



Heterogeneous preferences and equilibrium trading volume[☆]

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Abstract

The representative-agent Lucas model stresses aggregate risk and hence does not allow us to study the impact of agents' heterogeneity on the dynamics of equilibrium trading volume. In this paper, we investigate under what conditions non-informational heterogeneity, i.e., differences in preferences and endowments, leads to nontrivial trading volume in equilibrium. We present a non-informational no-trade theorem that provides necessary and sufficient conditions for zero equilibrium trading volume in a continuous-time Lucas market model with heterogeneous agents, multiple goods, and multiple securities. We explain in detail how no-trade equilibria are related to autarky equilibria, portfolio autarky equilibria, and peculiar financial market equilibria, which play an important role in the literature on international risk sharing.

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1. Introduction

The main focus of the representative-agent Lucas model (Lucas, 1978) is on equilibrium prices and returns. Agents in this model are identical by assumption. As a consequence, the equilibrium sharing rule is linear and can be implemented without trade in financial securities. The representative-agent Lucas model is therefore valuable as a tool to study aggregate market risk, but at the same time, does not provide any testable hypotheses for equilibrium trading volume.

In order to generate nontrivial trading volume in a Lucas-type model, one needs to model heterogeneity among agents. Heterogeneity can be introduced in terms of either information,¹ preferences or endowments. While it is well understood that in symmetric information models the degree of heterogeneity of endowments, preferences, and beliefs determines the equilibrium trading volume, necessary and sufficient conditions for trade in a dynamic model are still unavailable. The main result of this paper fills this gap. It comes in the form of a no-trade theorem that provides necessary and sufficient conditions for zero trading volume in a Pareto-efficient Lucas economy with multiple goods, multiple securities, symmetric information, and homogeneous beliefs. We illustrate this result in a number of examples that include most of the classical multi-good utility functions used in financial economics. These examples show that the existence of a no-trade equilibrium does not necessarily require that agents have identical preferences. In particular, we show that such an equilibrium can exist when agents have log-linear preferences but assign different weights to each good in their consumption bundle.

As shown by Cass and Pavlova (2004) in a continuous-time model with multiple stocks, markets are not necessarily complete in equilibrium even if the number of risky securities equals the number of sources of risk. In order to circumvent the difficulties arising in the study of inefficient equilibria, we restrict our attention to Pareto-efficient equilibria and use the resulting proportionality of the utility gradients to infer the characteristics of preferences and endowments that do not generate trade in equilibrium. In contrast, in finite dimensional models, it is possible to choose the aggregate dividend in such a way that markets are necessarily dynamically complete in equilibrium. Such a model is studied in Judd, Kubler, and Schmedders (2003), where the aggregate dividends are given as an irreducible, stationary Markov chain. They show that in this case, the optimal consumption policies inherit the time homogeneity of the aggregate dividend, and they conclude that no trading occurs after the initial period in equilibrium irrespective of the agents' preferences. This is a striking result, but one should bear in mind that stationarity and irreducibility are strong assumptions. In particular, they imply that all information about future dividends is revealed at the initial time and prevent the introduction of dividend growth into the model. Furthermore, Bossaerts and Zame (2005) show that the no-trade result of Judd, Kubler, and Schmedders (2003) fails to hold as soon as individual endowments are nonstationary, even if stationarity is preserved at the aggregate level. Our study complements this discussion by allowing for general arbitrarily growing dividend

¹See Pfleiderer (1984), Kyle (1985), Foster and Viswanathan (1990), and Wang (1994) for examples of models with asymmetric information. Note that to overcome the informational no-trade theorems of Milgrom and Stokey (1982) and Holmström and Myerson (1983), these models have to introduce exogenous liquidity traders or stochastic supply shocks.

processes in continuous time. However, this generalization comes at a cost, as it then becomes impossible to assume market completeness a priori.

A natural context in which our results can be applied is that of international finance, where each agent is interpreted as being representative of a country and the relative prices of goods represent the terms of trade. A very active area of research in this field is the analysis of international capital flows. In particular, [Souriounis \(2003\)](#) and [Hau and Rey \(2005\)](#) show that equity returns and portfolio rebalancings are an important source of exchange rate dynamics. Given these empirical findings, it is surprising that many of the theoretical asset pricing models in the international finance literature² consider preference specifications which satisfy the conditions of our theorem and thus fail to produce realistic international capital flows. Our result describes the structure of preferences for which a no-trade equilibrium prevails, and thus characterizes the minimal level of preference heterogeneity required to generate nontrivial portfolio rebalancings in equilibrium.

The no-trade equilibria introduced in this paper are related to autarky and portfolio autarky equilibria which are prominently featured in international financial economics. A no-trade equilibrium is an autarky equilibrium if initial endowments are individually optimal. [Lucas \(1982\)](#) uses such equilibria to study interest rates and currency prices in a general preference setting. He derives a perfectly pooled equilibrium assuming that investors have identical preferences and symmetric endowments. It follows from our main result and examples that such perfect pooling is not necessary: autarky equilibria are not necessarily symmetric and can exist even if agents are not identical.

In a multi-good Lucas model, intertemporal risk sharing occurs through two channels. First, as in a single-good economy, agents can trade Arrow-Debreu securities synthesized from risky assets to finance their consumption plans. At the same time, relative price movements and the possibility of trade in the spot market for goods provide additional means for consumption smoothing. The importance of this second channel for international trade has been stressed by [Cole and Obstfeld \(1991\)](#), [Zapatero \(1995\)](#), [Serrate \(2001\)](#), and [Pavlova and Rigobon \(2003\)](#). In particular, Cole and Obstfeld study the welfare gains associated with the existence of international financial markets and show that for identical Cobb–Douglas preferences, there exists a Pareto-efficient equilibrium for which optimal consumption plans can be financed without trades in financial assets. Such an equilibrium is referred to as a portfolio autarky equilibrium. Interestingly, and as demonstrated by our examples, for this class of preferences there exists a no-trade equilibrium which yields the same consumption allocation and prices. It turns out that, in general, the consumption allocation of an efficient no-trade equilibrium can be implemented with portfolio autarky if and only if all investors have unit elasticity of substitution. This condition is necessary and sufficient provided that the no-trade equilibrium is not already an autarky equilibrium which by definition is also a portfolio autarky equilibrium. If the same allocation can be achieved either by trading once in the financial markets and never after that, or by trading continuously in the goods market, financial markets are redundant and agents are indifferent with respect to their portfolio holdings. We formalize this intuition by showing that when efficient no-trade

²Cole and Obstfeld (1991), Backus, Kehoe, and Kydland (1992), Zapatero (1995), Baxter and Crucini (1995), Arvanitis and Mikkola (1996), Baxter and Jermann (1997), Kehoe and Perri (2002), and Heathcote and Perri (2002).

equilibria and efficient portfolio autarky equilibria coincide, the equilibrium is necessarily peculiar in the sense that all but one of the risky assets are redundant. Cass and Pavlova (2004) introduce peculiar equilibria and prove their existence in a model with log-linear preferences. Our results show that the property of logarithmic preferences which implies the existence of peculiar financial market equilibria is their unit elasticity of substitution. This property implies that the terms of trade are inversely proportional to the ratio of aggregate dividends, and thus stock prices are linearly dependent.

The rest of the paper is organized as follows. Section 2 introduces our multi-good economy and defines the different types of equilibria to be studied in the paper. Section 3 presents some simplifying notation and preliminary results. Section 4 contains our main result and shows its application in a number of examples prominently featured in international asset pricing. Section 5 discusses the economic relevance of no-trade equilibria and their relation to linear sharing rules, fund separation, and international risk sharing. Section 6 shows that no-trade equilibria are non-robust with regard to extensions of the basic model that introduce heterogeneous beliefs and random endowments. Proofs of all results are provided in the Appendix.

2. The economy

We consider a continuous-time, stochastic economy on the finite time interval $[0, T]$ modeled as follows.

2.1. Information structure

The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ on which is defined an n -dimensional Brownian motion B . The filtration \mathbf{F} is the usual augmentation of the filtration generated by the Brownian motion and we let $\mathcal{F} = \mathcal{F}_T$ so that the true state of nature is completely determined by the paths of the Brownian motion up to the terminal date of the model. All agents are endowed with the same information structure represented by \mathbf{F} and the same beliefs represented by the probability measure P .

All random processes to appear in the sequel are assumed to be progressively measurable with respect to the filtration \mathbf{F} , and all statements involving random quantities are understood to hold either almost surely or almost everywhere depending on the context.

2.2. Consumption space and goods markets

There is a finite number of perishable consumption goods indexed by $a \in \mathcal{A}$ for some finite set \mathcal{A} with $A := \text{card}(\mathcal{A})$. The consumption space \mathcal{C} is given by the set of nonnegative and integrable consumption rate processes.

Each of the A available consumption goods can be traded in a perfect spot market. We denote by p the A -dimensional vector process of relative prices of consumption goods. The first consumption good is assumed to serve as numéraire and therefore its relative price is normalized to one, i.e., $p_1^1 = 1$.

2.3. Securities

The financial market consists of a riskless savings account in zero net supply and \mathbf{n} risky stocks in positive net supply. Each stock represents a claim to an exogenously specified stream of dividends denominated in one of the A available consumption goods. More precisely, we assume that for each consumption good $a \in \mathcal{A}$ there is a number n_a , with

$$\mathbf{n} := \sum_{a \in \mathcal{A}} n_a \leq n, \tag{1}$$

of traded securities whose dividends are paid in consumption good a . The column vector of dividend rate processes associated with these securities is denoted by D^a and is assumed to be a nonnegative Itô process of the form

$$D_t^a = D_0^a + \int_0^t (\rho_s^a ds + \mathcal{G}_s^a dB_s) \tag{2}$$

for some vector-valued drift ρ^a and matrix-valued volatility \mathcal{G}^a . In what follows, we denote by D the \mathbf{n} -dimensional column vector obtained by stacking up the good-specific dividend vectors $(D^a)_{a \in \mathcal{A}}$ and assume that its volatility matrix \mathcal{G} has full row rank. This assumption appears natural given that the focus of this paper is to identify necessary and sufficient conditions for the existence of no-trade equilibria. Indeed, it allows us to exclude the non-generic cases where a no-trade equilibrium exists simply because the dividends are designed to pay exactly the agents' optimal consumption plans.

The initial value of the savings account is normalized to one and we assume that in equilibrium its price process is given by

$$S_t^0 = \exp\left(\int_0^t r_s ds\right) \tag{3}$$

for some instantaneous interest rate process r such that the above integral is well defined. For each consumption good $a \in \mathcal{A}$, we denote by S^a the vector of prices of the stocks whose dividends are paid in good a and assume that

$$S_t^a + \int_0^t p_s^a D_s^a ds = S_0^a + \int_0^t (\mu_s^a ds + \sigma_s^a dB_s) \tag{4}$$

for some vector-valued drift μ^a and matrix-valued volatility σ^a such that the above integrals are well defined. The security price coefficients $(r, \{\sigma^a, \mu^a\})$, or equivalently the security price processes $(S^0, \{S^a\})$, as well as the vector p of relative goods prices, are to be determined endogenously in equilibrium.

