ENDOGENOUS COMPLETENESS OF DIFFUSION DRIVEN EQUILIBRIUM MARKETS

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We study the existence of dynamic equilibria with endogenously complete markets in continuous-time, heterogenous agents economies driven by diffusion processes. Our main results show that under appropriate conditions on the transition density of the state variables, market completeness can be deduced from the primitives of the economy. In particular, we prove that a sufficient condition for market completeness is that the volatility of dividends be invertible and provide higher order conditions that apply when this condition fails as is the case in the presence of fixed income securities. In contrast to previous research, our formulation does not require that securities pay terminal dividends, and thus allows for both finite and infinite horizon economies.

KEYWORDS: Continuous-time asset pricing, dynamic market completeness, general equilibrium theory.

1. INTRODUCTION

EVER SINCE THE SEMINAL CONTRIBUTIONS of Kreps (1982), Duffie and Huang (1985), and Duffie (1986), the standard way to construct securities market equilibria in continuous-time economies with heterogenous agents has consisted of three steps. First, compute an Arrow–Debreu equilibrium. Second, define candidate prices for the traded risky securities by using the associated consumption price process as a pricing kernel and, third, verify that these prices give rise to dynamically complete markets. The last step in this program is crucial in establishing the existence of an equilibrium. Otherwise one cannot guarantee that the allocation of the Arrow–Debreu equilibrium can be implemented by dynamic trading in the given set of securities. This last step is also the most difficult one, since the candidate prices are given by conditional expectations which can rarely be computed explicitly.

In representative agent economies, market completeness does not matter for the existence of an equilibrium, but it is nonetheless important for two reasons. First, the microeconomic justification for such economies relies on aggregation results which require complete markets; see Constantinides (1982). Second, it is now quite common in asset pricing to start from a representative agent economy and then use the resulting equilibrium pricing kernel outside the model to price securities, such as derivatives, that were not included in the original menu of traded assets. Such an approach requires complete markets, since only in that case does the derived price give the amount necessary to replicate the cash flows by trading in the primitive securities.

Despite its importance, the question of endogenous completeness has not received much attention in the general equilibrium literature. In fact, most of the papers that study multiasset equilibrium models assume in one form or another that markets are complete, but do not actually prove it. A notable exception is Anderson and Raimondo (2008), who assumed that (i) the economy has a finite horizon and all risky securities pay dividends at the terminal time and (ii) the state variables are given by Brownian motions; they proved that the candidate prices generate complete markets as soon as the volatility matrix of the terminal dividends is nondegenerate.

Being the first of its kind, the result of Anderson and Raimondo (2008) is obviously very important. However, their assumptions are often too strong to be applicable in practice. In particular, (i) requires that all traded assets pay terminal dividends and hence does not allow for securities that pay only continuous dividends as is customary in the literature. Furthermore, this assumption implies that the menu of traded securities cannot include an instantaneously risk-free savings account.[2] Another obvious, but nonetheless important, limitation of (i) is that it does not allow for infinite horizon economies. While (ii) is satisfied in the benchmark case where dividends are modeled as correlated geometric Brownian motions, Anderson and Raimondo (2008) themselves remarked that this assumption is quite restrictive. In particular, this assumption does not allow for mean reversion in the state variables and thus excludes all of the standard equilibrium term structure models that assume mean reverting affine state variables (see, e.g., Vasicek (1977) and Cox, Ingersoll, and Ross (1985)).

In this paper, we extend the result of Anderson and Raimondo (2008) by removing both of their key assumptions. Specifically, we provide conditions for endogenous completeness in a continuous-time economy populated by heterogenous agents and driven by a multidimensional diffusion process that satisfies appropriate regularity conditions. In our formulation, the traded securities do not need to pay terminal dividends. As a result, the horizon of the economy can be either finite or infinite and we can include instantaneously risk-free bonds in the menu of traded assets as is customary in the asset pricing literature. In this setting, the main results of this paper show that dynamic market completeness can be deduced from the primitives of the economy in most standard continuous-time equilibrium models.

To highlight the intuition behind our results, consider a finite horizon economy and recall that, in continuous-time, market completeness is equivalent to the invertibility of the price volatility. Using a first order expansion, we show that this matrix is invertible in a neighborhood of the terminal time provided

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[2]This restriction is not in itself unnatural, as one could use a zero coupon bond in zero net supply in place of an instantaneously risk-free bond and then change the numéraire to obtain a riskless asset. However, most continuous-time models use a savings account as primitive security and it is therefore important to find conditions for completeness in such economies.

that the exogenous volatility of dividends is invertible and, given this, the question becomes that of knowing whether we may propagate this property to the whole time interval. As observed by Anderson and Raimondo (2008), who were the first researchers to use it in this context, the notion of real analyticity is uniquely suited to answer this question because a real analytic function is either identically equal to zero or almost everywhere different from zero.[3] Under our assumptions, the determinant of the price volatility is indeed real analytic as a function of time and the state variables, and combining this property with our expansion shows that markets are endogenously complete as soon as the exogenous volatility of dividends is invertible at least at one point of the state space.

The requirement that the volatility of the dividends be nondegenerate is sufficient for endogenous completeness, but it is not necessary. In particular, if some of the traded assets are fixed income securities, such as bonds or annuities, then this requirement fails, but markets may nonetheless be complete in equilibrium. To obtain sufficient conditions for market completeness in such cases, it is necessary to expand the price volatility to higher orders and we provide complete details for the second order expansion. Since some of the securities now draw their value solely from variations in the pricing kernel, the second order condition that we obtain depends not only on the dividends, but also on the agents' preferences and endowments through the equilibrium pricing kernel and might thus be difficult to apply. To circumvent this difficulty, we show that, surprisingly, the validity of the condition for one arbitrary agent in the economy is sufficient to guarantee the existence of equilibrium for a generic set of initial endowments.

In an infinite horizon economy there is no terminal time close to which the volatility of the candidate prices can be approximated. Instead, we expand the volatility of prices as a function of the agents' common discount rate and show that its determinant can be computed from the primitives of the economy in a neighborhood of infinity. Relying on this expansion, we show that an equilibrium exists as soon as the dividend volatility is invertible and provide a second order condition that applies when dividends are degenerate. In contrast to the finite horizon case, the existence result that we obtain holds for generic, rather than fixed, initial endowments and discount rates. The reason for this is that by varying the agents' discount rate, we are changing the initial distribution of wealth in the economy.

The rest of the paper is organized as follows. In Section 2, we present the model, state our assumptions, and recall some basic results about Arrow–Debreu equilibria. Section 3 contains our main conditions for endogenous market completeness in finite or infinite horizon economies. In Section 4, we

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[3]The observation that real analyticity is essential to prove dynamic completeness is also contained in Herzberg and Riedel (2010), who considered the same setting with terminal dividends as Anderson and Raimondo (2008), but allowed for diffusion state variables.

provide examples of models in which our assumptions on the primitives are satisfied. The proofs of our most important results are provided in the Appendix. More standard proofs as well as additional results are presented in the Supplemental Material (Hugonnier, Malamud, and Trubowitz (2012)).

