the funds divided by their respective fee rates are the same in equilibrium. As a result, the funds offer the same risk/return trade-off and it follows that the investor is indifferent between having access to one or more funds.[9] In particular, the total proportion of her wealth that the investor delegates to the funds is independent of the funds' characteristics and its allocation among the funds is arbitrary. This creates a role for marketing and corroborates the findings of recent empirical studies on the importance of media coverage and advertising in the mutual fund industry, see Barber et al. (2005), Gallaher et al. (2006), Hortaçsu and Syverson (2004) and Kaniel et al. (2007).

Finally, we note that the outcome of the Pareto dominant equilibrium is the same as that which would be reached if the managers decided to cooperate so as to maximize the present value of the total amount of fees paid by the investor.

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[9] A similar conclusion was reached independently by Cetin (2006) in a model with mean–variance preferences and a single risky asset.

7. Conclusion

In this paper we analyzed strategic competition between mutual funds in a dynamic setting where both the funds' portfolios and the fund flows are determined endogenously in equilibrium.

We first studied the investor's optimal reaction to the announced investment strategies of the mutual funds. We showed that the investor will choose to invest in both funds only in two cases: when both funds offer an equivalent risk/return trade-off, or when none dominates the other. In the second case, we showed that the interaction between the funds is such that, at any time, one of the fund managers can exclude the other from the market and thus increase the present value of the fees he collects, without changing the investor's welfare. As a result, we obtain that in equilibrium the funds offer the same risk/return trade-off. This implies that in equilibrium the investor is indifferent between the funds and would be equally well-off if there were only one of them. In particular, the investor's welfare and the market value of the total fees paid by the investor is the same in our model and in the single-fund model of Hugonnier and Kaniel (2010).

The model considered in this paper can be extended in various directions. First, it would be interesting to extend the framework in order to incorporate stochastic market coefficients. In this case, the optimal allocation of the mutual funds could be different and would probably incorporate a flow hedging component which might break the competition-irrelevance result stated above. Second, it would be natural to consider the case where mutual funds have access to different assets or face different investment constraints. In particular, it would be very interesting to study the case in which the funds are restricted from investing in the bond. Such a constraint would limit the ability of a fund to exclude the other from the investor's portfolio and is likely to change the competition-irrelevance result. We leave these challenging extensions to future research.

Appendix. Proofs

Proof of Proposition 2. Fix a pair $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$ of fund volatilities and let $\hat{\pi}$ be as in the statement. Furthermore, denote by $\hat{W}$ the wealth process associated with $(\hat{\pi}, \psi_1, \psi_2)$ and by W the wealth process associated with (π, ψ_1, ψ_2) for some arbitrary trading strategy $\pi \in \Pi(\psi)$.

Applying Itô's Lemma to Eq. (5) we obtain

$$d\left(\frac{W_t}{\hat{W}_t}\right) = \frac{1}{\hat{W}_t}dW_t + W_t d\left(\frac{1}{\hat{W}_t}\right) + d\left\langle W, \frac{1}{\hat{W}}\right\rangle_t$$
$$= \frac{W_t}{\hat{W}_t}\left(dY_t - dG_t - dH_t\right)$$

where Y is a stochastic integral with respect to the Brownian motion, hence a local martingale, and the processes (G, H) are defined by

$$G_t = \int_0^t \left(\pi_t' \psi_t \psi_t' \hat{\pi}_t - \pi_t' (\psi_t \xi - \gamma)\right) dt,$$

$$H_t = \int_0^t \left(\hat{\pi}_t' (\psi_t \xi - \gamma) - \|\psi_t' \hat{\pi}_t\|^2\right) dt.$$

We claim that dH_t is identically equal to zero while dG_t is non-negative. If $\hat{\pi}_t = 0$ then the result is obvious so assume that $\hat{\pi}_t \neq 0$. Three cases may arise depending on the fund volatilities. In region $\cup_{i=2}^4 \mathcal{C}_i$ it follows from footnote 7 that one of components of $\hat{\pi}_t$ is zero while the other equals λ_{it}. In region $\mathcal{C}_1$ the vector $\hat{\pi}_t$ equals the vector Λ_t and in region $\mathcal{C}_0$ both components of $\hat{\pi}_t$ are proportional to λ_{1t}. In all three cases the result follows upon inspection of G and H; we omit the details.

