Speculative behavior in decentralized markets*

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Abstract

This paper studies a decentralized financial market with short sales constraints in which a continuum of risk neutral agents have heterogenous beliefs regarding the quality of an asset and need to search for each other in order to trade. When agents cannot hold more than one unit of the asset there exists a unique monotone equilibrium. This equilibrium converges to a globally stable steady state that can be computed in closed form for any distribution of beliefs. The steady state trading volume is independent from the distribution of beliefs as long as it is atomless. When asset holdings are unrestricted there also exists a unique monotone equilibrium that can be computed in closed form but this equilibrium fails to converge to a meaningful steady state.

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1 Introduction

Over the counter markets are characterized by the existence of a cross-section of realized prices for a given asset at any point in time. Yet, in the leading model of such markets Duffie et al. (2005) consider a setting with only two types of agents so that only one price is realized in bilateral trades, and only one bid and one ask in trades intermediated by market makers. There are some examples of models with a finite number of types, see for example Lagos and Rocheteau (2009) or Feldhütter (2012), but in these contributions the market is not truly decentralized since all trades are intermediated.

In this paper I consider a version of the Duffie et al. (2005) model with an arbitrary distribution of types that arises through divergence in beliefs regarding the quality of the asset. When agents cannot hold more than one unit of the asset I show that there exists a unique monotone equilibrium that can be calculated in closed form with the use of an endogenously determined probability measure that summarizes the trading frictions in the economy. This equilibrium converges to a globally stable steady state that can be computed in closed form for any distribution of beliefs and which delivers a number of novel predictions. In particular, I show that the equilibrium is asymptotically efficient as frictions weaken, that trading volume is independent from the distribution of beliefs in the population as long as it is atomless, and that despite standard economic intuition an increase in market liquidity may lead to a decrease in welfare due to the conflicting effects of the meeting intensity on the intensive and extensive margins of trade.

The model that I consider can be seen as a search and matching version of the speculative behavior models of Harrison and Kreps (1978) and Morris (1996) in which agents are subject to a restriction on asset holdings. Capitalizing on this analogy I show that speculative behavior arises in the limit of perfect liquidity if and only if beliefs are time varying and the asset supply is low enough, and that it persists in the search market provided that the intensity of bilateral meetings is sufficiently high.

In the last part of the paper I generalize the model by allowing for unrestricted asset holdings. In this case I show that the economy still admits a unique monotone equilibrium that can be computed in closed form for any distribution of beliefs. But, unlike the restricted case, this equilibrium does not converge to a meaningfull steady state because, due to quasi-linear preferences, the supply of the asset tends to concentrate over time in the hands of a very small group of optimistic agents.

The remainder of the paper is organized as follows. Section 2 outlines the benchmark model. Section 3 defines equilibrium, solves for the unique equilibrium in which only those meetings between a buyer and a less optimistic seller lead to a trade, and shows that this monotone equilibrium converges to a unique globally stable steady state. Section 4 discusses various features of the equilibrium including asymptotic efficiency, trading volume and execution delays, the cross-section of trading prices and speculative behavior. Section 5 studies the unrestricted version of the model in which agents can hold any nonnegative number of units of the asset. Appendix A and B contain the proof of all results as well as technical arguments omitted from the text. Finally, Appendices C and D contain two extensions of the benchmark model that allow for the presence of market makers and the possibility of non stationary initial conditions.

2 The model

Time runs continuously and the economy is populated by a continuum (i.e. a non atomic finite measure space with mass one) of risk neutral agents who have access to a risk-free bank account with rate r, and to an over-the-counter market for an infinitely lived risky asset. The cash-flow from this asset are paid at rate $X_t dt$ for some strictly positive process that evolves according to

$$dX_t = X_t \left(\sigma dZ_t + \mu dt\right)$$

where $\mu, \sigma > 0$ are constants and the process Z_t is a Brownian motion under some probability measure P_0 that serves as a reference in the construction of the model.

To introduce heterogenous valuations I assume that the only information available to agents when forming expectations about the future is the history of the cash-flow process to date. In this setting the volatility parameter σ is common knowledge but agents cannot infer the growth rate μ from their observations and I assume that an agent's intrinsic type is characterized by his perception of this growth rate. Specifically, I assume that an agent's perceived growth rate evolves according to

$$dm_t = \int_{\mathbb{M}} (x - m_{t-}) N(dt, dx) \tag{1}$$

where the integrator is an agent-specific Poisson random measure whose predictable compensator is given by

$$Q(dt, \mathbb{B}) = \eta \int_{\mathbb{B}} dF(n) dt, \qquad \mathbb{B} \in \mathcal{B}(\mathbb{R}),$$

for some constant $\eta \geq 0$ that represents the arrival rate of changes in perception, and some right-continuous distribution function $F : \mathbb{R} \to [0,1]$ with

$$\inf_{m\in\mathbb{R}} F(m) = 1 - \sup_{m\in\mathbb{R}} F(m) = 0$$

that gives the distribution of the new perceived growth rate conditional on the arrival of a change. In what follows I further assume that the Poisson random measures are pairwise independent across agents and that the initial distribution of perceived growth rates across the population is $F(m)$ so that, by the law of large numbers, the same function also gives the distribution of perceived growth rates across the population at all subsequent dates. This stationarity assumption is relaxed in Appendix D where I allow for an arbitrary initial distribution of perceived growth rates.