2. THE ECONOMY

Information Structure: We consider a continuous-time economy on the time span $[0, T]$ for some horizon T that can be either finite or infinite. Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a Brownian motion $Z \in \mathbb{R}^d$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the usual augmentation of the filtration generated by the Brownian motion and we let $\mathcal{F} \equiv \mathcal{F}_T$.

Securities Markets: The financial market is frictionless and consists of $1 + d$ continuously traded securities: one locally riskless savings account in zero net supply and d dividend-paying stocks in positive supply of unit each.[4] The savings account pays no dividends and earns an endogenously determined rate of interest on deposits.[5] On the other hand, we assume that stock i pays dividends at rate[6] $g_i(X_t)$ for some nonnegative real analytic function g_i, where $X_t \in \mathbb{R}^n$ is a vector of state variables that evolves according to

$$(1) \qquad X_t = X_0 + \int_0^t \mu_X(\tau, X_\tau) \, d\tau + \int_0^t \sigma_X(\tau, X_\tau) \, dZ_\tau$$

for some $X_0 \in \mathbb{R}^n$, and some functions μ_X and σ_X with values in $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$. The key conditions we impose on the state variables are summarized in the following assumption.

ASSUMPTION A:
 (a) $n = d$ and $\mathrm{rank}(\sigma_X(t, x)) = d$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.
 (b) *The functions μ_X and σ_X are jointly real analytic in $(t, x) \in [0, T] \times \mathbb{R}^n$ and are time-independent if the economy has an infinite horizon.*
 (c) *The unique solution to equation* (1) *takes values in $\mathcal{X} \subseteq \mathbb{R}^d$ and admits a transition density $p(t, x, \tau, y)$ that is smooth for $t \neq \tau$.*

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[4]The market structure that we consider is standard in continuous-time asset pricing; see, for example, Duffie (2001, Chapter 9). While our analysis can be easily adapted to the setting of Anderson and Raimondo (2008), where none of the securities is locally riskless, we choose to focus on the standard formulation to facilitate the application of our results. The results for this alternative setting are similar to those we present and are available on request.

[5]One should think of the savings account as a series of instantaneous risk-free bonds. If an agent invests a at time t, then his investment grows to $a(1 + r_t \, dt)$ by time $t + dt$ and this amount is available for either consumption or reinvestment over the next infinitesimal time interval.

[6]The assumption that the stocks do not pay terminal dividends is adopted for simplicity and can be relaxed at the cost of more involved notation. See the Supplemental Material for details.

(d) *There are locally bounded functions (K, L), a metric d that is locally equivalent to the Euclidean metric, and constants ε, α, $\phi > 0$ such that $p(t, x, \tau, y)$ is analytic with respect to $t \neq \tau$ in the set*

$$\mathcal{P}_\varepsilon^2 \equiv \left\{ (t, \tau) \in \mathbb{C}^2 : \Re t \geq 0, 0 \leq \Re \tau \leq T, \text{ and } |\Im(\tau - t)| \leq \varepsilon \Re(\tau - t) \right\}$$

and satisfies

$$(2) \qquad |p(t, x, \tau, y)| \leq K(x) L(y) |\tau - t|^{-\alpha} e^{\phi|\tau - t| - d(x,y)^2/|\tau - t|} \equiv \overline{p}(t, x, \tau, y)$$

for all $(t, \tau, x, y) \in \mathcal{P}_\varepsilon^2 \times \mathcal{X}^2$.

The most important part in Assumption A is condition (d). This condition is meant to guarantee that the candidate equilibrium prices to be constructed below are real analytic functions of time and can be shown to hold in many different models. See Section 4 for various examples.

Preferences and Endowments: The economy is populated by $A \geq 1$ agents indexed by a. The preferences of agent a over lifetime consumption plans are represented by an expected utility index of the form $U_a(c) \equiv E_0 \int_0^T e^{-\rho \tau} u_a(c_\tau) \, d\tau$, where the constant $\rho \geq 0$ is a discount rate that is common to all agents and u_a is a utility function that is assumed to satisfy the following assumption.

ASSUMPTION B: *The function $u_a : (0, \infty) \to \mathbb{R}$ is real analytic, increasing, and strictly concave, and satisfies the Inada conditions $u_a'(0) = \infty$ and $u_a'(\infty) = 0$.*

Agent a is endowed with $\eta_{ai} \in [0, 1]$ units of stock i and receives income at rate $\ell_a(X_t)$ for some real analytic function $\ell_a : \mathcal{X} \to \mathbb{R}_+$. We let $\eta \in \mathbb{R}^{A \times d}$ denote the matrix of initial endowments and assume that $\eta^\top \mathbf{1}_A = \mathbf{1}_d$ so that markets clear.

Trading Strategies and Feasible Plans: A trading strategy is a predictable process $(\alpha, \pi) \in \mathbb{R}^{1+d}$, where α_t and π_{it} denote the number of units of the riskless asset and the number of units of stock i held in the portfolio at time t.

A consumption plan is an adapted process c that is almost surely locally integrable with respect to Lebesgue measure on $[0, T]$. A trading strategy (α, π) is said to finance the consumption plan c at cost w if the associated wealth process $W_t \equiv \alpha_t B_t + \pi_t^\top S_t$ satisfies the dynamic budget constraint

$$W_t = w + \int_0^t \alpha_\tau \, dB_\tau + \int_0^t \pi_\tau^\top \, d(S + D)_\tau - \int_0^t c_\tau \, d\tau,$$

where D and (B, S) denote, respectively, the vector of cumulative dividends and the endogenous securities price processes.

A consumption plan c is feasible for agent a given a consumption price process m if there exists a trading strategy that finances the net plan $c - \ell_a$ at an initial cost of $w_a \equiv \eta_a^\top S_0$ and is such that the process

$$m_t W_t + \int_0^t m_\tau (c_\tau - \ell_a(X_\tau)) \, d\tau$$

is a martingale with $W_T \geq 0$ if the horizon is finite and $\liminf_{t \to T} E[m_t W_t] \geq 0$ otherwise. The martingale property is a standard admissibility condition that excludes doubling strategies from the feasible set (see, e.g., Duffie (2001, Chapter 6)). On the other hand, the requirement on the behavior of wealth as the horizon approaches is meant to prevent agents from borrowing without ever paying back. Indeed, agents in the model are allowed to borrow against their future labor income and may therefore have negative wealth over some periods of time, but these interim debts must be repaid before the horizon of the model.

In what follows, we let $\mathcal{C}_a(m, B, S)$ denote the set of consumption plans that are feasible for agent a given the consumption and securities prices m, B, and S.