Since both W and $\hat{W}$ are non-negative, the above decomposition shows that their ratio is a non-negative local supermartingale and hence a true supermartingale by Fatou's lemma. In particular, we have

$$E\left[\frac{W_T}{\hat{W}_T}\right] \leq \frac{W_0}{\hat{W}_0} = 1.$$

Now, using this expression in conjunction with Jensen's inequality and the concavity of the investor's utility function we obtain that

$$U(\pi, \psi_1, \psi_2) - U(\hat{\pi}, \psi_1, \psi_2)$$
$$= E\left[\log \frac{W_T}{\hat{W}_T}\right] \leq \log\left(E\left[\frac{W_T}{\hat{W}_T}\right]\right) \leq 0$$

and the optimality of $\hat{\pi}$ will follow once we have shown that it satisfies Eq. (7). This is established in Lemma 8 below. $\square$

Proof of Proposition 3. Let $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$ be such that for some $u \in [0, T)$ and $\varepsilon > 0$ we have

$$A_k(\psi_{1t}, \psi_{2t}) = \frac{a_{kt}}{\|\psi_{jt}\|^2} > 0, \quad k \neq j \in \{1, 2\},$$

for all $t \in \mathscr{S} = [u, u + \varepsilon]$. This implies that $\psi_t \psi_t'$ is non-singular on $\mathscr{S}$ and it thus follows from Proposition 2 that the corresponding reaction function of the investor is given by

$$\hat{\pi}_t(\psi_1, \psi_2) = \frac{1}{\Delta_t(\psi_1, \psi_2)}\begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

where $\Delta_t(\psi_1, \psi_2)$ is the determinant of the matrix $\psi_t \psi_t'$. Now, let

$$\epsilon_t := \frac{\gamma_i a_{it}}{\gamma_i a_{it} + \gamma_j a_{jt}} \in (0, 1),$$

and consider the fund volatility ψ_i^ϵ defined by

$$\psi_{it}^\epsilon = \begin{cases} \epsilon_t \psi_{it} + (1 - \epsilon_t)\dfrac{\gamma_i}{\gamma_j}\psi_{jt}, & \text{if } t \in \mathscr{S}, \\ \psi_{it}, & \text{otherwise.} \end{cases}$$

For this fund volatility we have

$$A_i(\psi_{it}^\epsilon, \psi_{jt}) = \frac{\epsilon_t a_{it}}{\|\psi_{jt}\|^2} > 0$$

$$A_j(\psi_{it}^\epsilon, \psi_{jt}) = \frac{\epsilon_t}{\|\psi_{it}^\epsilon\|^2}\left(\epsilon_t a_{jt} - (1 - \epsilon_t)\frac{\gamma_i}{\gamma_j}a_{it}\right) = 0,$$

$$(\psi_{it}^\epsilon)' \xi - \gamma_i = \frac{\epsilon_t}{a_{it}}\frac{\|a_{1t}\psi_{1t} + a_{2t}\psi_{2t}\|^2}{\Delta_t(\psi_1, \psi_2)} > 0,$$

$$\Delta_t(\psi_i^\epsilon, \psi_j) = \epsilon_t^2 \Delta_t(\psi_1, \psi_2) \neq 0$$

for all $t \in \mathscr{S}$ while $A_{kt}(\psi_i^\epsilon, \psi_j) = A_{kt}(\psi_i, \psi_j)$ for $k \in \{1, 2\}$ and all $t \notin \mathscr{S}$. Therefore, if manager i uses the fund volatility ψ_i^ϵ, then Proposition 2 implies that the investor's reaction changes to

$$\hat{\pi}_{it}(\psi_i^\epsilon, \psi_j) = \frac{1}{\epsilon_t}\hat{\pi}_{it}(\psi_i, \psi_j),$$

$$\hat{\pi}_{jt}(\psi_i^\epsilon, \psi_j) = 0$$

on $\mathscr{S}$. Since $0 < \epsilon < 1$, this implies that on $\mathscr{S}$ the fund volatility ψ_i^ϵ produces a higher instantaneous fee to manager i than ψ_i. On the other hand,