2.4. Trading strategies

Trading takes place continuously and there are no frictions such as transaction costs or taxes. Given the security prices, a trading strategy is a collection of share holdings

$$\theta := (\theta^0, \{\theta^a : a \in \mathcal{A}\}), \tag{5}$$

where θ^0 represents the number of shares of the savings account held in the portfolio and, for each consumption good $a \in \mathcal{A}$, the vector process θ^a represents the number of shares held in the portfolio of each of the stocks paying dividends in good a . A trading strategy θ

is said to be admissible if the associated wealth process, which is defined by

$$W_t := \theta_t^0 S_t^0 + \sum_{a \in \mathcal{A}} (\theta_t^a)^\top S_t^a, \quad (6)$$

is uniformly bounded from below by a constant. In what follows, we denote by Θ the set of all admissible trading strategies. The requirement that the wealth process of an admissible trading strategy be bounded from below is standard in the literature. It rules out the possibility of doubling strategies and thus implies that the set Θ is free of arbitrage opportunities, see [Dybvig and Huang \(1988\)](#).

2.5. Preferences and endowments

The economy is populated by two price-taking agents indexed by $i \in \{1, 2\}$. The preferences of agent i over consumption plans in \mathcal{C} are represented by a time-additive expected utility functional

$$U_i(c) := \mathbb{E} \left[\int_0^T u_i(t, c_t) ds \right]. \quad (7)$$

Throughout the paper we assume that the utility function u_i satisfies textbook regularity, monotonicity, and concavity assumptions as well as a multi-good version of the Inada conditions. In order to guarantee that certain expectations can be differentiated under the integral sign, we further assume that the utility function satisfies a rather weak technical condition which is stated and discussed in the Appendix. We note for later use that, as a result of the above assumptions, the utility gradient ∇u_i is a one-to-one mapping and hence admits an inverse function which we denote by f_i .

Agent i is initially endowed with a portfolio consisting of $v_i^{ak} \geq 0$ shares of each of the available stocks, where v_i^{ak} refers to the k th stock paying dividends in good a . We assume, without loss of generality, that the agents' initial portfolios verify the identity

$$v_1^{ak} + v_2^{ak} = 1, \quad (a, k) \in \mathcal{A} \times \{1, \dots, n_a\} \quad (8)$$

so that the net supply of each of the stocks is normalized to one unit. For further reference we also denote by v_i^a the vector of the number of shares of stocks paying dividends in good a in the initial portfolio of agent i . As observed by [Cass and Pavlova \(2004\)](#), agent i 's endowment maybe negative in some stocks as long as the initial market value of the portfolio is nonnegative. Given that our prime interest is to establish the existence of no-trade equilibria, we restrict our analysis to nonnegative endowments.

2.6. Feasible consumption plans

A consumption plan $c \in \mathcal{C}$ is said to be feasible for agent i if there exists an admissible trading strategy $\theta \in \Theta$ whose associated wealth process satisfies agent i 's dynamic budget constraint

$$W_0 = \theta_0^0 + \sum_{a \in \mathcal{A}} (\theta_0^a)^\top S_0^a = W_0^i := \sum_{a \in \mathcal{A}} (v_i^a)^\top S_0^a, \quad (9)$$

$$dW_t = \theta_t^0 dS_t^0 + \sum_{a \in \mathcal{A}} ((\theta_t^a)^\top (dS_t^a + p_t^a D_t^a dt) - p_t^a c_t^a dt) \quad (10)$$

and for which terminal wealth is nonnegative. We denote by \mathcal{C}_i the set of feasible consumption plans for agent i . Note that this set is not empty as agent i is initially endowed with a long position in each of the stocks.

2.7. Definitions of equilibrium

In what follows we denote by $\mathcal{E} := ((\Omega, \mathcal{F}, \mathbf{F}, P), \{u_i, v_i^a\}, \{D^a\})$ the primitives for the above continuous-time economy. Our concept of equilibrium is similar to that of the equilibrium of plans, prices, and expectations introduced by Radner (1972) and is defined in the following:

Definition 1 (*Financial market equilibrium*). An equilibrium for the continuous-time economy \mathcal{E} is a set of security prices $(S^0, \{S^a\})$, a relative price process p , and a set of consumption plans and admissible trading strategies $\{c_i, \theta_i\}$ such that:

1. The consumption plan c_i maximizes U_i over the feasible set \mathcal{C}_i and is financed by the admissible trading strategy $\theta_i \in \Theta$.
2. The securities and goods markets clear in the sense that

$$\theta_{1t}^0 + \theta_{2t}^0 = 0, \tag{11}$$

$$\theta_{1t}^a + \theta_{2t}^a = \mathbf{1}_a, \quad \text{and} \tag{12}$$

$$c_{1t}^a + c_{2t}^a = \mathbf{1}_a^\top D_t^a \tag{13}$$

hold for all $a \in \mathcal{A}$ and $t \in [0, T]$ where $\mathbf{1}_a$ denotes an n_a -dimensional column vector of ones.

In our model the dividend processes of the traded securities are linearly independent since their volatility matrix has full rank. However, because there are fewer traded securities than there are Brownian motions, the equilibria for the economy \mathcal{E} have incomplete financial markets in general. Furthermore, and as demonstrated by Cass and Pavlova (2004), the equilibrium may very well have incomplete financial markets even if there are as many traded securities as there are Brownian motions. Given this observation, and in order to facilitate our study, we further restrict ourselves to equilibria that are efficient in the sense that they yield Pareto-optimal allocations given the asset structure.³ While the full set of equilibria is in general very hard to characterize (see, e.g., Cuoco and He, 1994), that of efficient equilibria is more easily analyzed. Indeed, by the Pareto optimality of equilibrium allocations, there exists a strictly positive constant λ such that

$$\nabla u_1(t, c_{1t}) = \lambda \nabla u_2(t, c_{2t}), \tag{14}$$

where ∇u_i denotes the gradient of agent i 's utility function. Along with the goods market-clearing condition, the above restriction implies that the individual consumption allocations solve the maximization problem

$$u(t, \lambda, \delta_t) := \max_{c_1 + c_2 = \delta_t} \{u_1(t, c_1) + \lambda u_2(t, c_2)\}, \tag{15}$$

³In static models this notion is also referred to as constrained Pareto optimality or Pareto optimality in the Diamond sense (see Diamond, 1967; or Magill and Quinzii, 1996).

where the process $\delta = (\delta^a)$ with $\delta_t^a := \mathbf{1}_a^\top D_t^a$ denotes the vector of good-specific aggregate dividends. As a result, every efficient equilibrium can be supported by a representative agent endowed with the aggregate supply of securities and with utility function $u(t, \lambda, \cdot)$ even though the resulting financial markets might be incomplete. In order to facilitate the presentation of our main results, we briefly review this characterization in the next section.

Our main objective is to present necessary and sufficient conditions for the existence of an efficient no-trade equilibrium. Therefore, we next define the notion of no-trade equilibrium that we shall be using throughout the paper.

Definition 2 (No-trade equilibrium). An equilibrium for the economy \mathcal{E} is a no-trade equilibrium if it satisfies the following three assertions:

1. Portfolio shares of risky assets are constant over time: $\theta_{it}^a = \theta_{i0}^a$.
2. Agents do not hold the riskless asset: $\theta_{i0}^0 = \theta_{it}^0 = 0$.
3. There is no activity in the spot market for goods: $c_{it}^a = (\theta_{i0}^a)^\top D_t^a$.

In a no-trade equilibrium, trading volume is zero after the initial period in any of the financial assets. Furthermore, there is no activity in the spot markets in the sense that the optimal consumption plans coincide with the dividend payments that the agents receive from their portfolios. As the dividends in each good vary stochastically over time, optimal consumption varies as well. But in contrast to the equilibria in Definition 1, the resulting fluctuations in consumption are not smoothed by trades in either of the open markets.

In the context of a multi-country Lucas model, where our results most naturally apply, a no-trade equilibrium does not guarantee the absence of geographical trade. Indeed, if the domestic agent holds a fraction of a foreign asset, the dividend paid in a foreign good must be shipped to the domestic country for consumption.⁴ We further discuss the implications of our results for international finance models in Section 5.4.

3. Preliminary results

3.1. A useful notation

In order to simplify the presentation of our results, we now introduce a vector notation which will be used repeatedly in what follows. For an arbitrary collection $(x^a)_{a \in \mathcal{A}}$ of vectors with $x^a \in \mathbf{R}^{n_a}$, we use the shorthand notation $\Phi(x^a)$ to denote the

⁴Basak and Croitoru (2004) have recently shown that this kind of financial shipping can relieve market imperfections causing segmentations of good markets, as it substitutes for physical shipping of consumption goods. It is important to note that the substitutability of physical and financial shipping is an idiosyncratic feature of the Lucas tree model. As emphasized by Cass and Pavlova (2004), initial endowments in Lucas tree models are given in terms of property rights on the dividend streams risky assets pay in the future and not in terms of commodities as in the benchmark real asset model from financial equilibrium theory, introduced by Magill and Shafer (1990).

rectangular matrix

$$\Phi(x^a) := \begin{bmatrix} x^{1*} & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & x^{2*} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & x^{A*} & \dots \end{bmatrix} \in \mathbf{R}^{A \times n}. \tag{16}$$

The linear operator Φ enables us to transform a collection of good-specific vectors into a matrix. This simplifies the notation in consumption and portfolio computations. In particular, the vector of good-specific aggregate dividend processes is given by $\delta_t = \mathbb{1}D_t$ where $\mathbb{1} := \Phi(\mathbf{1}_a)$.

3.2. Individual optimality

Let the security and goods prices be given and assume that there are no arbitrage opportunities, for otherwise the market could not be in equilibrium. As is well known (see, e.g., Karatzas and Shreve, 1998), this assumption implies that there exists an n -dimensional process κ , which is referred to as a relative risk premium, such that

$$\mu_t^a - r_t S_t^a = \sigma_t^a \kappa_t, \quad a \in \mathcal{A}. \tag{17}$$

Let \mathcal{K} denote the set of relative risk premiums, and for every such process consider the nonnegative process defined by

$$\xi_t^\kappa := \exp\left(-\int_0^t r_s ds - \int_0^t \kappa_s^\top dB_s - \frac{1}{2} \int_0^t \|\kappa_s\|^2 ds\right). \tag{18}$$

The following proposition shows that the set $\mathcal{S} := \{\xi^\kappa : \kappa \in \mathcal{K}\}$ coincides with the set of arbitrage-free state price densities and provides a convenient necessary and sufficient condition for the optimality of a given consumption plan.