Equilibrium: The concept of equilibrium that we use is that of equilibrium of plans, prices, and expectations introduced by Radner (1972):

DEFINITION 1: An equilibrium is a set of price processes (m, B, S), a consumption allocation $(c_a)_{a=1}^A$, and a set of strategies $(\alpha_a, \pi_a)_{a=1}^A$ such that the following statements hold:
    (a) The plan c_a maximizes U_a over $\mathcal{C}_a(m, B, S)$ and is financed by (α_a, π_a).
    (b) All markets clear.
An equilibrium with consumption price m has dynamically complete markets if any plan c such that $w_c \equiv E \int_0^T m_\tau c_\tau \, d\tau < \infty$ can be financed at cost w_c.

The rest of the paper is devoted to finding conditions under which there exists an equilibrium with dynamically complete markets. The starting point of our analysis is a static Arrow–Debreu equilibrium defined as a consumption price process m and a consumption allocation $(c_a)_{a=1}^A$ such that c_a maximizes U_a over the set of consumption plans which satisfy the static budget constraint

$$E \int_0^T m_\tau (c_\tau - \ell_a(X_\tau) - \eta_a^\top g(X_\tau)) \, d\tau \leq 0,$$

and the consumption good market clears. To guarantee that such an equilibrium exists, we impose the following assumption.[7]

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[7]Alternative sets of sufficient conditions for the existence of an Arrow–Debreu equilibrium in a setting similar to ours can be found in Dana (1993).

ASSUMPTION C: *There are constants $R \leq \rho$ and $\nu > 1$ such that*

$$\int_0^T \sum_{a=1}^A \left(\int_{\mathcal{X}} e^{-R\tau} u_a'(\overline{g}(y)/A)\overline{g}(y)\overline{p}(0, x, \nu\tau, y)\, dy \right) d\tau < \infty$$

for all $x \in \mathcal{X}$, where $\overline{p}$ is defined as in equation (2) *and $\overline{g} \equiv g^\top \mathbf{1}_d + \ell^\top \mathbf{1}_A$ denotes the aggregate consumption.*

Our first result establishes the existence of an Arrow–Debreu equilibrium and characterizes the corresponding consumption price process:

PROPOSITION 1: *The set of Arrow–Debreu equilibria is nonempty. In any such equilibrium, the consumption price process is given by*

$$m_t = m(t, X_t) \equiv e^{-\rho t} \frac{\partial u}{\partial c}(\lambda, \overline{g}(X_t))$$

for some $\lambda \in \mathcal{S}_+$, where $u(\lambda, c) = \max_{s \in \mathcal{S}} \sum_{a=1}^A \lambda_a u_a(s_a c)$ and $\mathcal{S} \subseteq \mathbb{R}^A$ denotes the unit simplex. In particular, the equilibrium consumption price function is jointly real analytic in $(t, \lambda, x) \in (0, T) \times \mathcal{S}_{++} \times \mathcal{X}$.

REMARK 1: We focus on a formulation with homogenous discount rates and time-independent aggregate consumption because it covers most of the cases of interest and allows us to guarantee that equilibrium prices are real analytic under simple conditions. While it is possible to find conditions on the primitives of the model under which that property holds with heterogenous discount rates and/or time-dependent aggregate consumption, these conditions are a lot more involved and become very difficult to interpret. See Appendix 2 for details.

3. ENDOGENOUS COMPLETENESS

To find conditions under which our economy admits an equilibrium with dynamically complete markets, we follow the path set by Kreps (1982), Duffie and Huang (1985), Duffie (1986), and Huang (1987). Namely, we start from an Arrow–Debreu equilibrium, then construct candidate prices for the traded securities by using the consumption price process as a state price deflator, and finally check whether these prices deliver complete markets.

3.1. *Candidate Price Functions*

Fix an Arrow–Debreu equilibrium and let $m_t \equiv m(t, X_t)$ denote the corresponding consumption price. Appealing to Proposition 1 for the required

smoothness and applying Itô's lemma shows that

$$-A_t \equiv \int_0^t E_\tau \left[\frac{dm_\tau}{m_\tau} \right] = \int_0^t \frac{\mathcal{D}(m(\tau, X_\tau))}{m(\tau, X_\tau)} \, d\tau,$$

where the second order differential operator

$$\mathcal{D} \equiv \frac{\partial}{\partial t} + \mathcal{A} = \frac{\partial}{\partial t} + \mu_X(t, x)^\top \frac{\partial}{\partial x} + \frac{1}{2} \operatorname{Tr} \left[\sigma_X(t, x) \sigma_X(t, x)^\top \frac{\partial^2}{\partial x^2} \right]$$

denotes the extended infinitesimal generator of the state variables. Accordingly, we take $B_t \equiv \exp(A_t)$ as our candidate for the price of the riskless asset. On the other hand, a natural candidate for the stock price is the fundamental value of dividends computed at the equilibrium consumption price, namely

$$(3) \qquad S_t = S(t, X_t) \equiv E_t \int_t^T \frac{m(\tau, X_\tau)}{m(t, X_t)} g(X_\tau) \, d\tau.$$

The following result establishes some properties of this candidate price function that prove crucial for the existence of an equilibrium.

PROPOSITION 2: *The function S is jointly real analytic in $(t, x) \in (0, T) \times \mathcal{X}$ and belongs to $C^\infty((0, T] \times \mathcal{X})$.*

As is well known, to assert that the candidate prices give rise to an equilibrium, it suffices to prove that, given these prices, markets are dynamically complete. In a continuous-time model, the latter is closely related to the properties of the stock volatility. Specifically, it can be shown that markets are complete if and only if the stock volatility is almost everywhere invertible (see Duffie (2001, Chapter 6)). Since S is smooth, an application of Itô's formula shows that the volatility of the candidate stock prices is given by $\sigma_S(t, X_t) = \frac{\partial S}{\partial x}(t, X_t) \sigma_X(t, X_t)$. Combining this with the fact that a real analytic function is either identically zero or almost everywhere different from zero delivers the following proposition.[8]

PROPOSITION 3: *If $\det(\sigma_S(t, x)) \neq 0$ for some $(t, x) \in (0, T) \times \mathcal{X}$, then there exists an equilibrium with dynamically complete markets.*

The main obstacle one encounters when trying to apply Proposition 3 is that unless $d \equiv 1$ or the candidate price function can be computed in closed form, it is in general very difficult to check that the price volatility is invertible even

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[8]See Riedel (2001) for a related result in a single stock economy with incomplete information.

at a single point.[9] To circumvent this difficulty, we show in the next section that it may be sufficient to check that the volatility of the intermediate dividends, rather than that of the candidate prices, is nondegenerate.

3.2. *Conditions for Market Completeness*

In this section, we present conditions that are sufficient to guarantee that the price volatility is invertible and, hence, that there exists an equilibrium with complete markets. To highlight the intuition behind our approach, we start by considering the finite horizon case before we turn to infinite horizon economies.