$$\gamma' \hat{\pi}_t(\psi_1, \psi_2) = \frac{\gamma_1 a_{1t} + \gamma_2 a_{2t}}{\Delta_t(\psi_1, \psi_2)}$$
$$= \frac{\gamma_i a_{it}}{\epsilon_t \Delta_t(\psi_1, \psi_2)} = \gamma_i \hat{\pi}_{it}(\psi_i^\epsilon, \psi_j),$$

$$\hat{\pi}_{it}(\psi_i^\epsilon, \psi_j)\psi_i^\epsilon + \hat{\pi}_{jt}(\psi_i^\epsilon, \psi_j)\psi_{jt}$$

$$= \frac{a_{it}}{\Delta_t(\psi_1, \psi_2)} \left(\psi_{it} + \left(\frac{\gamma_i}{\gamma_j} \right) \left(\frac{1 - \epsilon_t}{\epsilon_t} \right) \psi_{jt} \right)$$

$$= \frac{a_{it} \psi_{it} + a_{jt} \psi_{jt}}{\Delta_t(\psi_1, \psi_2)}$$

$$= \hat{\pi}_{it}(\psi_i, \psi_j)\psi_{it} + \hat{\pi}_{jt}(\psi_i, \psi_j)\psi_{jt},$$

so that the total fees paid by the investor and the volatility of her wealth are the same under both (ψ_i, ψ_j) and $(\psi_i^\epsilon, \psi_j)$. In conjunction with Eq. (9), this implies that the investor's wealth process is the same under both strategies and it follows that we have

$$R_i\left(\psi_i^\epsilon, \psi_j \right) - R_i\left(\psi_i, \psi_j \right)$$

$$= E_Q \left[\int_{\mathcal{S}} \gamma_i e^{-rt} W_t(\psi_i, \psi_j) \left(\hat{\pi}_{it}^\epsilon - \hat{\pi}_{it} \right) dt \right]$$

$$= E_Q \left[\int_{\mathcal{S}} \gamma_i \left(\frac{1}{\epsilon_t} - 1 \right) e^{-rt} W_t(\psi_i, \psi_j) \hat{\pi}_{it}(\psi_i, \psi_j) dt \right] > 0$$

which completes the proof. $\quad\square$

Corollary 7. *For any $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$ the non-negative process $\hat{\pi}_i(\psi_i, \psi_j)$ is uniformly bounded from above by $\eta_i = \|\xi\|^2 / 4\gamma_i$.*

Proof. Combining the proof of the previous proposition with the definition of $\hat{\pi}$ we have that for any (ψ_i, ψ_j) there exists an x such that

$$\hat{\pi}_{it}(\psi_i, \psi_j) \le \hat{\pi}_{it}(x, \psi_j) = \frac{x'\xi - \gamma_i}{\|\psi\|^2} = f_i(x).$$

The desired result now follows by observing that the global maximizer of the function f_i is given by $x_i^* = 2\gamma_i\xi/\|\xi\|^2$ with $f_i(x_i^*) = \eta_i$, see Lemma 10 below for details. $\quad\square$

Lemma 8. *The portfolio process $\hat{\pi} = \hat{\pi}(\psi_1, \psi_2)$ belongs to the set $\Pi(\psi_1, \psi_2)$ for any $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$.*

Proof. Let $\hat{W} \ge 0$ be the wealth process associated with $(\hat{\pi}, \psi_1, \psi_2)$. Using Girsanov's theorem in conjunction with the non-negativity of $\hat{\pi}$ and Fatou's lemma we deduce that $e^{-rt}\hat{W}_t$ is a supermartingale under Q. Combining this property with Fubini's theorem and Corollary 7 we obtain

$$E_Q \left[\int_0^T \gamma' \hat{\pi}_t e^{-rt} \hat{W}_t dt \right] \le (\gamma_1\eta_1 + \gamma_2\eta_2) \int_0^T E_Q \left[e^{-rt} \hat{W}_t \right] dt$$