Proposition 1. *If security and goods prices are given, then the following assertions hold:*

1. *A consumption plan is feasible for agent i if and only if it satisfies the static budget constraint*

$$\mathbb{E} \left[\int_0^T \xi_t^\kappa p_t^\top c_t dt \right] \leq W_0^i \tag{19}$$

for all market price of risk processes $\kappa \in \mathcal{K}$ where the initial wealth W_0^i is defined as in Eq. (9).

2. *A feasible consumption plan is optimal for agent i if and only if*

$$c_t = f_i(t, y_i p_t \xi_t^{\kappa_i}) \tag{20}$$

for some strictly positive constant y_i and some process $\kappa_i \in \mathcal{K}$ such that Eq. (19) holds as an equality.

The results of the proposition can be summarized as follows. The first part shows that a consumption plan is feasible if and only if it satisfies a static budget constraint with respect

to each of the arbitrage-free state price densities. The second part establishes that a feasible consumption plan is optimal if and only if its marginal utility defines an arbitrage-free state price density process for which the static budget is saturated.

3.3. A characterization of efficient equilibria

Assume that there exists an efficient equilibrium for the economy \mathcal{E} , denote by $\{c_i\}$ the corresponding consumption allocations, and let the representative-agent utility function be defined as in (15).

Since consuming the good-specific aggregate dividends must be optimal for the representative agent, it follows from the second part of Proposition 1 that the process of marginal rates of substitution

$$p_t \xi_t := \frac{\nabla u(t, \lambda, \delta_t)}{\nabla_1 u(0, \lambda, \delta_0)} = \frac{\nabla u_i(t, c_{it})}{\nabla_1 u_i(0, c_{i0})} \tag{21}$$

identifies the vector of good-specific equilibrium state prices. Moreover, Pareto optimality implies that the consumption allocations solve the representative agent’s optimization problem and it follows that

$$c_{1t} := f_1(t, \nabla u(t, \lambda, \delta_t)), \quad c_{2t} := \delta_t - c_{1t} = f_2(t, \nabla u(t, \lambda, \delta_t)/\lambda), \tag{22}$$

where $f_i(t, \cdot)$ denotes the inverse of agent i ’s gradient mapping. On the other hand, the definition of the set of state price densities and the absence of arbitrage opportunities imply that the process

$$N_t^a := \xi_t S_t^a + \int_0^t \xi_s p_s^a D_s^a ds \tag{23}$$

is a martingale under the objective probability measure.⁵ It follows that in any efficient equilibrium the security prices satisfy

$$S_t^a = E \left[\int_t^T \frac{\xi_s p_s^a D_s^a ds}{\xi_t} \middle| \mathcal{F}_t \right], \quad a \in \mathcal{A}, \tag{24}$$

where the good-specific state price $p^{a\xi}$ is defined as in (21) for some strictly positive λ such that the first agent’s budget constraint holds as an equality.

As the financial market is in general incomplete, it is very difficult to check directly that Eq. (22) defines a pair of feasible consumption plans given the security prices in Eq. (24). As a result, the above characterization cannot be used to construct an efficient equilibrium unless we can verify a priori that Eq. (24) defines a complete financial market. However, if we restrict our attention to efficient equilibria that have no trade, then the situation becomes much simpler. Indeed, for such equilibria the consumption plans in Eq. (22) are linear functions of the dividends and are thus feasible by construction. In the next section, we use this argument to obtain sufficient conditions on the primitives of the economy for the existence of an efficient no-trade equilibrium.

⁵Strictly speaking, the absence of arbitrage opportunities and the definition of \mathcal{S} only imply that the process N^a is a local martingale. The technical argument required to show that this process is a real martingale is provided in the Appendix after the proof of Proposition 1.

4. Equilibrium trading volume

4.1. The no-trade theorem

We now turn to this paper’s main topic and investigate conditions under which an efficient equilibrium generates trade. While it is well understood that heterogeneity among agents should generate trade in both goods and financial markets, necessary and sufficient conditions for trade in a dynamic model are lacking. Our main result fills this gap and comes in the form of a no-trade theorem.

Theorem 1. *The following assertions are equivalent:*

1. *There exists an efficient no-trade equilibrium.*
2. *There exists an efficient equilibrium in which the individual consumption allocation satisfies*

$$\frac{c_{it}^a}{c_{i0}^a} = \frac{\mathbf{1}_a^\top D_t^a}{\mathbf{1}_a^\top D_0^a} = \frac{\delta_t^a}{\delta_0^a}, \quad (a, t, i) \in \mathcal{A} \times [0, T] \times \{1, 2\}, \tag{25}$$

where δ_t^a denotes the aggregate output of good $a \in \mathcal{A}$ at time $t \in [0, T]$.

3. *There exists a diagonal matrix w with strictly positive, constant diagonal elements $w^a \in (0, 1)$ such that*

$$\frac{\nabla u_1(t, w\delta_t)}{\nabla_1 u_1(0, w\delta_0)} = \frac{\nabla u_2(t, \delta_t - w\delta_t)}{\nabla_1 u_2(0, \delta_0 - w\delta_0)}, \quad t \in [0, T] \tag{26}$$

and

$$\mathbb{E} \left[\int_0^T \nabla u_1(t, w\delta_t)^\top (w\mathbb{1} - \Phi(v_1^a)) D_t dt \right] = 0, \tag{27}$$

where $\mathbb{1}$ is the rectangular matrix defined in Section 3.1 and $\delta = \mathbb{1}D$ denotes the vector of good-specific aggregate dividends.

The results of the above theorem can be summarized as follows. Assertion 2 shows that in an efficient no-trade equilibrium, the consumption policies of each of the agents must exhibit the same growth rate as the corresponding good-specific aggregate output and that, given the existence of an efficient equilibrium, this property is also sufficient for the existence of a no-trade equilibrium. The third assertion states that a no-trade equilibrium exists if and only if the utility gradients of the agents can be aligned along a linear sharing rule that satisfies either of the static budget constraints. This third assertion is the most important from a practical point of view, as it provides necessary and sufficient conditions for the existence of an efficient no-trade equilibrium in terms of the model primitives. In the next section we review most of the classic forms of multi-good utility functions and use the third assertion to determine the minimal level of heterogeneity needed to generate nontrivial trading volume.

The conclusions of Theorem 1 share some close connections with classical results on linear sharing rules and fund separation. These connections, as well as the implications of Theorem 1 for international finance models, are discussed in Section 5.

4.2. Examples

In this section, we illustrate the implications of Theorem 1 for some common classes of utility functions. For simplicity of exposition we assume throughout this section that there are only two consumption goods ($A = 2$) and that there is only one security paying out in each of the two available consumption goods ($n_1 = n_2 = 1$).

4.2.1. Constant elasticity of substitution (CES)

As a first example, we consider the class of CES utility functions. Agents' utility functions display constant elasticity of substitution if they take the parametric form

$$u_i(t, c) := e^{-kt} [\alpha_{i1}(c^1)^{\rho_i} + \alpha_{i2}(c^2)^{\rho_i}]^{\gamma_i/\rho_i}, \quad (28)$$

where k is a nonnegative constant subjective discount rate which we assume to be equal across agents. The preference parameters, $\rho_i \in (0, 1)$, $\gamma_i < 1$, and $\alpha_{ia} > 0$, are fixed constants. As is easily seen, such preferences have constant relative risk aversion γ_i and constant elasticity of substitution $1/(1 - \rho_i)$ determined by the parameter ρ_i .

Using the equivalent assertions of Theorem 1 we now show that, given such preferences, an efficient no-trade equilibrium exists if and only if the agents have cardinally identical preferences orderings.

Corollary 1. *Assume that agents have constant elasticity of substitution utility functions (28). Then an efficient no-trade equilibrium exists if and only if $\rho_1 = \rho_2$, $\gamma_1 = \gamma_2$ and $\frac{\alpha_{11}}{\alpha_{12}} = \frac{\alpha_{21}}{\alpha_{22}}$.*

Next, we consider two classes of preferences for which a closed form characterization of an efficient no-trade equilibrium can be obtained for general dynamics of dividends.

4.2.2. Nonseparable Cobb–Douglas preferences

Agents have nonseparable Cobb–Douglas preferences if their utility functions take the parametric form

$$u_i(t, c) := e^{-kt} a_i (c^1)^{\alpha_{i1}} (c^2)^{\alpha_{i2}}, \quad (29)$$

where k is a nonnegative constant subjective discount rate which we assume equal across agents, a_i is a positive constant, and $\alpha_{ia} \in (0, 1)$ are constants such that $\alpha_{i1} + \alpha_{i2} < 1$. Note that this parametric form is the limit of the constant elasticity of substitution specification as the coefficient ρ_i goes to zero. Thus, nonseparable Cobb–Douglas preferences are homogeneous of degree $\alpha_{i1} + \alpha_{i2}$ and have unit elasticity of substitution.

Corollary 2. *Assume that agents have nonseparable Cobb–Douglas preferences. Then an efficient no-trade equilibrium exists if and only if $\alpha_{1a} = \alpha_{2a}$. In this case, the equity holdings and consumption shares of the first agent satisfy*

$$\theta_{1t}^a = \frac{c_{1t}^a}{D_t^a} = w := \frac{\alpha_{11}v_1^1 + \alpha_{12}v_1^2}{\alpha_{11} + \alpha_{12}}, \quad a \in \mathcal{A}, \quad (30)$$

where $v_1^a \in [0, 1]$ is the number of shares of the stock paying in good a in the initial portfolio of the first agent.

Corollary 2 shows that a necessary and sufficient condition for the existence of an efficient no-trade equilibrium is that all agents have equal preference weights for consumption goods in the nonseparable Cobb–Douglas utility function. Even though

agents with such preferences have identical elasticities of substitution by definition, an efficient no-trade equilibrium only exists if the weights for each good are the same. The only heterogeneity allowed is given by the constant a_i . As von Neumann–Morgenstern preferences are unique up to affine transformations, it follows that an efficient no-trade equilibrium with nonseparable Cobb–Douglas preferences exists if and only if agents have cardinally identical preference orderings.

The results of Section 3.3 and Corollary 2 allow us to bring to light a striking property of efficient equilibria with nonseparable Cobb–Douglas preferences. Indeed, using Eq. (21) we find that in any such equilibrium the relative price is proportional to the ratio of aggregate dividends and given by

$$p_t^2 = \frac{\alpha_{12} D_t^1}{\alpha_{11} D_t^2}. \quad (31)$$

This implies that the dividends of the two stocks are linearly dependent when expressed in units of the numéraire consumption good and it now follows from Eq. (24) that the equilibrium stock prices satisfy

$$S_t^2 = \frac{\alpha_{12}}{\alpha_{11}} S_t^1. \quad (32)$$

As a consequence, the volatility matrix of the risky assets is singular and it follows that, in any efficient equilibrium, both stocks represent the same investment opportunity. This provides an intuitive explanation for the existence of a no-trade equilibrium with cardinally equivalent Cobb–Douglas preferences. As the volatility matrix of the risky assets is singular in any efficient equilibrium, the trading strategies that implement the equilibrium consumption allocation are not uniquely defined. The special form of the relative price process then implies that these strategies can be chosen in such a way that there is no trade in the financial markets after the initial period as well as no activity in the goods markets and it follows that an efficient no-trade equilibrium exists.