3.2.1. *Finite Horizon Economies*

Let $T < \infty$, fix an Arrow–Debreu equilibrium, and denote by S the corresponding candidate price function. Using equation (3) in conjunction with Proposition 2 and standard martingale arguments shows that the candidate prices solve the partial differential equation

$$(4) \qquad -\frac{\partial}{\partial t}(m(t, x)S(t, x)) = \mathcal{A}(m(t, x)S(t, x)) + m(t, x)g(x)$$

subject to $S(T, x) = \mathbf{0}_d$, where $\mathcal{A}$ denotes the infinitesimal generator of the state variables. Differentiating with respect to x on both sides and using the continuity of derivatives established in Proposition 2 then gives

$$\frac{\partial S}{\partial x}(t, x) = (T - t)g'(x) + o(T - t),$$

and it follows that

$$\sigma_S(t, x) = (T - t)\sigma_g(T, x) + o(T - t),$$

where the matrix-valued function $\sigma_g(t, x)$ denotes the volatility of dividends. This simple expansion shows that the determinant of the price volatility is proportional to the determinant of the dividend volatility in a neighborhood of the terminal time and leads to the following theorem.

THEOREM 1: *If* $\det(\sigma_g(T, x)) \neq 0$ *for at least one* $x \in \mathcal{X}$, *then there exists an equilibrium with dynamically complete markets.*

[9]An important case in which this property can be checked quite easily is that of finite horizon economies where all risky securities pay terminal dividends as in Anderson and Raimondo (2008). Indeed, in such a case, the price volatility coincides with that of the terminal dividends at time T and it suffices to assume that the latter is nondegenerate to obtain the existence of a complete markets equilibrium. See the Supplemental Material for details.

The conclusion of Theorem 1 is quite intuitive. Indeed, it simply states that under our assumptions, nondegeneracy of the exogenous volatility of dividends is automatically transmitted to the endogenous volatility of the prices and thus ensures market completeness. A striking feature of this result is that it only depends on the dividends: Changing the utility functions, initial endowments, and/or labor income has no effect on the existence of an equilibrium.

The condition of Theorem 1 is sufficient, but it is not necessary. In particular, if some of the traded assets are fixed income securities, then this condition fails but markets may nonetheless be complete as illustrated by Example 1 below. To find sufficient conditions for market completeness in such cases, we perform a second order expansion of the volatility. Differentiating equation (4) with respect to (t, x) and using the continuity of the derivatives of the candidate prices shows that

$$(5) \qquad \sigma_S(t, x) = (T - t)\sigma_g(T, x) + \frac{1}{2}(T - t)^2 H(x) + o(T - t)^2,$$

where we have set

$$(6) \qquad H(x) \equiv \frac{\partial}{\partial x}\left(\frac{\mathcal{D}(m(T, x)g(x))}{m(T, x)}\right)\sigma_X(T, x) - 2\frac{\partial\sigma_g}{\partial t}(T, x).$$

Combining this expansion with well known results on determinants then leads to a second order condition for the existence of an equilibrium with dynamically complete markets. Specifically, defining

$$B_i(x) = \sigma_g(T, x) + \mathbf{e}_i\mathbf{e}_i^\top(H(x) - \sigma_g(T, x)),$$

where $\mathbf{e}_i$ is the ith vector in the orthonormal basis of $\mathbb{R}^d$, it can be shown that an equilibrium with dynamically complete markets exists provided that $\det(B_1(x)) + \cdots + \det(B_d(x)) \neq 0$ for some $x \in \mathcal{X}$. In contrast to that of Theorem 1, this condition depends not only on dividends, but also on preferences through the pricing kernel and might thus be difficult to apply since m can rarely be computed in closed form. To circumvent this difficulty, we show below that an equilibrium exists for generic initial endowments if the above condition holds when m is replaced by the marginal utility of a single agent. The reason why we need to consider generic endowments is that to get the result, we have to further expand the price volatility around the case where the economy is populated by a single agent.

To state the result, let $m_a(t, x) \equiv e^{-\rho t}u_a'(\overline{g}(x))$ denote the discounted marginal utility of agent a evaluated at the aggregate consumption and set

$$(7) \qquad B_{a,i}(x) = \sigma_g(T, x) + \mathbf{e}_i\mathbf{e}_i^\top(H_a(x) - \sigma_g(T, x)),$$

where the vector $\mathbf{e}_i \in \mathbb{R}^d$ is defined as before and the function H_a is defined as in equation (6), but with the function m replaced by m_a.

THEOREM 2: *Assume that the relative risk aversion of all agents is bounded between γ_1 and γ_2 for some $0 < \gamma_1 \leq \gamma_2$. If $\det(B_{a,1}(x)) + \cdots + \det(B_{a,d}(x)) \neq 0$ for some a and some $x \in \mathcal{X}$, then an equilibrium with dynamically complete markets exists for all matrix η of initial endowments outside of a closed set of measure zero.*

The following example illustrates how one can apply Theorem 2 to establish the existence of a complete markets equilibrium in an economy where Theorem 1 fails due to the presence of a fixed income security.

EXAMPLE 1: Consider a finite horizon economy where at least one agent, say agent 1, has constant relative risk aversion $\gamma > 0$ and let $d \equiv 2$. Assume that

$$dX_{1t} = (X_{2t} - \|\sigma_1\|^2/2)\, dt + \sigma_1^\top\, dZ_t,$$
$$dX_{2t} = \kappa(\theta - X_{2t})\, dt + \sigma_2^\top\, dZ_t$$

for some constants $(\kappa, \theta, \sigma_i)$ such that $\det(\sigma_1, \sigma_2) \neq 0$, and that the dividends rates are given by $g_1(x) \equiv 1$ and $g_2(x)$ for some real analytic function g_2.

Using Proposition 4 in conjunction with well known results on Gaussian processes we have that Assumptions A, B, and C hold. However, Theorem 1 cannot be used here because the dividend volatility is degenerate. To circumvent this difficulty, we use Theorem 2. Since agent 1 has power utility, a direct calculation shows that

$$\sum_{i=1}^{2} \det(B_{1,i}(x)) = \frac{\gamma g_2'(x)\overline{g}'(x)}{\overline{g}(x)} \det(\sigma_1, \sigma_2)$$

is nonzero as soon as $\overline{g}' \neq 0$ and it follows from Theorem 2 that an equilibrium with dynamically complete markets exists for generic initial endowments.

REMARK 2: (a) Theorems 1 and 2 show that in the finite horizon case, completeness can be deduced from the primitives of the economy provided that the prices are real analytic functions of both time and the state variables. We prove in the Supplemental Material that real analyticity in space can be dispensed with provided that the conditions of the theorems hold for almost every $x \in \mathcal{X}$ rather than for at least one $x \in \mathcal{X}$. We also show there that the requirement of real analyticity in time cannot be relaxed by providing examples of representative agent economies that fail to admit a complete markets equilibrium despite nondegenerate dividends because the candidate prices are not real analytic.