$$= \frac{\|\xi\|^2}{2} \int_0^T E_Q \left[e^{-rt} \hat{W}_t \right] dt \le \frac{\|\xi\|^2}{2} W_0 T$$

and the proof is complete. $\quad\square$

Lemma 9. *Fix a pair $(\psi_i, \psi_j) \in \Psi_i \times \Psi_j$ of fund volatilities such that $\psi_i \in \Psi_i^*(\psi_j)$. Then the process*

$$\hat{M}_t = \hat{M}_t(\psi_i, \psi_j) = e^{\int_0^t \gamma' \hat{\pi}_s(\psi_i, \psi_j) ds} \frac{\hat{W}_t(\psi_i, \psi_j)}{W_0 S_t^0}$$

is a strictly positive, uniformly integrable martingale under the risk-neutral probability measure.

Proof. Using Girsanov's theorem in conjunction with Eq. (3) we have that the process $Z_t = B_t + \xi t$ is a standard Brownian motion under the risk-neutral probability measure. Using this property in conjunction with Eq. (5) and Itô's lemma we obtain

$$d\hat{M}_t = \hat{M}_t \hat{\pi}_t(\psi_i, \psi_j)\psi_t dZ_t$$

and it follows that $\hat{M}$ is a non-negative local martingale under the risk-neutral measure. On the other hand, since $(\psi_{1t}, \psi_{2t}) \notin \mathcal{C}_1$ it

follows from Proposition 2 that

$$0 \le \hat{\pi}_{it}(\psi_i, \psi_j) \le \frac{(\psi_{it}'\xi - \gamma_i)^+}{\|\psi_{it}\|^2}.$$

Using this property in conjunction with the Cauchy–Schwartz inequality and the fact that $\|x + y\|^2 \le 4\|x\|^2 + 4\|y\|^2$ we obtain

$$\left\| \psi_t' \hat{\pi}_t(\psi_i, \psi_j) \right\|^2 = \left\| \hat{\pi}_{it}(\psi_i, \psi_j)\psi_{it} + \hat{\pi}_{jt}(\psi_i, \psi_j)\psi_{jt} \right\|^2$$

$$\le \sum_{k=1}^2 4\|\hat{\pi}_{kt}\psi_{kt}\|^2 \le \sum_{k=1}^2 4 \left(\frac{(\psi_{kt}'\xi - \gamma_k)^+}{\|\psi_{kt}\|} \right)^2$$

$$\le \sum_{k=1}^2 4 \left(\frac{\psi_{kt}'\xi}{\|\psi_{kt}\|} \right)^2 \le \sum_{k=1}^2 4 \frac{\|\psi_{kt}\|^2 \|\xi\|^2}{\|\psi_{kt}\|^2} = 8\|\xi\|^2.$$

This shows that the volatility of the local martingale $\hat{M}$ is uniformly bounded and hence satisfies Novikov's condition. $\quad\square$

Proof of Proposition 4. Let $(\psi_1, \psi_2) \in \Psi_1^d \times \Psi_2^d$ be such that for some $u \in [0, T)$ and $\varepsilon > 0$ we have

$$\hat{\pi}_{it}(\psi_1, \psi_2) = 0$$

for all $t \in \mathcal{S} = [u, u + \varepsilon]$. According to Proposition 2, this implies that λ_{jt} is non-negative and combining this property with the Cauchy–Schwartz inequality we conclude that

$$b_t = \begin{cases} \dfrac{\gamma_i}{2\gamma_j\lambda_{jt}\|\xi\|^2} \left[\|\xi\| + \sqrt{\|\xi\|^2 - 4\gamma_j\lambda_{jt}} \right], & \text{if } \lambda_{jt} > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is well defined and non-negative for all $t \in \mathcal{S}$. Now define a fund volatility by setting

$$\psi_{it}^\upsilon = \begin{cases} \upsilon_t\xi, & \text{if } t \in \mathcal{S}, \\ \psi_{it}, & \text{otherwise,} \end{cases}$$

for some arbitrary υ such that

$$\upsilon_t \ge \max \left\{ b_t, \frac{\gamma_i}{\|\xi\|^2}, \frac{\gamma_i}{\gamma_j} \frac{\psi_{jt}'\xi}{\|\xi\|^2} \right\} > 0.$$