Cass and Pavlova (2004) show that if the agents have log-linear preferences, then the stock prices will be linearly dependent in any efficient equilibrium, and they label this situation *Peculiar Financial Equilibrium*. The result of Corollary 2 and the above discussion show that this type of equilibrium also occurs if the agents have identical Cobb–Douglas utility functions.

4.2.3. Log-linear preferences

Agents have log-linear preferences if their utility functions take the parametric form

$$u_i(t, c) := e^{-kt} (\alpha_{i1} \log(c^1) + \alpha_{i2} \log(c^2)), \quad (33)$$

where k is a nonnegative constant subjective discount rate which we assume equal across agents and the α_{ia} are strictly positive, agent-specific constants. This specification of preferences is popular for its tractability and has been used in numerous studies including Zapatero (1995) and Cass and Pavlova (2004).

Corollary 3. *Assume that agents have log-linear preferences (33). Then an efficient no-trade equilibrium exists for all $(\alpha_{ia}) \in (0, \infty)^4$. In this equilibrium the equity holdings and*

consumption shares of the first agent satisfy

$$\theta_{1t}^1 = \frac{c_{1t}^1}{D_t^1} = w^1 := \frac{\alpha_{11}(\alpha_{21}v_1^1 + \alpha_{22}v_1^2)}{\alpha_{12}\alpha_{21}(1 - v_1^2) + \alpha_{11}(\alpha_{21} + \alpha_{22}v_1^2)}, \tag{34}$$

$$\theta_{1t}^2 = \frac{c_{1t}^2}{D_t^2} = w^2 := \frac{\alpha_{12}(\alpha_{21}v_1^1 + \alpha_{22}v_1^2)}{\alpha_{11}\alpha_{22}(1 - v_1^1) + \alpha_{12}(\alpha_{22} + \alpha_{21}v_1^1)}, \tag{35}$$

where $v_i^a \in [0, 1]$ is the number of shares of the stock paying in good a in agent i 's initial portfolio.

As in the previous example, we can recover the relative price and stock prices from the results of Section 3.3. Indeed, Eq. (21) identifies the vector of good-specific state prices as

$$p_t \xi_t = \frac{\nabla u_i(t, c_{it})}{\nabla_1 u_i(0, c_{i0})} = \left[e^{-kt} \frac{D_0^1}{D_t^1}, \frac{\alpha_{12}w^1 D_t^1}{\alpha_{11}w^2 D_t^2} \right]^\top \tag{36}$$

and plugging this back into the pricing relations (24) shows that the equilibrium prices of the risky securities satisfy

$$S_t^1 = \frac{1}{k}(1 - e^{-k(T-t)})D_t^1 = \frac{\alpha_{11}w^2}{\alpha_{12}w^1}S_t^2. \tag{37}$$

With this particular form of utility function, the price of the first stock is a linear function of the first dividend, and as the relative price of the second good is inversely proportional to the ratio of dividends, the price of the second stock is also a linear function of the first dividend. It follows that the stock volatility matrix is degenerate and hence that the no-trade equilibrium is a peculiar financial equilibrium as was the case for identical Cobb–Douglas preferences in the previous example.

While efficient no-trade equilibria and peculiar financial equilibria coincide for log-linear and Cobb–Douglas preferences, it is important to note that this is not the case in general. In Section 5.4 we establish that a sufficient condition for an efficient no-trade equilibrium to be a peculiar financial equilibrium is that agents have unit-elastic utility functions. This condition is in turn equivalent to the fact that the relative price is proportional to the ratio of dividends and we provide an example to show that if this property fails the efficient no-trade equilibrium need not be of the peculiar type.

4.2.4. Separable Cobb–Douglas preferences

Agents have separable Cobb–Douglas preferences if their utility function takes the parametric form

$$u_i(t, c) := \sum_{a \in \mathcal{A}} e^{-kt} \frac{1}{\alpha_{ia}} (c^a)^{\alpha_{ia}}, \tag{38}$$

where k is a nonnegative constant subjective discount rate which we assume equal across agents and the $\alpha_{ia} \in (-\infty, 1)$ are nonzero constants that determine the agent's relative risk aversion in each of the goods. Note that, contrary to the three other examples of this section, separable Cobb–Douglas preferences have a non-constant elasticity of substitution between goods.

Using the equivalent assertions of Theorem 1, we now show that an efficient no-trade equilibrium exists if and only if the agents' preferences exhibit the same relative risk aversion for consumption in each of the two goods.

Corollary 4. *Assume that agents have separable Cobb–Douglas preferences (38). Then an efficient no-trade equilibrium exists if and only if $\alpha_{1a} = \alpha_{2a}$ for all $a \in \mathcal{A}$.*

In the absence of non-traded goods, this specification of the utility functions is a special case, with identical discount rates, of that employed in Serrat (2001). In his Section 3.2 Serrat claims that in the absence of non-traded goods agents follow buy-and-hold strategies in equilibrium. Using the above results, we note that this claim is only valid provided that the subjective discount rate is the same for the two agents.

5. The relevance of no-trade equilibria

In this section we discuss the link between the existence of no-trade equilibria and linear risk tolerance, elasticities of substitution, fund separation, discrete-time stationary Markov equilibria, and international asset pricing.

5.1. No-trade equilibria and linear sharing rules

Borch (1962), Wilson (1968), and Huang and Litzenberger (1985) have shown that a necessary and sufficient condition for the generic optimality of linear sharing rules in single-good, static economies is that all agents have linear risk tolerance with identical cautiousness parameters. Our results can be viewed as a generalization of theirs to the case of dynamic economies with multiple consumption goods.

To see this, consider the single-good case with time-independent utility functions. We start by observing that, since consumption must be positive at all times, market clearing implies that any linear sharing rule must have a zero intercept in order to be feasible. Thus, it follows from the second assertion of Theorem 1 that given the existence of an efficient equilibrium, the generic optimality of linear sharing rules is equivalent to the generic existence of an efficient no-trade equilibrium. Using Assertion 3, this is in turn equivalent to the fact that for any aggregate dividend process and any initial allocation the budget constraint (27) holds, and in addition, there exists a constant $w \in (0, 1)$ such that

$$\tau_1(wD) = \tau_2((1-w)D), \quad D \in (0, \infty), \quad (39)$$

where τ_i denotes the relative risk tolerance of agent i . For these equations to admit a solution in $(0, 1)$ regardless of the aggregate dividend and the initial allocations, it is necessary and sufficient that both agents have the same constant relative risk tolerance parameter. We thus conclude that in a single-good, continuous-time economy a necessary and sufficient condition for the generic optimality of linear sharing rules is that both agents have the same constant relative risk aversion utility function.

In the multi-good setting, the situation is less simple. The generic optimality of linear sharing rules is still equivalent to the generic existence of an efficient no-trade equilibrium. However, the latter property can no longer be characterized explicitly in terms of the agents' utility functions unless we assume that agents have separable utility functions of

the form

$$u_i(t, c) = \sum_{a \in \mathcal{A}} e^{-kt} u_i^a(c^a) \quad (40)$$

for some nonnegative subjective discount rate which is common to both agents. In this case, it follows from the third assertion of Theorem 1 that the generic existence of an efficient no-trade equilibrium is equivalent to the generic existence of strictly positive constants $w^a < 1$ such that

$$\tau_1^a(w^a x) = \tau_2^a((1 - w^a)x), \quad (x, a) \in (0, \infty) \times \mathcal{A}, \quad (41)$$

and the budget constraint (27) holds where τ_i^a denotes the relative risk tolerance of agent i for consumption in good a . For these equations to admit a solution for all aggregate dividends and initial allocations, it is necessary and sufficient that both agents have the same constant relative risk tolerance parameter for consumption in each of the goods. We thus conclude that in a multi-good, continuous-time economy with separable preferences, a necessary and sufficient condition for the generic optimality of linear sharing rules is that agents have identical separable Cobb–Douglas preferences.

5.2. No-trade equilibria and fund separation

In the single good case, it is well known from Hakanson (1967) and Cass and Stiglitz (1970), that the optimality of linear sharing rules, and hence also the existence of an efficient no-trade equilibrium, is related to fund separation. In order to explore this connection, assume first that an efficient no-trade equilibrium exists. By the second assertion of Theorem 1, this implies that the equilibrium sharing rule is linear and since there is no activity in the goods market, it follows that the wealth of the agents are given by

$$W_t^1 = \sum_{a \in \mathcal{A}} w^a M_t^a \quad \text{and} \quad W_t^2 = \sum_{a \in \mathcal{A}} (1 - w^a) M_t^a \quad (42)$$

for some strictly positive constants w^a where the process $M^a := \mathbf{1}_a^\top S^a$ denotes the value of the market portfolio of assets which pay their dividends in good a . Thus, we conclude that a sufficient condition for A -fund separation to hold in a continuous-time economy with A consumption goods is that there exists an efficient no-trade equilibrium.

If there is a single consumption good, then this condition is also necessary. To see this, consider the single-good case and assume that one-fund separation holds so that each agent holds a constant fraction of the market portfolio. This implies that there is no trading on the financial market and, since consumption cannot be smoothed by any other means, it follows that the equilibrium sharing rule is linear. The second assertion of Theorem 1 then implies that there exists a no-trade equilibrium and we conclude that one-fund separation is necessary and sufficient for the existence of an efficient no-trade equilibrium.

In the multi-good setting the situation is less simple and it is no longer possible to show that the existence of an efficient no-trade equilibrium is necessary for A -fund separation to hold. The reason for this impossibility is twofold. First, nothing guarantees that the mutual funds correspond to static portfolios of the underlying risky assets. Second, even if the mutual funds do correspond to buy-and-hold portfolios, this does not imply that the

equilibrium sharing rule is linear because of the possibility of trading in the spot market for goods.

5.3. No-trade equilibria in discrete time economies

A careful inspection of the proof of Theorem 1 reveals that the only place where the assumptions of continuous time and Itô process dynamics are used is in the proof of the fact that Assertion 1 implies Assertion 2. It follows that, after suitable modifications of the basic model, the conditions of Assertion 3 are still sufficient for the existence of an efficient no-trade equilibrium in a discrete-time economy with multiple goods and finite or infinite horizon.