(b) Herzberg and Riedel (2010) showed that in the setting of Anderson and Raimondo (2008), real analyticity of the candidate price function is sufficient to establish the existence of a complete markets equilibrium if the volatility matrix of terminal dividends is invertible. However, the conditions they impose are

not sufficient for the candidate prices to be real analytic. Indeed, to guarantee that this property holds, it is necessary to impose bounds on the transition density in a complex neighborhood of $[0, T]$ as in equation (2).

3.2.2. *Infinite Horizon Economies*

In an infinite horizon economy, there is no terminal time close to which the price volatility can be approximated and, as result, the approach of the previous section cannot be used. Instead, we expand the volatility of the candidate prices as a function of the agents' common discount rate and use this expansion to derive conditions for the generic existence of a complete markets equilibrium.

Using equation (3), it can be shown (see Lemmas 4, 5, and 6 in the Appendix) that the volatility matrix of the candidate prices is real analytic in ρ and satisfies[10]

$$\sigma_S(x, \rho) = \frac{1}{\rho}\sigma_g(x) + \frac{1}{2\rho^2}H(x) + o(1/\rho)^2,$$

which is the direct analog of equation (5) for the infinite horizon case. Combining this expansion with some generic determinacy arguments then delivers the following counterpart to Theorems 1 and 2.

THEOREM 3: *Assume that the relative risk aversion of all agents is bounded between γ_1 and γ_2 for some $0 < \gamma_1 \leq \gamma_2$. If either $\det(\sigma_g(x)) \neq 0$ or $\det(B_{a,1}(x)) + \cdots + \det(B_{a,d}(x))$ for some a and some $x \in \mathcal{X}$, then an equilibrium with dynamically complete markets exists for all η and $\rho > R$ outside of a closed set of measure zero.*

REMARK 3: A close inspection of the proof shows that Theorem 3 remains valid if we only require equation (2) to hold for real, rather than complex, values of the time arguments. The reason for this important simplification is that with an infinite horizon, the volatility of the candidate prices is automatically real analytic as a function of ρ and this is all that is needed to deduce the generic existence of an equilibrium with complete markets from the primitives of the model. The result of Theorem 3 also extends to the case of heterogenous discount rates, provided that we make appropriate changes in Assumption C.

REMARK 4: To facilitate the presentation, we have assumed that there are as many risky assets as Brownian motions, but this assumption is not necessary for the validity of our main results. In particular, the conclusions of Theorems 1, 2, and 3 remain valid if there are more risky securities than Brownian motions provided that the stated conditions hold for a fixed set of d risky securities.

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[10]In the infinite horizon case, the drift and diffusion of the state variables are assumed to be time-independent. As a result, the volatility of dividends is also time-independent and the expressions in equations (6) and (7) simplify.

4. APPLICATIONS

In this section, we provide examples of classes of models that satisfy the conditions of Assumption A.

Vector Autoregressive Processes: Assume that the state variables follow a vector autoregressive process of the form

$$dX_t = (b(t) - A(t)X_t)\,dt + \sigma_X(t)\,dZ_t,$$

where $Z \in \mathbb{R}^d$ is a Brownian motion, and $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$, and $\sigma_X \in \mathbb{R}^{d \times d}$ are real analytic functions. This class of models includes as a special case the model studied by Anderson and Raimondo (2008), where the state variables coincide with the underlying Brownian motions, but it is significantly more flexible as it allows for arbitrary mean reverting Gaussian processes.

PROPOSITION 4: *If* $\mathrm{rank}(\sigma_X(t)) = d$ *for all* $t \geq 0$, *then Assumption* A *holds provided that the economy has a finite horizon.*[11]

Autonomous Diffusion Processes: Assume that each of the coordinates of the vector of state variables follows an autonomous diffusion process of the form

$$(8) \qquad dX_{it} = \mu_i(X_{it})\,dt + \sigma_i(X_{it})\,dZ_{it}$$

for some real analytic drift and volatility functions. Let $\mathcal{X}_i \equiv (l_i, r_i)$ with $-\infty \leq l_i < r_i \leq \infty$ denote the state space of the ith coordinate and assume that the solution to equation (8) does not reach the boundaries of $\mathcal{X}_i$ in finite time. For this class of models, the existence of a transition density follows from well known results (see, e.g., Itô and McKean (1965, Paragraph 4.11) and we have the following proposition:

PROPOSITION 5: *Assume that equation* (2) *holds for real values of* $t \neq \tau$. *Then Assumption* A *holds*.

Since the bound only needs to hold for real, rather than complex, values of time, it is much easier to check. For example, relying on the above result it can be shown that Assumption A holds, provided that each coordinate follows an arithmetic Brownian motion or a square root process

$$dX_{it} = (\mu_i - \kappa_i X_{it})\,dt + \xi_i \sqrt{|X_{it}|}\,dZ_{it}$$

for some $\xi_i^2 < 2\mu_i$, or a constant elasticity of variance process

$$dX_{it} = \mu_i X_{it}\,dt + \xi_i |X_{it}|^{\beta_i}\,dZ_{it}$$

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[11]If the economy has an infinite horizon, then we assume that the coefficients of the driving process are time-independent and, in that case, additional conditions on the eigenvalues of the matrix A are required for the validity of the result. See the Supplemental Material for details.

for some $\beta_i \geq 1$, or a general one-dimensional diffusion whose drift and volatility coefficients μ_i and $\sigma_i \equiv 1/\phi_i'$ are such that $G_i(y) \equiv (\mu_i \phi_i' + \frac{1}{2}\sigma_i^2 \phi_i'') \circ \phi_i^{-1}(y)$ satisfies a linear growth condition (see Quian and Zheng (2004, Theorem 3.2)).

General Diffusion Processes: To obtain a general class of models in which the conditions of Assumption A are satisfied, assume that the drift μ_X and volatility σ_X are bounded real analytic functions with bounded derivatives, and that the volatility is uniformly elliptic in the sense that

$$\sup_{(t,x)\in(0,T)\times\mathcal{X}} \|\sigma_X(t,x)^\top \xi\|^2 \geq \varepsilon \|\xi\|^2$$

for all $\xi \in \mathbb{R}^d$ and some $\varepsilon > 0$. Under these assumptions, it follows from the general theory of heat kernel bounds that the state variables admit a transition density which satisfies Assumption A(d). See, for example, Eidelman (1969, Theorem 8.1), Lunardi (1995, Chapters 3, 5, 6, and 8), and Auscher (1996).