Using these definitions we readily deduce that

$$\left(\psi_{it}^\upsilon \right)' \xi - \gamma_i = \upsilon_t\|\xi\|^2 - \gamma_i > 0,$$

$$A_j\left(\psi_{it}^\upsilon, \psi_{jt} \right) = \frac{\gamma_i\psi_{jt}'\xi - \gamma_j\upsilon_t\|\xi\|^2}{\upsilon_t\|\xi\|^2} < 0,$$

hold for all $t \in \mathcal{S}$ so that, according to Proposition 2, $\hat{\pi}_{it}(\psi_i^\upsilon, \psi_j) > 0$ and $\hat{\pi}_{jt}(\psi_i^\upsilon, \psi_j) = 0$ on the set $\mathcal{S}$. As a consequence, the instantaneous fees compare as follows on $\mathcal{S}$

$$\gamma' \left[\hat{\pi}_t(\psi_i, \psi_j) - \hat{\pi}_t(\psi_i^\upsilon, \psi_j) \right]$$

$$= \gamma_j \left(\frac{\psi_{jt}\xi - \gamma_j}{\|\psi_{jt}\|^2} \right) - \gamma_i \left(\frac{\upsilon_t\|\xi\|^2 - \gamma_i}{\upsilon_t^2\|\xi\|^2} \right)$$

$$= \frac{1}{(\upsilon_t\|\xi\|)^2} \left(\gamma_j\upsilon_t^2\lambda_{jt} - \gamma_i\upsilon_t\|\xi\|^2 + \gamma_i^2 \right) \ge 0$$

where the last equality follows from the fact that $\upsilon_t \ge b_t$. This implies that the state variable x is larger under the pair of fund volatilities $(\psi_i^\upsilon, \psi_j)$. Combining this with the definition of the objective function yields

$$J_i(\psi_i^\upsilon, \psi_j) - J_i(\psi_i, \psi_j) = \int_{\mathcal{S}} \gamma_i x_t(\psi_{it}^\upsilon, \psi_{jt})\hat{\pi}_{it}(\psi_{it}^\upsilon, \psi_{jt}) dt$$

$$+ \int_{u+\varepsilon}^T \gamma_i(x_t(\psi_{it}^\upsilon, \psi_{jt}) - x_t(\psi_{it}, \psi_{jt}))\hat{\pi}_{it}(\psi_{it}, \psi_{jt}) dt \ge 0$$

and the proof is complete. $\quad\square$

Lemma 10. *The function* $f_i(x) = (x'\xi - \gamma_i)/\|x\|^2$ *is strictly quasi-concave on* $D_i = \{x \in \mathbb{R}^n : x'\xi > \gamma_i\}$ *and admits a maximum at* $x_i^* = 2\gamma_i\xi/\|\xi\|^2$

Proof. Consider $(x, z) \in D_i^2$ such that $f_i(z) \geq f_i(x)$ or equivalently,

$$\|x\|^2 (z'\xi - \gamma_i) \geq \|z\|^2 (x'\xi - \gamma_i).$$

Then we have

$$\nabla f_i(x) \cdot (z - x) = \frac{\|x\|^2 \xi' - 2(x'\xi - \gamma_i)x'}{\|x\|^4} \cdot (z - x)$$

$$= \frac{\|x\|^2 (z'\xi - \gamma_i) + \|x\|^2 (x'\xi - \gamma_i)}{\|x\|^4} - \frac{2(x'z)(x'\xi - \gamma_i)}{\|x\|^4}$$

$$\geq \frac{\|z\|^2 (x'\xi - \gamma_i) + \|x\|^2 (x'\xi - \gamma_i)}{\|x\|^4} - \frac{2(x'z)(x'\xi - \gamma_i)}{\|x\|^4}$$

$$= \frac{(x'\xi - \gamma_i)}{\|x\|^2} (\|z\|^2 + \|x\|^2 - 2(x'z))$$

$$= \frac{(x'\xi - \gamma_i)\|z - x\|^2}{\|x\|^2} > 0$$

which proves the strict quasiconcavity of f_i. As is easily seen we have that $x_i^* \in D_i$ satisfies $\nabla f_i(x_i^*) = 0$ and the desired result now follows from the first part of the proof. □