While sufficient for the existence of an efficient no-trade equilibrium, Assertion 3 is far from being necessary in a discrete-time economy with multiple goods. Indeed, it can easily be shown that the conditions of Assertion 3 remain sufficient if we replace (26) by the weaker requirement that

$$\nabla u_1(t, \Phi(\beta_a)D_t) = \lambda \nabla u_2(t, (\mathbf{1} - \Phi(\beta_a))D_t) \tag{43}$$

for some collection of nonnegative vectors $(\beta^a)_{a \in \mathcal{A}}$ and some strictly positive constant λ which represents the weight of the second agent in the construction of the representative agent’s utility function.

In a recent paper, Judd, Kubler, and Schmedders (2003) show that trading volume is generically zero in a discrete-time, single-good economy populated by heterogeneous agents. This seems to contradict our theorem. As they remark, however, their result relies on the strong distributional assumption of a time-homogeneous stationary Markov chain for the aggregate dividend. In that case, equilibrium consumption allocations inherit the time-homogeneity properties of the dividend process and it follows that there always exists a solution to Eq. (43) irrespective of the choice of the utility functions. To illustrate this let us consider a single good economy with N states of the world and N stocks paying linearly independent dividends modeled as a time homogeneous stationary Markov chain. The financial market in this discrete-time economy does not include a locally riskless savings account; to replace it we assume, as in Judd, Kubler, and Schmedders (2003), that one of the risky assets is a fixed-income security such as a coupon bond. In such a model, Eq. (43) may be rewritten as

$$u'_1(t, \beta^\top D_n) = \lambda u'_2(t, (\mathbf{1} - \beta)^\top D_n), \quad n \in \{1, \dots, N\}, \tag{44}$$

for some $\beta \in [0, 1]^N$ and some strictly positive constant λ where D_n denotes the vector of dividends of the risky assets in state n . When utility functions are time separable and discount rates are identical across agents, as in Judd, Kubler, and Schmedders (2003), this system may be rewritten without the time dependency as

$$u'_1(\beta^\top D_n) = \lambda u'_2((\mathbf{1} - \beta)^\top D_n), \quad n \in \{1, \dots, N\}. \tag{45}$$

The linear independence of the dividends and the fact that the marginal utilities are strictly decreasing imply that, for each strictly positive λ , this system admits a unique solution $\beta(\lambda)$ with $\beta(\infty) = 0$ and $\beta(0) = \mathbf{1}$. As a result, the constant λ can be chosen in such a way that the budget constraint (27) holds and it follows that an efficient no-trade equilibrium can be constructed irrespective of the choice of the agents’ utility functions.

5.4. International risk sharing

A natural context in which to apply our general results is that of international finance, where each agent is interpreted as being representative of a country and the relative prices of goods define the terms of trade.

Examples of studies which analyze the properties of international equilibria include Lucas (1982), Cole and Obstfeld (1991), and Zapatero (1995), who all use models with two countries, two consumption goods, one risky asset in each country, and time-separable utility functions of the form

$$u_i(t, c) = e^{-kt} v_i(c) \quad (46)$$

for some nonnegative subjective discount rate k which is assumed to be equal across agents.⁶ In order to facilitate comparison we use a similar setting throughout this section. In our notation, such a model corresponds to $A = 2$, $n_1 = n_2 = 1$ and hence coincides with the two-good model underlying the examples of the previous section.

In order to connect the results of this paper to those of the international finance literature, we start by defining two concepts of equilibrium that are commonly used in that literature.

Definition 3 (Autarky equilibrium). An autarky equilibrium for \mathcal{E} is an equilibrium with no activity in goods markets and financial markets, that is, a no-trade equilibrium with $\theta_{it}^a = v_i^a$ for all $(a, t, i) \in \mathcal{A} \times [0, T] \times \{1, 2\}$.

In a classic paper, Lucas (1982) shows that an efficient autarky equilibrium exists for the model of this section if both investors are identical in terms of preferences and endowments. The intuition for this result is straightforward: because agents are identical, each of them consumes half of the aggregate output in the two goods and no trade on either of the open markets is necessary as the initial portfolios of the agents produce exactly the equilibrium allocation.

The results of Theorem 1 show that efficient autarky equilibria can exist even if the agents are not identical in preferences and endowments. Indeed, it easily follows from the third assertion of Theorem 1 that if a no-trade equilibrium exists for a given preference structure, then an autarky equilibrium can be constructed by defining $v_1^a = w^a$ to be the initial endowment of the first agent. As illustrated in the first example of the previous section, this requires neither $v_1 = v_2$ nor that the agents be endowed with half of the risky assets.

Definition 4 (Portfolio autarky equilibrium). A portfolio autarky equilibrium is an equilibrium in which there is no activity in the financial markets: agents hold on to their initial stock allocations and optimal consumptions are attained by trading only in the spot market for goods.

In contrast to a no-trade equilibrium where there is no activity in the goods markets and where dividend payments from equity holdings finance the optimal consumption plans, in

⁶More sophisticated models have been studied by, among others, Serrat (2001), who uses a two-country model with two traded goods and two non-traded goods; Pavlova and Rigobon (2003), who analyze a two-country model with log-linear utilities and agent-specific stochastic discount rates; and Pavlova and Rigobon (2005), who consider a similar model with three countries, three goods, three risky assets, and portfolio constraints.

a portfolio autarky equilibrium agents exclusively use the spot markets to smooth their consumption. As can be seen from the definition, an efficient autarky equilibrium is an efficient no-trade equilibrium since it entails no trading in either of the open markets. On the other hand, the concept of no-trade equilibrium introduced in this paper is very different from that of portfolio autarky since it requires that there be no activity on the spot market for goods.

Cole and Obstfeld (1991) study the welfare gains from international risk sharing by analyzing conditions under which a portfolio autarky equilibrium is efficient. They start from an economy where agents are restricted to hold their domestic financial assets but can trade on the goods market. They show that if the agents have identical Cobb–Douglas utility then there exists an efficient portfolio autarky equilibrium. In other words, for some specific preference structure, international financial markets do not improve welfare in their model. They show, however, that simple perturbations such as non-tradable goods restore the need for international financial markets in reaching an efficient allocation. Surprisingly, and as can be seen from Corollary 2, for this preference structure an efficient no-trade equilibrium also exists and yields the same consumption allocation. In order to reconcile these findings, one needs to think in terms of the risk spanned by the domestic and foreign financial markets. If the same allocation can be achieved either by trading once in the financial markets and never after that, or by trading continuously in the goods market, this suggests that financial markets are somewhat redundant and that agents are indifferent with respect to their portfolios. In other words, we expect that in this situation the domestic and foreign financial markets represent identical investment opportunities and thus that the equilibrium is peculiar in the sense of Cass and Pavlova (2004). The following result formalizes this intuition and clarifies the relations between no-trade and portfolio autarky equilibria.

Proposition 2. *Assume that an efficient no-trade equilibrium exists. Then its consumption allocation can be implemented in portfolio autarky if and only if one of the following conditions holds:*

1. *It is an autarky equilibrium.*
2. *Agents have utility functions with unit elasticity of substitution.*

In the latter case, one of the two stocks is redundant and it follows that the equilibrium is peculiar in the sense of Cass and Pavlova (2004).

The conclusions of the above proposition are twofold. First, it shows that an efficient no-trade equilibrium that is not an autarky equilibrium can be implemented in portfolio autarky if and only if both agents have unit elasticity of substitution. In particular, and as illustrated in the first and last examples of Section 4.2, the economy can admit an efficient no-trade equilibrium even if it does not admit an efficient portfolio autarky equilibrium.

Second, it shows that whenever the two types of equilibrium coexist for a given economy, then the equilibrium is necessarily peculiar in the sense that one of the stocks is redundant. Interestingly, most of the tractable international models in the literature assume unit-elastic preferences and admit an efficient no-trade equilibrium. For example, Cole and Obstfeld (1991) consider the case of identical nonseparable Cobb–Douglas utility functions and Zapatero (1995) assumes log-linear preferences. Such models do not provide a role for international financial markets since the domestic and foreign assets span the

same risk. Therefore, while tractable, these models might not be economically relevant except to demonstrate the existence of efficient portfolio autarky equilibria.

Different approaches have been proposed to circumvent this difficulty while maintaining tractability. For example, Pavlova and Rigobon (2003) introduce individual specific preference shocks in an equilibrium model with log-linear preferences; and Cole and Obstfeld (1991) suggest introducing country specific non-traded goods while maintaining the assumptions of Cobb–Douglas utility for the traded goods.⁷ Another way to maintain tractability while relaxing the conditions of Proposition 2 is to use the concept of no-trade equilibrium. Indeed, since no-trade equilibria can exist without the restriction of unit elasticity of substitution, Theorem 1 makes it possible to build tractable international equilibrium models where the financial markets are not perfectly correlated. In order to construct an example of such a model, consider the case where the two agents have identical utility functions of the form

$$v_i(c) = \sum_{a \in \mathcal{A}} \frac{1}{\alpha_a} (c^a)^{\alpha_a} \quad (47)$$

for some nonzero constants $\alpha_a < 1$ and assume that the dividend processes of the two risky securities are independent Markov processes. As the agents have identical separable Cobb–Douglas utility functions, it follows from Corollary 3 that an efficient no-trade equilibrium exists. In this equilibrium, the optimal consumption of the first agent is given by $c_{1t} = wD_t$ for some diagonal matrix w with coordinates in $(0, 1)$ and it follows that

$$p_t^2 = \frac{(w^2 D_t^2)^{\alpha_2 - 1}}{(w^1 D_t^1)^{\alpha_1 - 1}} \quad (48)$$

identifies the relative price process. In particular, the corresponding allocation cannot be implemented in portfolio autarky since none of the conditions in Proposition 2 are satisfied. In order to show that this equilibrium does not have perfectly correlated financial markets, let us now turn to the stock prices. Given that the utility function is separable in the two goods and the dividends are Markov, the price of stock one does not involve the relative price process and is thus a function

$$S_t^1 = \varphi(t, D_t^1) \quad (49)$$

of time and the first dividend process. On the other hand, the price of the second stock involves the relative price process and is thus a function

$$S_t^2 := \psi(t, D_t^1, D_t^2) \quad (50)$$

of time and the two dividend processes. Under our assumptions, this implies that the volatility matrix of the stock price process is invertible and hence that international financial markets are not perfectly correlated.

Cole and Obstfeld (1991) numerically assess the importance of international financial markets for risk sharing by calculating the welfare loss induced by forcing portfolio autarky when agents have identical CES utility functions. It is of interest to note that the efficient equilibrium that they numerically compute is a no-trade equilibrium and that no particular restriction on the dividend processes are needed to ensure analytical tractability.

⁷This suggestion was studied further by Serrat (2001) who considered a continuous-time model with two countries, two traded goods, two stocks and one non-traded good in each country.

In fact, it follows from Corollary 1 that for CES utility functions, a no-trade equilibrium exists if and only if utility functions are cardinally identical across agents. Kollmann (2006) provides a discussion on the volume of trade obtained by considering agents with different elasticity of substitution.