Analytic Semigroups: Relying on the semigroup approach to diffusion processes, it can be shown that the bound of Assumption A(d) holds with $d(x, y) = C \sum_i |x_i - y_i|^{1/2}$ for some $C > 0$, provided that the operator $\mathcal{A}$ generates an analytic semigroup on an appropriate space of functions. Very general sufficient conditions that cover many important cases can be found in Lunardi (1995), Gozzi, Monte, and Vespri (2002), and Grigor'yan (1994, 2003, 2006), among others.

APPENDIX 1: PROOFS

PROOF OF PROPOSITION 1: The existence result follows by a slight modification of the arguments in Malamud (2008) and is reported in the Supplemental Material. The characterization of the consumption price and its real analyticity follow from Huang (1987, Propositions 3.1 and 3.2), the market clearing condition and the real analytic implicit function theorem (see, e.g., Krantz and Parks (2002, Section 2.3)).   *Q.E.D.*

LEMMA 1: *The transition density* $p(t, x, \tau, y)$ *is jointly real analytic in* (t, x) *for* $\tau \neq t$. *Furthermore, for any constant* $\varepsilon > 0$ *and any vector* $k \in \mathbb{N}_+^d$, *there exists a locally bounded function* $A(x) = A(x; \varepsilon, |k|) > 0$ *such that*

$$\left| \frac{\partial^{k_0+|k|} p(t,x,\tau,y)}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_d^{k_d}} \right| \leq A(x)L(y)|\tau-t|^{-(\alpha+k_0+|k|)}$$

$$\times e^{\phi|\tau-t|(1+\varepsilon)-(d(x,y))^2/(2(1+\varepsilon)|\tau-t|)}$$

for all $k_0 \in \mathbb{N}$ *and* $(t, \tau, x, y) \in \mathcal{P}_\varepsilon^2 \times \mathcal{X}^2$ *such that* $d(x, y) > \varepsilon$.

PROOF: Since the transition density is smooth for $\tau \neq t$, it follows from standard results that it solves the backward Kolmogorov equation

$$\mathcal{D}(p(t, x, \tau, y)) = \frac{\partial p(t, x, \tau, y)}{\partial t} + \mathcal{A}(p(t, x, \tau, y)) = 0$$

for $t \neq \tau$. Combining this property with Eidelman (1969, Theorem 6.2) shows that p is real analytic in $x \in \mathcal{X}$ for $\tau \neq t$. Furthermore, its radius of analyticity is bounded away from zero when (t, x) vary in a compact subset of $[0, \tau) \times \mathcal{X}$ by Assumption A(d), and joint real analyticity follows from Siciak (1969); see also Eidelman (1969, Theorem 8.1).

The second part follows from Assumption A and estimates for solutions of parabolic partial differential equations. See Davies (1997), Eidelman (1969, pp. 23–28), and the Supplemental Material for details.   *Q.E.D.*

LEMMA 2: *We have $0 < \mathbf{m}(x) < C_m \sum_{a=1}^{A} u_a'(\bar{g}(x)/A))$ for some constant $C_m > 0$ where $\mathbf{m}(x) \equiv e^{\rho t} m(t, x)$.*

For the proof, see the Supplemental Material.

LEMMA 3: *The function S is well defined and real analytic in $t \in (0, T)$.*

PROOF: Fix an arbitrary $t_0 \in (0, T)$. By Proposition 1 and Lemma 2, we know that the function m is strictly positive and real analytic with respect to $t \in [0, T]$, so it suffices to prove that for each i, the function

$$(9) \qquad Q_i(z, x) \equiv m(z, x) S_i(z, x) = \int_z^T \int_{\mathcal{X}} p(z, x, \theta, y) m(\theta, y) g_i(y) \, dy \, d\theta$$

is well defined and analytic in a complex neighborhood $\mathcal{P}_0 \supset t_0$. Choosing the neighborhood appropriately, we may assume that any segment connecting points $z \in \mathcal{P}_0$ with T lies in the set $\mathcal{P}_\varepsilon^2$ of Assumption A. Therefore, it follows from Lemma 2 and the second part of Assumption C that the integrand in (9) has an integrable majorant and it follows that Q_i is well defined in $\mathcal{P}_0$. Furthermore, the integrand being analytic in z by Assumption A and Proposition 1, it follows from the Morera theorem (see, e.g., Shabat (1992, Theorem 2) and Eidelman (1969, p. 223)) that Q_i is analytic with respect to $z \in \mathcal{P}$, and the proof is complete.   *Q.E.D.*

PROOF OF PROPOSITION 2: Let $\pi_i(t, x, \tau) \equiv E_t[\mathbf{m}(X_\tau) g(X_\tau)]$. By Proposition 1 and Lemma 2, we know that the function $\mathbf{m}$ is strictly positive and real analytic with respect to $x \in \mathcal{X}$, so it suffices to show that the result holds for the function

$$Q_i(t, x) \equiv m(t, x) S_i(t, x) = \int_t^T e^{-\rho \tau} \pi_i(t, x, \tau) \, d\tau.$$

Fix an $x \in \mathcal{X}$, pick two open sets $\mathcal{X}_1 \subset \mathcal{X}_2 \subseteq \mathcal{X}$, such that $x \in \mathcal{X}_1$, and let $h_i \equiv n\mathbf{m}g_i$, where $n: \mathcal{X} \to [0, 1]$ is a smooth function that is equal to 1 on $\mathcal{X}_1$ and to 0 outside of $\mathcal{X}_2$. Using this notation, we have $\pi_i = F_i + H_i$, where

$$F_i(t, x, \tau) \equiv \int_{\mathcal{X}_1^c} p(t, x, \tau, y)(\mathbf{m}g_i - h_i)(y)\, dy,$$

$$H_i(t, x, \tau) \equiv \int_{\mathcal{X}_2} p(t, x, \tau, y)h_i(y)\, dy,$$

and $\mathcal{X}_1^c = \mathcal{X} \setminus \mathcal{X}_1$. A direct calculation based on Lemmas 1 and 2, Assumption C, and the fact that $d(x, y) > \varepsilon$ for some $\varepsilon > 0$ and all $y \in \mathcal{X}_1^c$ shows that for every $k \in \mathbb{N}^d$, there exists an integrable function $f_{k,x}$ such that

$$(10) \qquad \left| \frac{\partial^k p(t, x, t + \theta, y)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} (\mathbf{m}g_i - h_i)(y) \right| \le f_{k,x}(\theta, y)$$

for all $(t, \theta, y) \in [0, T] \times [0, T - t] \times \mathcal{X}_1^c$. Therefore, it follows from the dominated convergence theorem that F_i is smooth in x and satisfies

$$(11) \qquad \frac{\partial^k F_i(t, x, \tau)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} = \int_{\mathcal{X}_1^c} \frac{\partial^k p(t, x, \tau, y)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} (\mathbf{m}g_i - h_i)(y)\, dy$$

for all $(t, \tau) \in [0, T]$. On the other hand, since the function h_i is smooth and compactly supported, it follows from Eidelman (1969, Theorem 5.3) that H_i is smooth with respect to x and satisfies

$$(12) \qquad \frac{\partial^k H_i(t, x, \tau)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} = \int_{\mathcal{X}_1} \frac{\partial^k p(t, x, \tau, y)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} h_i(y)\, dy,$$

as well as

$$(13) \qquad \lim_{t \to T} \int_t^T e^{-\rho\tau} \frac{\partial^k H_i(t, x, \tau)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}\, d\tau = 0.$$