Proof of Proposition 5. Fix an arbitrary $\psi_j \in \Psi_j^d$. Combining the results of Propositions 3 and 4 we have that the manager's best response can be restricted to the set $\mathcal{B}_i(\psi_j)$ of Eq. (11). Now, for an arbitrary fund volatility ψ_i in this set we have from Proposition 2 that

$$\hat{\pi}_{it}(\psi_i, \psi_j) = \varepsilon f_i(\psi_{it}) = \varepsilon \frac{\psi_{it}'\xi - \gamma_i}{\|\psi_{it}\|^2},$$

$$\hat{\pi}_{jt}(\psi_i, \psi_j) = (1 - \varepsilon)\frac{\gamma_i}{\gamma_j} f_i(\psi_{it}) = (1 - \varepsilon)\frac{\gamma_i}{\gamma_j}\frac{\psi_{it}'\xi - \gamma_i}{\|\psi_{it}\|^2},$$

$$\gamma'\hat{\pi}_t(\psi_i, \psi_j) = \gamma_i f_i(\psi_{it}) = \gamma_i \frac{\psi_{it}'\xi - \gamma_i}{\|\psi_{it}\|^2}$$

for some constant $\varepsilon \in (0, 1]$ where the function f_i is defined as in Lemma 10. As a result, the objective function of manager i is given by

$$J_i(\psi_i, \psi_j) = \varepsilon(1 - x_T(\psi_i, \psi_j))$$

$$= \varepsilon\left(1 - e^{-\gamma_i \int_0^T f_i(\psi_{it})dt}\right)$$

and it follows that his best response can be obtained by maximizing the function $f_i(p)$ on the set

$$\mathcal{X}_{it}(\psi_{jt}) = \left\{p \in \mathbb{R}^n : p'\xi > \gamma_i \text{ and } A_{jt}(p, \psi_{jt}) \leq 0\right\} \subseteq D_i$$

for each $t \in [0, T]$. Lemma 10 implies that f_i is strictly quasiconcave on $\mathcal{X}_{it}(\psi_{jt})$ and it follows that the Kuhn–Tucker conditions are both necessary and sufficient to characterize its maximum. We now prove the three assertions of the statement.

Assertion 1. If $\psi_{jt}'\xi < 2\gamma_j$ then it is easy to check that the unconstrained maximum given by Lemma 10 belongs to the set $\mathcal{X}_{it}(\psi_{jt})$.

Assertion 2. If $\psi_{jt} = c_t\xi$ for some $c_t \geq 2\gamma_j/\|\xi\|^2$ then the Kuhn–Tucker conditions

$$\nabla f_i(p) = -\mu \frac{\partial}{\partial p} A_j(p, \psi_{jt})$$

$$= \mu\left(-c_t\xi f_i(p) - c_t p'\xi \nabla f_i(p)\right),$$

$$0 = A_j(p, \psi_{jt})$$

$$= (c_t\|\xi\|^2 - \gamma_j) - c_t \frac{p'\xi}{\|p\|^2}(p'\xi - \gamma_i),$$

hold for $p = \gamma_i(\psi_{jt}/\gamma_j)$ and the multiplier $\mu = (\psi_{jt}'\xi - 2\gamma_j)/(\gamma_i\|\psi_{jt}\|^2)$. Using the definition of c_t allows to check that this solution lies in the set $\mathcal{X}_{it}(\psi_{jt})$.

Assertion 3. Assume that $\psi_{jt}'\xi \geq \gamma_j$ and ψ_{jt} is not collinear with ξ. If the best response of manager i is such that $(\psi_{it}, \psi_{jt}) \in \mathcal{C}_0$ then it must be the case that $\psi_{it} = \gamma_i(\psi_{jt}/\gamma_j)$ and plugging this into the Kuhn–Tucker conditions we obtain

$$\left(\frac{1}{\|\psi_{jt}\|^2} + \mu\right)\xi$$

$$= \psi_{jt}(\psi_{jt}'\xi - \gamma_j)\left[\frac{2\left(\frac{\gamma_j}{\gamma_i}\right)^3}{\|\psi_{jt}\|^4} + \frac{\mu\left(\frac{\gamma_j}{\gamma_i}\right)^2}{\|\psi_{jt}\|^2}\right]$$

for some Lagrange multiplier μ. This implies that the fund volatility ψ_{jt} is collinear with ξ and yields the desired contradiction.