6. Extensions

As demonstrated in Section 4.2, an efficient no-trade equilibrium can exist even if the agents do not have identical preferences. Given this result, one naturally wonders what other sources of heterogeneity could generate nontrivial equilibrium trading volume. In order to partially answer this question, we now briefly discuss two extensions of the basic model: one incorporating heterogeneous beliefs and one where the agents receive random flows of endowments through time.

To accommodate such extensions of the model we assume throughout this section that there are multiple goods but only one traded security per good. This is sufficient to illustrate that no-trade equilibria are generically not robust to the introduction of heterogenous beliefs or to the addition of other income.

6.1. Heterogeneous beliefs

Throughout the paper we have maintained the assumption that agents differ only through their utility functions and initial portfolios. In particular, we have assumed that the two agents share the same beliefs. Standard economic intuition suggests that heterogeneity in beliefs is likely to increase exchanges among agents. To clarify this point, assume that agent i 's beliefs are fully described by the density process Z^i of his subjective probability measure P^i relative to P and that his preferences are represented by a time additive expected utility functional

$$U_i(c) := E^i \left[\int_0^T u_i(t, c_t) dt \right] = E \left[\int_0^T Z_t^i u_i(t, c_t) dt \right], \tag{51}$$

where E^i is the expectation operator under P^i . Now assume that there exists an efficient no-trade equilibrium. In any such equilibrium, the Pareto optimality of the consumption allocations and the necessity of linear sharing rules for no trade, imply that

$$\nabla u_1(t, w\delta_t)Z_t^1 = \lambda \nabla u_2(t, \delta_t - w\delta_t)Z_t^2 \tag{52}$$

holds almost everywhere for some strictly positive constant λ and diagonal matrix w with strictly positive, constant diagonal elements $w^a < 1$. For this relation to hold, the divergence in beliefs must exactly compensate the potential divergence in marginal utilities. While it might be possible to construct such beliefs structures, they are non-generic and their economic relevance seems doubtful.⁸

⁸Pavlova and Rigobon (2003) consider an economy with multiple goods, log-linear preferences, and heterogeneous beliefs. For this economy, they show the existence of a complete market equilibrium in which the optimal portfolio strategies are buy-and-hold. The associated consumption shares, which depend on the divergence in beliefs, are stochastic and time varying. Therefore, the implementation of the equilibrium allocation requires continuous trading in the spot market for goods.

6.2. Random endowments

Let us now assume that, in addition to an initial portfolio of the traded securities, agents receive a random flow of endowment in each of the available goods. Denote by e_i^a the rate at which agent i receives his endowment in good a and assume that the corresponding vector of good-specific endowments is a bounded Itô process of the form

$$e_{it}^a = e_{i0}^a + \int_0^t (\zeta_{is}^a ds + (\tau_{is}^a)^\top dB_s) \tag{53}$$

for some exogenously given drift process ζ_i and volatility matrix τ_i . In such a setting, a consumption plan is said to be feasible for agent i if there exists an admissible trading strategy θ which implies a wealth process that satisfies the dynamic budget constraint

$$W_0 = \theta_0^0 + \sum_{a \in \mathcal{A}} \theta_0^a S_0^a = W_0^i := \sum_{a \in \mathcal{A}} v_i^a S_0^a, \tag{54}$$

$$dW_t = \theta_t^0 dS_t^0 + \sum_{a \in \mathcal{A}} (\theta_t^a (dS_t^a + p_t^a D_t^a dt) - p_t^a (c_t^a - e_t^a) dt), \tag{55}$$

with a nonnegative terminal value. In the following corollary we provide necessary and sufficient conditions for the existence of a no-trade equilibrium for the above continuous-time economy with random endowments. We state the results without proof, as they are simply obtained by replacing c_i^a by $c_i^a - e_i^a$ in the proof of Theorem 1.

Corollary 5. *The following assertions are equivalent:*

1. *There exists an efficient no-trade equilibrium.*
2. *There exists an efficient equilibrium in which the individual consumption policies satisfy*

$$\frac{c_{it}^a - e_{it}^a}{c_{i0}^a - e_{i0}^a} = \frac{D_t^a}{D_0^a}, \quad i \in \{1, 2\}, \tag{56}$$

where D^a is the dividend process associated with the only security paying out in good $a \in \mathcal{A}$.

3. *There exists a diagonal matrix ϕ with strictly positive, constant diagonal elements $(\phi^a)_{a \in \mathcal{A}}$ such that*

$$\frac{\nabla u_1(t, \phi D_t + e_{1t})}{\nabla_1 u_1(0, \phi D_0 + e_{10})} = \frac{\nabla u_2(t, D_t - \phi D_t + e_{2t})}{\nabla_1 u_2(0, D_0 - \phi D_0 + e_{20})}, \tag{57}$$

and

$$E \left[\int_0^T \nabla u_1(t, \phi D_t + e_{1t})^\top ((\phi - \Phi(v_1^a)) D_t - e_{1t}) dt \right] = 0, \tag{58}$$

where D denotes the vector of good-specific dividend processes and Φ is the linear operator defined in Section 3.1.

Note that it is always possible to construct the agents' endowment processes in such a way that, given the other primitives of the economy, there exists an efficient no-trade equilibrium. An example of such a construction, albeit in a slightly different setup, can be found in Constantinides and Duffie (1996). However, and as already observed for peculiar

financial equilibria by Cass and Pavlova (2004), the resulting endowment processes generally lie in a set of measure zero.

7. Conclusion

In this paper we investigate under what conditions non-informational heterogeneity among agents leads to positive trading volume in equilibrium. We provide necessary and sufficient conditions for the existence of an efficient no-trade equilibrium in a continuous-time economy with multiple goods, multiple securities, symmetric information, and homogeneous beliefs. We illustrate our results with numerous examples that include most of the classic multi-good utility functions. Relations with linear sharing rules, fund separation, autarky, and portfolio autarky equilibria are also addressed.

No-trade equilibria are computationally tractable and thus attractive for future empirical studies of the connections between financial markets, exchange rates, and spot markets for goods. In contrast to portfolio autarky equilibria they cannot necessarily be implemented by trades only in the spot market of consumption goods. Financial markets are non-redundant and, contrary to peculiar financial market equilibria, asset volatilities are non-degenerate in general. If extended to an overlapping generation setting where one cohort of investors is always in their initial period, these equilibria can potentially be used to derive tractable efficient equilibria with trade in both the spot market for consumption goods and the financial markets. The study of such a model would overcome some of the deficiencies of the classic international asset pricing models and is left for future research.

Appendix

Proof of Proposition 1. The first assertion follows directly from Theorem 8.5 in Karatzas, Lehoczky, Shreve, and Xu (1991) after some straightforward modifications to accommodate the presence of multiple goods and intermediate consumption.

The second assertion can be established in the same way as Theorem 9.3 of Karatzas, Lehoczky, Shreve, and Xu (1991) provided that the utility functional can be differentiated under the integral sign. To guarantee that this is indeed the case we assume that the utility function satisfies the growth condition

$$\limsup_{b \rightarrow \infty} \sup_{c \in \mathcal{C}(b)} \frac{c^\top \nabla_a u_i(t, c)}{u_i(t, c)} < 1, \quad (a, t) \in \mathcal{A} \times [0, T], \tag{59}$$

where $\mathcal{C}(b)$ denotes the set of nonnegative vectors whose lowest coordinate is larger than the nonnegative constant b . This condition is referred to as reasonable asymptotic elasticity and has proved crucial in the resolution of incomplete markets portfolio and consumption choice problems; see Kramkov and Schachermayer (1999) for the single-good case and Kamizono (2001) for the multi-good case. \square

Characterization of efficient equilibria. All there is to prove is that for each consumption good $a \in \mathcal{A}$ the local martingale N^a defined in (23) is a martingale. To this end, we start by observing that the wealth process of agent i along the equilibrium path is given by

$$W_t^i := E \left[\int_t^T \frac{\xi_s p_s^\top c_{is} ds}{\xi_t} \middle| \mathcal{F}_t \right], \tag{60}$$

where the vector p^ξ of good-specific state prices is defined as in (21). Summing the above expressions over i and using the goods market-clearing conditions, we deduce that the aggregate wealth in the economy is given by

$$M_t = W_t^1 + W_t^2 = E \left[\int_t^T \frac{\xi_s p_s^\top \delta_s ds}{\xi_t} \middle| \mathcal{F}_t \right] \tag{61}$$

and it follows that the nonnegative process defined by

$$Q_t = \xi_t M_t + \int_0^t \xi_s p_s^\top \delta_s ds \tag{62}$$

is a martingale of class \mathbb{D} under the objective probability measure. Now let $a \in \mathcal{A}$ be given and fix an arbitrary $k \in \{1, \dots, n_a\}$. The absence of arbitrage opportunities and the definition of the vector of aggregate dividends imply that we have $0 \leq N^{ak} \leq Q$. It follows that the local martingale N^{ak} is of class \mathbb{D} , and hence it is a martingale.

Proof of Theorem 1. To establish the implication $1 \Rightarrow 2$, assume that there exists an efficient no-trade equilibrium. First note that the optimal holding of the money market account must be zero. Indeed, as in the absence of arbitrage opportunities $S_T^a = \mathbf{0}_{n_a}$, it follows from the individual optimality that

$$0 = W_T^i = \theta_{iT}^0 S_T^0 + \sum_{a \in \mathcal{A}} (\theta_{iT}^a)^\top S_T^a = \theta_{iT}^0 S_T^0. \tag{63}$$

Observing that S_T^0 is strictly positive, we conclude that $\theta_{i0}^0 = \theta_{iT}^0 = 0$. On the other hand, applying Itô’s lemma to (6) and using the dynamic budget constraint, we obtain

$$p_t^\top (\Phi(\beta_a) D_t - c_{1t}) = 0, \tag{64}$$

where $\beta_a := \theta_{10}^a \in (0, 1)^{n_a}$. By definition of a no-trade equilibrium, there is no activity on the goods markets, so the above identity and the goods market-clearing conditions imply that the equilibrium consumption policies are linear in the dividends and given by

$$c_{1t} = \Phi(\beta_a) D_t, \quad c_{2t} = (\mathbb{1} - \Phi(\beta_a)) D_t. \tag{65}$$

Now, Pareto optimality of the equilibrium consumption allocations implies that the marginal utilities of the two agents are aligned in the sense that there exists a strictly positive constant λ such that

$$\nabla u_1(t, \Phi(\beta_a) D_t) = \lambda \nabla u_2(t, (\mathbb{1} - \Phi(\beta_a)) D_t). \tag{66}$$