Adding (11) and (12) shows that π_i is smooth with respect to x, and it immediately follows that Q_i is smooth with respect to x and satisfies

$$(14) \qquad \frac{\partial^k Q_i(t, x)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} = \int_t^T e^{-\rho(\tau-t)} \frac{\partial^k \pi(t, x, \tau)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}\, d\tau$$

as well as

$$\lim_{t \to T} \frac{\partial^k Q_i(t, x)}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} = 0,$$

where the limit follows from (13), (10), and the monotone convergence theorem. Since Q_i is real analytic in time by Lemma 3, we know that it is smooth with respect to t and it thus follows from standard martingale arguments that

$$(15) \qquad -\frac{\partial Q_i(t, x)}{\partial t} = \mathcal{A}(Q_i(t, x)) + m(t, x)g_i(x).$$

As a result, any mixed derivative of Q_i can be expressed in terms of derivatives with respect to x and it follows that $Q_i \in C^k((0, T] \times \mathcal{X})$. Finally, since the coefficients of (15) are real analytic in x, we know from Eidelman (1969, Theorem 6.2) that Q_i is real analytic in x with a radius of analyticity that is uniformly bounded away from zero when (t, x) vary in compact subsets of $(0, T) \times \mathcal{X}$, and joint real analyticity now follows from Lemma 3 and the results of Siciak (1969).   Q.E.D.

PROOF OF THEOREMS 1 AND 2: The validity of (5) follows by differentiating (4) and using the smoothness of S, which is provided by Proposition 2. Furthermore, a direct but tedious calculation provided in the Supplemental Material shows that

$$(16) \qquad \det(\sigma_S(t, x)) = \theta^d \det(\sigma_g(t, x)) + \frac{\theta^{1+d}}{2} \sum_{i=1}^{d} \det(B_i(x)) + o(\theta)^{1+d},$$

where $\theta = T - t$. Under the assumption of Theorem 1, this shows that the function $\det(\sigma_S(t, x))$ is not identically zero. Since it is real analytic by Proposition 2 and Krantz and Parks (2002, Proposition 2.2.3), it follows from Anderson and Raimondo (2008, Theorem B.3) that $\det(\sigma_S(t, x))$ is almost everywhere nonzero on $(0, T) \times \mathcal{X}$. Combining this with Proposition 3 shows that an equilibrium with complete markets exists and completes the proof of Theorem 1.

To prove Theorem 2, we argue as follows: By the real analytic implicit function theorem, we have that $m(t, x, \lambda)$ is real analytic in $(t, x, \lambda) \in [0, T] \times \mathcal{X} \times \mathcal{S}_{++}$ and satisfies $m(t, x, \mathbf{e}_a) = m_a(t, x)$. Since the drift and volatility of the state variables are real analytic, this implies that $\det(B_i(x, \lambda))$ is real analytic in $(x, \lambda) \in \mathcal{X} \times \mathcal{S}_{++}$ and satisfies $\det(B_i(x, \mathbf{e}_a)) = \det(B_{a,i}(x))$. Combining this with (16) shows that under the assumption of Theorem 2, the function $\det(\sigma_S(t, x, \lambda))$ is almost everywhere nonzero on $(0, T) \times \mathcal{X} \times \mathcal{S}_{++}$ and it only remains to prove that generic Pareto weights λ correspond to generic initial endowments η. As we show in the proof of Theorem 3, our assumptions guarantee that the equilibrium is determinate for almost every η. Therefore, in a small ball $\mathcal{B}(\eta')$ around a generic initial endowment η', there exists a C^1 bijection between η and λ, and the desired result follows.   Q.E.D.

Consider now an infinite horizon economy and let $S(x, \rho, \lambda)$ denote the candidate prices of the risky assets, which are seen as functions of the state variables, the agents' common discount rate, and the Pareto weights.

LEMMA 4: *Under the assumptions of Theorem 3, we have*

$$0 = \lim_{\rho \to \infty} \left(\rho \frac{\partial S}{\partial x}(x, \rho, \lambda) - g'(x) \right)$$

$$= \lim_{\rho \to \infty} \left(\rho^2 \frac{\partial S}{\partial x}(x, \rho, \lambda) - \rho g'(x) - H(x) \right).$$

PROOF: Since the coefficients of the state variables are time-independent, we have that $\pi_i(t, x, t + \theta) = \bar{\pi}_i(x, \theta)$ depends only on (x, θ). It follows from the proof of Proposition 2 that $\bar{\pi}_i \in C^\infty(\mathcal{X} \times [0, \infty))$. Standard martingale arguments imply that $\bar{\pi}_i$ is a solution to $(\partial/\partial \theta)\bar{\pi}_i = \mathcal{A}(\bar{\pi}_i)$ with the initial condition $\bar{\pi}_i(x, 0) = \mathbf{m}(x)g_i(x)$, and the desired result now follows from Lemma 5 (below), the definition of the candidate price function, and (14).   Q.E.D.

LEMMA 5: *If $f \in C^\ell(\mathbb{R}_+)$ is a function such that $e^{-\rho t}f(t)$ is integrable for all $k \le \ell$ and some $\rho > 0$, then we have*

$$\lim_{\rho \to \infty} \rho^k \left(\rho \int_0^\infty e^{-\rho t} f(t) \, dt - \sum_{i=0}^k \rho^{-i} f^{(i)}(0) \right) = 0, \quad k \le \ell - 1.$$

The proof follows from a standard induction argument based on the dominated convergence theorem and is presented in the Supplemental Material.

LEMMA 6: *Under the assumptions of Theorem 3 and for almost every vector of Pareto weights $\lambda \in \mathcal{S}_{++}$, we have that the volatility matrix of the candidate prices is almost surely nondegenerate for almost every $\rho > R$.*

PROOF: Combining the result of Lemma 5 with an argument similar to that which we used in the proof of Theorems 1 and 2 shows that

$$\det(\sigma_S(x, \rho, \lambda))$$

$$= \frac{1}{\rho^d} \det(\sigma_g(x)) + \frac{1}{2\rho^{1+d}} \sum_{i=1}^d \det(B_i(x, \lambda)) + o(1/\rho)^{1+d}.$$

Since S is automatically real analytic in $\rho > R$, we have that $\det(\sigma_P(x, \rho, \lambda))$ is also real analytic in $\rho > R$. It follows that the required assertion holds for those $\lambda \in \mathcal{S}_+$, for which $|\det \sigma_g(x)| + |\det(B_i(x, \lambda)) + \cdots + \det(B_d(x, \lambda))|$ is not identically equal to zero. Since $\det(B_i(x, \mathbf{e}_a)) = \det(B_{a,i}(x))$ and the function $\det(B_i(x, \lambda))$ is real analytic in (x, λ), the assumption of the statement guarantees that this holds for generic $\lambda \in \mathcal{S}_+$ and the proof is complete.
          Q.E.D.