Proof of Proposition 6. Among the available equilibria, the managers will choose the one allowing them to extract the highest initial value of fees, which corresponds to choosing $c_t = c^*$ at all times. Applying this value to Eqs. (12)–(15) and (16)–(17) yields the result. □

References

Barber, B., Odean, T., & Zheng, L. (2005). Out of sight, out of mind: the effect of expenses on mutual fund flows. *Journal of Business*, 78, 2095–2120.

Basak, S., Pavlova, A., & Shapiro, A. (2007). Optimal asset allocation and risk shifting in money management. *Review of Financial Studies*, 20, 1683–1721.

Boudoukh, , Richardson, M., Stanton, R., & Whitelaw, R. (2004). The economics of asset management. *Working paper*. New York University.

Carpenter, J. (2000). The optimal dynamic investment policy for a fund manager compensated with an incentive fee. *Journal of Finance*, 55, 2311–2331.

Carpenter, J., Dybvig, P., & Farnsworth, H. (2010). Portfolio performance and agency. *The Review of Financial Studies*, 23, 1–23.

Cetin, C. (2006). Delegated dynamic portfolio management under mean–variance preferences. *Journal of Applied Mathematics and Decision Sciences*, 1–22.

Cuoco, D., & Kaniel, R. (2001). Equilibrium prices in the presence of delegated Portfolio management. *Working paper*. Duke University.

Das, S., & Sundaram, R. (2002). On the regulation of mutual fund fee structures. In M. Avellaneda (Ed.)., *Quantitative analysis in financial markets, Volume III*. World Scientific.

Das, S., & Sundaram, R. (2002). Fee speech: signaling, risk-sharing and the impact of fee structures on investor welfare. *Review of Financial Studies*, 15, 1465–1497.

Gallaher, S., Kaniel, R., & Starks, L. (2006). Madison avenue meets wall street: mutual fund families, competition and advertising. Working Paper. Duke University.

Goetzmann, W., Ingersoll, J., & Ross, S. (2003). High water marks and hedge fund management contracts. *Journal of Finance*, 58, 1685–1717.

Golec, J. (1999). Regulation and the rise in asset-based mutual fund management fees. *Journal of Financial Research*, 26, 19–30.

Golec, J., & Starks, L. (2004). Performance fee contract change and mutual fund risk. *Journal of Financial Economics*, 73, 93–118.

Grinblatt, M., & Titman, S. (1989). Adverse risk incentives and the design of performance based contracts. *Management Science*, 35, 807–822.

Gruber, M. (1996). Another puzzle: the growth in actively managed mutual funds. *Journal of Finance*, 51, 783–810.

Hortaçsu, A., & Syverson, C. (2004). Product differentiation, search costs and competition in the mutual fund industry: a case study of S&P 500 index funds. *Quarterly Journal of Economics*, 119, 403–456.

Hugonnier, J., & Kaniel, R. (2010). Mutual fund portfolio choice in the presence of dynamic flows. *Mathematical Finance*, 20(2), 187–227.

Jain, P., & Wu, J. (2000). Truth in mutual fund advertising: evidence on future performance and fund flows. *Journal of Finance*, 55, 937–958.

Kaniel, R., Starks, L., & Vasudevan, V. (2007). Headlines and bottom lines: attention and learning effects from media coverage of mutual funds. Working Paper. Duke University.

Lynch, A., & Musto, D. (1997). Understanding fee structures in the asset management business. Working Paper. New York University.

Massa, M. (1998). Why so many mutual funds? Mutual funds families, market segmentation and financial performance. Working Paper. INSEAD.

Ou-Yang, H. (2003). Optimal contracts in a continuous-time principal-agent problem. *Review of Financial Studies*, *16*, 173–208.

Roll, R. (1992). A mean–variance analysis of tracking error. *Journal of Portfolio Management*, 13–22.

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