Applying Itô’s lemma to both sides of the above equation and identifying the volatility coefficients, we obtain that

$$(\mathcal{H} u_1(t, \Phi(\beta_a) D_t) \Phi(\beta_a) - \lambda \mathcal{H} u_2(t, (\mathbb{1} - \Phi(\beta_a)) D_t) (\mathbb{1} - \Phi(\beta_a))) \mathfrak{D}_t = \mathbf{0} \tag{67}$$

almost surely for all $t \in [0, T]$ where \mathcal{H} denotes the Hessian matrix of second derivatives and \mathfrak{D} is the volatility matrix of the dividend processes. As the volatility matrix of the dividend processes has full row rank by assumption, this in turn implies

$$\mathcal{H} u_1(t, \Phi(\beta_a) D_t) \Phi(\beta_a) = \lambda \mathcal{H} u_2(t, (\mathbb{1} - \Phi(\beta_a)) D_t) (\mathbb{1} - \Phi(\beta_a)). \tag{68}$$

Using the definition of the mapping Φ in conjunction with the fact that the above equation holds almost everywhere, we easily deduce that

$$\mathcal{W}_a^{-1} \beta_a = \mathcal{W}_a^2 \mathbf{1}_a, \quad a \in \mathcal{A}, \tag{69}$$

where the constants \mathcal{W}_a^i denote the diagonal elements of the A -dimensional, negative definite square matrices defined by

$$\mathcal{W}^2 := \mathcal{H}u_2(0, (\mathbb{1} - \Phi(\beta_a))D_0), \tag{70}$$

$$\mathcal{W}^1 := \mathcal{H}u_1(0, \Phi(\beta_a)D_0) + \lambda\mathcal{W}^2. \tag{71}$$

Using the fact that the agents' utility functions are strictly concave, we can easily deduce that the constants \mathcal{W}_a^i satisfy $\mathcal{W}_a^1 < \mathcal{W}_a^2 < 0$ and it follows that we have $\beta_a = w^a \mathbf{1}_a$ for some strictly positive constant in $(0, 1)$. Plugging this result back into Eq. (65) gives the condition in Assertion 2 after some straightforward simplifications.

To establish the implication $2 \Rightarrow 3$, assume that there exists an efficient equilibrium satisfying the conditions of Assertion 2 and let w denote the diagonal matrix with strictly positive diagonal elements defined by

$$w^a := \frac{c_{10}^a}{\mathbf{1}_a^\top D_0^a}, \quad a \in \mathcal{A}. \tag{72}$$

Using the Pareto optimality of the equilibrium allocations in conjunction with the assumed form of the consumption plans, we obtain that there exists a strictly positive constant λ such that

$$\nabla u_1(t, w\delta_t) = \lambda \nabla u_2(t, \delta_t - w\delta_t). \tag{73}$$

Writing the first coordinate of this vector identity at time zero allows us to identify the Negishi weight as

$$\lambda = \frac{\nabla_1 u_1(0, w\delta_0)}{\nabla_1 u_2(0, \delta_0 - w\delta_0)} \tag{74}$$

and plugging this expression back into Eq. (73) gives the first condition in Assertion 3. On the other hand, the assumed form of the equilibrium allocations and the second part of Proposition 1 imply that

$$\nabla u_i(t, c_{it}) = y_i p_t \zeta_t^i \tag{75}$$

for some strictly positive constant y_i and some arbitrage-free state price density process $\zeta^i := \zeta^{k_i} \in \mathcal{S}$ such that

$$W_0^i = \Phi(v_t^i)S_0 = \mathbb{E} \left[\int_0^T \zeta_t^i p_t^\top w \delta_t dt \right], \tag{76}$$

where S denotes the \mathbf{n} -dimensional column vector obtained by stacking up the good-specific securities price vectors $(S^a)_{a \in \mathcal{A}}$. Using (75) in conjunction with the Pareto optimality of the equilibrium allocations, we deduce that

$$p_t \zeta_t^1 = p_t \zeta_t^2 = \frac{1}{y_1} \nabla u_1(t, w\delta_t) = \frac{1}{y_2} \nabla u_2(t, \delta_t - w\delta_t). \tag{77}$$

Using the first coordinate of the above equation at time zero to identify the constant y_1 and plugging the result into (76) with $i = 1$ then gives

$$\Phi(v_1^a)S_0 = \mathbb{E} \left[\int_0^T \frac{\nabla u_1(t, w\delta_t)^\top w \delta_t}{\nabla_1 u_1(0, w\delta_0)} dt \right]. \tag{78}$$

As it is efficient, the equilibrium can be supported by a representative agent with utility function $u(t, \lambda, \cdot)$ as in Eq. (15) even if the resulting markets are incomplete. Thus, it follows from the definition of the representative agent’s marginal utility and the results of Section 3.3 that the equilibrium securities prices satisfy

$$S_0^a = E \left[\int_0^T \frac{\nabla_a u_1(s, w\delta_t) D_t^a}{\nabla_1 u_1(0, w\delta_0)} dt \right]. \tag{79}$$

Plugging this expression back into Eq. (78) and rearranging the terms gives the second condition in Assertion 3.

In order to establish the implication $3 \Rightarrow 1$, and thus complete the proof of the theorem, we have to show that given a matrix w satisfying the conditions of Assertion 3 we can construct an efficient no-trade equilibrium. To this end, consider the trading strategies and consumption rates defined by

$$\theta_{1t}^0 = \theta_{2t}^0 = 0, \tag{80}$$

$$\theta_{1t}^a = w^a \mathbf{1}_a = \mathbf{1}_a - \theta_{2t}^a, \tag{81}$$

$$c_{1t}^a = w^a \delta_t^a = w^a \mathbf{1}_a^\top D_t^a = \delta_t^a - c_{2t}^a, \tag{82}$$

and let the securities and relative goods prices be given by

$$p_t := \frac{\nabla u_1(t, c_{1t})}{\nabla_1 u_1(t, c_{1t})} = \frac{\nabla u_2(t, c_{2t})}{\nabla_1 u_2(t, c_{2t})}, \tag{83}$$

$$S_t^a := E \left[\int_t^T \frac{\nabla_1 u_1(s, c_{1s})}{\nabla_1 u_1(t, c_{1t})} p_t^a D_t^a dt \middle| \mathcal{F}_t \right]. \tag{84}$$

As (i) all markets clear, (ii) there is no trading volume on any of the open markets and (iii) the marginal utilities of the two agents are aligned, in order to establish that the collection $(p, \{S^a\}, \{c_i, \theta_i\})$ constitutes an efficient no-trade equilibrium only requires proving that the consumption allocations are optimal given the securities prices. To this end, let ξ be the process defined by

$$\xi_t := \frac{\nabla_1 u_1(t, c_{1t})}{\nabla_1 u_1(0, c_{10})} = \frac{\nabla_1 u_2(t, c_{2t})}{\nabla_1 u_2(0, c_{20})} \geq 0. \tag{85}$$

Using the definition of c_i in conjunction with the definition of the securities prices and the second condition in Assertion 3, we have

$$E \left[\int_0^T \xi_t p_t^\top \{c_{it} - \Phi(v_i^a) D_t\} dt \right] = 0. \tag{86}$$

On the other hand, using the fact that for each $a \in \mathcal{A}$ the process

$$\xi_t S_t^a + \int_0^t \xi_s p_s^a D_s^a ds \tag{87}$$

is a martingale, we deduce that the process ξ belongs to the set \mathcal{S} of state price densities and since c_i is feasible by construction, it follows from the second part of Proposition 1 that the consumption plan c_i is optimal for agent i . \square

Proof of Corollary 1. According to the third assertion of Theorem 1, we have that an efficient no-trade equilibrium exists if and only if there are strictly positive constants

$w^a \in (0, 1)$ such that

$$\frac{\left(\frac{\alpha_{11}(w^1 D_t^1)^{\rho_1} + \alpha_{12}(w^2 D_t^2)^{\rho_1}}{\alpha_{11}(w^1 D_0^1)^{\rho_1} + \alpha_{12}(w^2 D_0^2)^{\rho_1}}\right)^{\frac{\gamma_1}{\rho_1}-1}}{\left(\frac{\alpha_{21}((1-w^1)D_t^1)^{\rho_2} + \alpha_{22}((1-w^2)D_t^2)^{\rho_2}}{\alpha_{21}((1-w^1)D_0^1)^{\rho_2} + \alpha_{22}((1-w^2)D_0^2)^{\rho_2}}\right)^{\frac{\gamma_2}{\rho_2}-1}} = \left(\frac{D_t^1}{D_0^1}\right)^{\rho_2-\rho_1} = \left(\frac{D_t^2}{D_0^2}\right)^{\rho_2-\rho_1} \tag{88}$$

and the static budget constraint (27) holds true. Since the dividend processes are linearly independent by assumption, the above equation holds if and only if $\rho_1 = \rho_2 = \rho$ and

$$\frac{\left(\frac{\alpha_{11}(w^1 D_t^1)^\rho + \alpha_{12}(w^2 D_t^2)^\rho}{\alpha_{11}(w^1 D_0^1)^\rho + \alpha_{12}(w^2 D_0^2)^\rho}\right)^{\frac{\gamma_1}{\rho}-1}}{\left(\frac{\alpha_{21}((1-w^1)D_t^1)^\rho + \alpha_{22}((1-w^2)D_t^2)^\rho}{\alpha_{21}((1-w^1)D_0^1)^\rho + \alpha_{22}((1-w^2)D_0^2)^\rho}\right)^{\frac{\gamma_2}{\rho}-1}} = 1. \tag{89}$$

Using once again the linear independence of the dividends, we deduce that the above equation admits a solution if and only if $\gamma_1 = \gamma_2 = \gamma$. Using these restrictions in the third assertion of Theorem 1 yields the following system of equations:

$$\frac{\alpha_{11}}{\alpha_{21}} \left(\frac{w_1}{1-w_1}\right)^{\rho-1} = \frac{\frac{\alpha_{11}(w_1 D_{10})^{\rho-1}(\alpha_{11}(w_1 D_{10})^\rho + \alpha_{12}(w_2 D_{20})^\rho)^{\frac{\gamma}{\rho}-1}}{\alpha_{21}((1-w_1)D_{10})^{\rho-1}(\alpha_{21}((1-w_2)D_{10})^\rho + \alpha_{22}((1-w_2)D_{20})^\rho)^{\frac{\gamma}{\rho}-1}}}{\frac{(\alpha_{11}(w_1 D_{1t})^\rho + \alpha_{12}(w_2 D_{2t})^\rho)^{\frac{\gamma}{\rho}-1}}{(\alpha_{21}((1-w_1)D_{1t})^\rho + \alpha_{22}((1-w_2)D_{2t})^\rho)^{\frac{\gamma}{\rho}-1}}}, \tag{90}$$

$$\frac{\alpha_{12}}{\alpha_{22}} \left(\frac{w_2}{1-w_2}\right)^{\rho-1} = \frac{\frac{\alpha_{11}(w_1 D_{10})^{\rho-1}(\alpha_{11}(w_1 D_{10})^\rho + \alpha_{12}(w_2 D_{20})^\rho)^{\frac{\gamma}{\rho}-1}}{\alpha_{21}((1-w_1)D_{10})^{\rho-1}(\alpha_{21}((1-w_2)D_{10})^\rho + \alpha_{22}((1-w_2)D_{20})^\rho)^{\frac{\gamma}{\rho}-1}}}{\frac{(\alpha_{11}(w_1 D_{1t})^\rho + \alpha_{12}(w_2 D_{2t})^\rho)^{\frac{\gamma}{\rho}-1}}{(\alpha_{21}((1-w_1)D_{1t})^\rho + \alpha_{22}((1-w_2)D_{2t})^\rho)^{\frac{\gamma}{\rho}-1}}}. \tag{91}$$