PROOF OF THEOREM 3: Fix the initial endowments for all but the first stock and consider the vector-valued mapping defined by

$$G_a(\rho, \lambda) \equiv E \int_0^\infty \frac{e^{-\rho t} \mathbf{m}(X_t, \lambda)}{S_1(x, \rho, \lambda)}$$
$$\times \left(I_a(\lambda_a^{-1} \mathbf{m}(X_t, \lambda)) - \ell_a(X_t) - \sum_{i=2}^d \eta_{ai} g_i(X_t) \right) dt,$$

where I_a denotes the inverse marginal utility of agent a. With this notation, we have that the Negishi equation that allows us to determine the equilibrium Pareto weights from the agents' initial endowments is

(17) $G(\rho, \lambda) = (\eta_{11}, \ldots, \eta_{A1})^\top \equiv \eta_1 \in \mathcal{S}.$

By the existence part of Proposition 1, we know that G is onto. Since the relative risk aversions are bounded from above and away from zero, it can be shown (see the Supplemental Material for details) using Assumption C and Lemma 2 that G is C^1 with respect to $(\rho, \lambda) \in (R, \infty) \times \mathcal{S}_{++}$. Therefore, by Sard's theorem (see, e.g., Sternberg (1964, Theorem II.3.1)), we get that for each fixed $\rho > R$, almost every $\eta_1 \in \mathcal{S}$ is regular for G in the sense that any solution λ to (17) satisfies $\det \frac{\partial G(\rho, \lambda)}{\partial \lambda} \neq 0$. Fix such a regular (ρ, η_1^*), let λ^* be a solution to (17), and let $\mathcal{B}$ be a small open neighborhood of (ρ, λ^*) such that the map $(\rho, \lambda) \to (\rho, G(\rho, \lambda))$ is C^1 bijection from $\mathcal{B}$ to some $\mathcal{B}'$. By Lemma 6, the candidate prices volatility matrix is nondegenerate for almost every $(\rho, \lambda) \in \mathcal{B}$ and the standard change of variables formula implies that the image of such (ρ, λ) has full measure in $\mathcal{B}'$. As a result, an equilibrium with dynamically complete markets exists for all (ρ, η_1) in this image and the desired conclusion now follows from the arbitrariness of the initial endowments in stocks $2, \ldots, d$. Q.E.D.

The proofs of Propositions 3, 4, and 5 are provided in the Supplemental Material.

APPENDIX 2: TIME-DEPENDENT CONSUMPTION AND HETEROGENOUS DISCOUNT RATES

In this Appendix, we provide conditions on the dividends, labor income rates, and preferences that allow us to extend the validity of our finite horizon results to cases in which the aggregate consumption $\overline{g}(t, x) = \sum_i g_i(t, x) + \sum_a \ell_a(t, x)$ is time-dependent and the discount rates can be heterogenous across agents. The main difficulty in dealing with such cases is that to define the candidate price

$$S_i(t, x) = \int_t^T \int_\mathcal{X} p(t, x, \tau, y) m(\tau, y) g_i(\tau, y) \, dy \, d\tau$$

for complex values of t, we need the radius of complex analyticity of the integrand $p(t, x, \tau, y)m(\tau, y)g_i(\tau, y)$ to be bounded away from zero *uniformly in* y. When discount rates are homogeneous and dividends are time-independent, this is not a problem, as in this case $m(\tau, y) = e^{-\rho\tau}\mathbf{m}(y)$ for some time-independent function $\mathbf{m}$. Otherwise, showing the uniform analyticity of m becomes a highly nontrivial problem. In particular, in that case the conditions imposed by Anderson and Raimondo (2008) are not sufficient to guarantee this uniform analyticity even in their simpler setting. To solve this problem, one needs to impose the following stringent conditions:

ASSUMPTION D: *Consider a finite horizon economy and make the following assumptions*:

(a) *For any $\varepsilon > 0$, there exists a complex neighborhood $\mathcal{O} \supset [0, T]$ such that $\overline{g}(t, x)$ can be analytically continued to $\mathcal{O}$,*

$$|\Im(\overline{g}(t, x))| \leq \varepsilon\Re(\overline{g}(t, x)) \quad and$$
$$K^{-1}\overline{g}(\Re t, x) \leq \Re(\overline{g}(t, x)) \leq K\overline{g}(\Re t, x)$$

for all $(t, x) \in \mathcal{O} \times \mathcal{X}$ and some constant $K = K(\varepsilon, \mathcal{O}) > 0$.

(b) *There exists a constant $\varepsilon > 0$ such that the inverse marginal utilities $I_a(z)$ are analytic in the sector $\{z \in \mathbb{C} : |\Im z| < \varepsilon\Re z\}$ and*

$$\lim_{z \in \mathbb{C}_\varepsilon, |z| \to \infty} \left(z^{b_a^\infty} I_a(z) - c_a^\infty\right) = \lim_{z \in \mathbb{C}_\varepsilon, |z| \to 0} \left(z^{b_a^0} I_a(z) - c_a^0\right) = 0$$

for each $a \leq A$ and some constants $b_a^0, b_a^\infty > 0$ and $c_a^0, c_a^\infty > 0$.

(c) *There are constants $R \leq \min_a \rho_a$ and $\nu > 1$ such that*

$$\int_0^T \int_{\mathcal{X}} \sum_{a=1}^A e^{-R\tau} u_a'(\overline{g}(\tau, x)/A)\overline{g}(\tau, x)\,\bar{p}(0, x, \nu\tau, y)\,dy\,d\tau < \infty$$

for all $x \in \mathcal{X}$, where the function $\overline{p}$ is defined as in Assumption A.

REMARK 5: A simple case where condition (a) holds is $g_i(t, x) = e^{\delta_i t}g_i(x)$ and $\ell_a(t, x) = e^{\kappa_a t}\ell_a(x)$ for some real analytic $g_i, \ell_a \geq 0$ and some constants δ_i, κ_a. Condition (b) requires that the utility functions behave like power functions close to zero and infinity, and can be shown to hold for most standard utility functions.

The following theorem shows that our results remain valid in this more general setting and concludes this Appendix.

THEOREM 4: *Assume that the aggregate consumption is time-dependent and that the agents' discount rates are heterogenous. Then Propositions 1, 2, and 3 and Theorems 1 and 2 remain valid, provided that Assumption C is replaced by Assumption D.*

See the Supplemental Material for the proof.

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