Manipulating this system yields the following set of equations:

$$\frac{\alpha_{11}}{\alpha_{21}} \left(\frac{w_1}{1-w_1}\right)^{\rho-1} = \frac{\alpha_{12}}{\alpha_{22}} \left(\frac{w_2}{1-w_2}\right)^{\rho-1}, \tag{92}$$

$$\frac{\alpha_{11}}{\alpha_{21}} \left(\frac{w_1}{1-w_1}\right)^\rho = \frac{\alpha_{12}}{\alpha_{22}} \left(\frac{w_2}{1-w_2}\right)^\rho, \tag{93}$$

which admits a solution if and only if

$$\frac{\alpha_{11}}{\alpha_{12}} = \frac{\alpha_{21}}{\alpha_{22}}, \tag{94}$$

in which case

$$w^1 = g(w^2) := \left(1 + \frac{1 - w^2}{w^2}\right)^{-1}. \tag{95}$$

Plugging this relation back into the static budget constraint (27), we obtain that an efficient no-trade equilibrium exists if and only if there exists a strictly positive constant $\phi \in (0, 1)$ such that

$$h(\phi) := \mathbb{E} \left[\int_0^T \nabla u_1(t, GD_t)^\top (G - \Phi(v_1^q)) D_t dt \right] = 0, \tag{96}$$

where G denotes the diagonal matrix with elements $g(\phi)$ and ϕ . Using well-known analytic arguments, as found for example in Detemple and Serrat (2003), it can be shown that under our assumptions the function h is continuous on the interval $(0, 1)$ with

$$h(0+) := \lim_{\phi \rightarrow 0} h(\phi) < 0 < h(1-) := \lim_{\phi \rightarrow 1} h(\phi). \tag{97}$$

This implies the existence of a point ϕ such that $h(\phi) = 0$ and it follows that there exists an efficient no-trade equilibrium. \square

Proof of Corollary 2. According to the third assertion of Theorem 1, an efficient no-trade equilibrium exists if and only if there are strictly positive constants $w^a \in (0, 1)$ such that

$$\alpha_{11} \left(\frac{D_t^1}{D_0^1}\right)^{\alpha_{11} - \alpha_{21}} = \alpha_{21} \left(\frac{D_t^2}{D_0^2}\right)^{\alpha_{22} - \alpha_{12}}, \tag{98}$$

$$\frac{\alpha_{12} w^1}{\alpha_{11} w^2} \left(\frac{D_t^1}{D_0^1}\right)^{\alpha_{11} - \alpha_{21}} = \frac{\alpha_{22} (1 - w^1)}{\alpha_{21} (1 - w^2)} \left(\frac{D_t^2}{D_0^2}\right)^{\alpha_{22} - \alpha_{12}} \tag{99}$$

and the static budget constraint (27) holds true. Since the dividend processes are linearly independent by assumption, we can deduce that the above equation holds if and only if $\alpha_{1a} = \alpha_{2a}$ and thus $w^1 = w^2 = w$. The static budget constraint (27) then gives

$$\frac{\alpha_{11}(w - v_1^1) + \alpha_{12}(w - v_1^2)}{w} = 0. \tag{100}$$

Solving this linear equation, we conclude that

$$w = \frac{\alpha_{11} v_1^1 + \alpha_{12} v_1^2}{\alpha_{11} + \alpha_{12}}. \tag{101}$$

As $v_1^q \in (0, 1)$ we have by assumption that $w \in (0, 1)$, and thus, it follows that an efficient no-trade equilibrium exists. \square

Proof of Corollary 3. Using the equivalent assertions of Theorem 1 and the log-linear structure of the utility functions, we deduce that an efficient no-trade equilibrium exists if and only if the two-dimensional system

$$\left(1 + \frac{\alpha_{12} \alpha_{21}}{\alpha_{11} \alpha_{22}} \left(\frac{1 - w^2}{w^2}\right)\right)^{-1} = w^1, \tag{102}$$

$$\alpha_{11} \frac{w^1 - v_1^1}{w^1} + \alpha_{12} \frac{w^2 - v_1^2}{w^2} = 0, \tag{103}$$

admits a solution in $(0, 1)^2$. Using the first equation to express w^1 as a function of w^2 and plugging the result into the second equation, we obtain that the above system admits a unique solution, which is explicitly given by

$$w^1 = \frac{\alpha_{11}(\alpha_{21}v_1^1 + \alpha_{22}v_1^2)}{\alpha_{12}\alpha_{21}(1 - v_1^2) + \alpha_{11}(\alpha_{21} + \alpha_{22}v_1^2)}, \tag{104}$$

$$w^2 = \frac{\alpha_{12}(\alpha_{21}v_1^1 + \alpha_{22}v_1^2)}{\alpha_{11}\alpha_{22}(1 - v_1^1) + \alpha_{12}(\alpha_{22} + \alpha_{21}v_1^1)}. \tag{105}$$

As $v_1^a \in (0, 1)$ by assumption, we obtain that this solution lies in $(0, 1)^2$ and it follows that an efficient no-trade equilibrium exists. \square

Proof of Corollary 4. According to the third assertion of Theorem 1, we have that there exists an efficient no-trade equilibrium if and only if there are strictly positive constants $w^a \in (0, 1)$ such that

$$(D_t^1)^{\alpha_{11}-\alpha_{21}} = (D_0^1)^{\alpha_{11}-\alpha_{21}}, \tag{106}$$

$$\frac{(w^2 D_t^2)^{\alpha_{12}-1}}{((1 - w^2) D_t^2)^{\alpha_{22}-1}} = \frac{(w^1 D_t^1)^{\alpha_{11}-1}}{((1 - w^1) D_t^1)^{\alpha_{21}-1}}, \tag{107}$$

and the static budget constraint (27) holds. Since the dividend processes are stochastic and linearly independent, the above equations admit a solution if and only if $\alpha_{1a} = \alpha_{2a}$. Assuming that this is the case and solving the second equation for the nonnegative constant w^1 , we find

$$w^1 = g(w^2) := \left(1 + \left(\frac{1 - w^2}{w^2} \right)^{\frac{\alpha_{12}-1}{\alpha_{11}-1}} \right)^{-1}. \tag{108}$$

Plugging this relation back into the static budget constraint of agent 1 and invoking an argument similar to that used in the proof of Corollary 1 then gives the existence of an efficient no-trade equilibrium. \square

The following easy lemma provides a characterization of the class of unit-elastic utility functions and will be useful in the proof of Proposition 2.

Lemma 1. Assume that there are two consumption goods so that $A = 2$. Then the utility function v_i has unit elasticity of substitution if and only if

$$v_i(c) = F_i(c_2(c_1^{\frac{1}{m_i}})) \tag{109}$$

for some constant $m_i \in (0, \infty)$ and some strictly increasing, strictly concave, and continuously differentiable function F_i that satisfies the Inada conditions.

Proof of Proposition 2. Assume that there exists an efficient no-trade equilibrium which is not an autarky equilibrium and denote by

$$c_{1t} = \begin{pmatrix} w^1 D_t^1 \\ w^2 D_t^2 \end{pmatrix} \tag{110}$$

the optimal consumption of the first agent. As is easily seen from the definition, it is possible to implement this consumption allocation in portfolio autarky if and only if the

required net transfers of goods have zero value in the sense that

$$(w^1 - v_1^1)D_t^1 + (w^2 - v_2^1)p_t^2 D_t^2 = 0, \quad (111)$$

where v_1^a denotes the initial portfolio of the first agent. This is in turn equivalent to the fact that the equilibrium spot price is given by

$$p_t^2 = \frac{w^1 - v_1^1}{v_2^1 - w^2} \left(\frac{D_t^1}{D_t^2} \right) = m \left(\frac{D_t^1}{D_t^2} \right). \quad (112)$$

In order to complete the first part the proof, we need to show that this condition is equivalent to that of unit-elastic utility functions. To this end, assume that Eq. (112) holds. Since the equilibrium allocation is efficient, we have

$$\frac{\nabla_2 v_1(w^1 D_t^1, w^2 D_t^2)}{\nabla_1 v_1(w^1 D_t^1, w^2 D_t^2)} = \frac{\nabla_2 v_2((1 - w^1)D_t^1, (1 - w^2)D_t^2)}{\nabla_1 v_2((1 - w^1)D_t^1, (1 - w^2)D_t^2)} = m \frac{D_t^1}{D_t^2}. \quad (113)$$

These equations and the fact that the dividend processes are unbounded imply that the agents' utility functions satisfy

$$\frac{\nabla_2 v_i(c)}{\nabla_1 v_i(c)} = m_i \frac{c^1}{c^2}, \quad c \in (0, \infty)^2, \quad (114)$$

for some strictly positive constants m_i . Solving this differential equation shows that the iso-utility curves of the utility functions are given by $c^2 = B_i(c^1)^{-1/m_i}$ for some nonnegative constants B_i and it follows that

$$v_i(c) = v_i(1, c^2((c^1)^{\frac{1}{m_i}})). \quad (115)$$

In particular, the utility functions satisfy the conditions of Lemma 1 with the functions $F_i(x) = v_i(1, x)$ and hence have unit elasticity of substitution.

Conversely, assume that the utility function u_i satisfies Eq. (109) for some (m_i, F_i) such that an efficient no-trade equilibrium exists and let the optimal consumption of the first agent be given by Eq. (110). Since the allocation is efficient we have that the relative price process is given by

$$p_t^2 = m_1 \frac{w^1 D_t^1}{w^2 D_t^2} = m_2 \frac{(1 - w^1)D_t^1}{(1 - w^2)D_t^2}. \quad (116)$$

Plugging this back into the static budget constraint (27) and using the definition of the agent's consumption allocation, we obtain

$$m_1 \frac{w^1}{w^2} = \frac{w^1 - v_1^1}{v_2^1 - w^2}. \quad (117)$$

In particular, Eq. (112) holds and it follows that the efficient allocation can be implemented in portfolio autarky.

To complete the proof, we now need to show that for unit-elastic preferences, one of the stocks is redundant. This easily follows from the expression of the equilibrium stock prices and Eq. (112). \square

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