The model outlined above implies that from the point of view of a generic agent the cash-flows from the asset evolve according to

$$dX_t = X_t(\sigma dW_t + m_t dt) \tag{2}$$

where the process

$$W_t = Z_t + \int_0^t (1/\sigma)(\mu - m_s)ds$$

is a standard Brownian motion under the subjective probability measure associated with his perceived growth rate. To guarantee that asset values are finite across of the population I impose the following parametric assumption.

Assumption 1 *It holds that:* $r > \overline{m} = \inf\{m \in \mathbb{R} : F(m) = 1\}$.

A fraction $s \in [0,1]$ of the population is initially endowed with one unit of the asset, and I assume that short sales are prohibited and that agents can hold at most one unit of the asset. As a result, an agent's type is characterized by a pair (q_t, m_t) where $q_t \in \{0,1\}$ records his ownership status and $m_t \in \mathbb{M}$ represents his perception of the growth rate. The restriction on holdings allows for a simple characterization of the equilibrium and will be relaxed in Section 5 below where I consider a model in which agents can hold an arbitrary number of units of the asset subject to a short sale constraint.

Following Duffie et al. (2005) I assume that agents meet randomly and pairwise independently at the jump times of a Poisson process with arrival rate $\lambda \geq 0$. Once they have made contact, an owner and a non owner can trade the asset in exchange for some mutually agreeable price. I assume that agents truthfully reveal their beliefs to each other upon meeting and that the terms of trade are determined through Nash bargaining with bargaining powers $\theta_1 \in [0,1]$ for the seller and $\theta_0 = 1-\theta_1$ for the buyer. This assumption rules out adverse selection but allows for a simple analysis of the bargaining problem and the induced equilibrium. See Duffie et al. (2002, 2005, 2007), Lagos and Rocheteau (2009), Lagos et al. (2011) and Weill (2008) among others for similar assumptions.

Remark 1 (Potential and realized types) Transient state

Remark 2 (Model specification) Since all agents are risk neutral the volatility σ and the nature of the source of risk driving cash flows will not play any role in the analysis. This implies that under appropriate conditions all results carry over to the case where the perceived evolution of cash flows is given by

$$dX_t = X_{t-}\left(dM_t + m_t dt\right)$$

for some observed martingale M_t that is independent from the evolution of the agent's beliefs and which could be even be discontinuous.

3 Dynamic search equilibrium

3.1 Individual value functions

Let $F_{1,t}(m)$ denote the distribution of perceived growth rates in the population of asset owners and denote by $F_{0,t}(m)$ the corresponding distribution in the population of non asset owners. These distributions are exogenously given at the initial date and have to be determined endogenously at all subsequent dates subject to

$$0 = F_{1,t}(m) + F_{0,t}(m) - F(m) = s - F_{1,t}(\overline{m}), \qquad (t,m) \in [0,\infty) \times \mathbb{M}. \tag{3}$$

The first constraint reflects the fact that the distribution of perceived growth rates in the population remains constant through time. The second constraint is a market clearing condition which requires that the total mass of owners be equal to the asset supply at all times. In view of these constraints it is clear that it suffices to determine one of the distributions, and I will concentrate on $F_{1,t}(m)$.

When two agents make contact they have to decide whether to trade and then to bargain over the terms of this trade. To describe the result of this process let $V_{q,t}(m,x)$

denote the value function of an agent of type (q, m), define

$$G_t(m, x) = V_{1,t}(m, x) - V_{0,t}(m, x) \tag{4}$$

to be the gain from becoming an owner and consider a meeting between an owner with perceived growth rate m and a non owner with perceived growth rate n.

If a trade occurs at price P then the owner becomes a non owner and receives $V_{0,t}(m, x) + P$, while the non owner receives $V_{1,t}(n, x) - P$ and becomes an owner. If no trade occurs then the agents part ways with their value functions unchanged. In this setting the assumption of Nash bargaining implies that the meeting results in a trade if and only if the agents' perceived growth rates are such that $G_t(m, x) \leq G_t(n, x)$ in which case the realized price is given by

$$P = \operatorname*{argmax}_{p \in \mathbb{R}} (G_t(n, x) - p)^{\theta_0} (p - G_t(m, x))^{\theta_1} = \theta_0 G_t(m, x) + \theta_1 G_t(n, x).$$

Using this realized price and taking as given the cumulative distributions of perceived growth rates among the populations of owners and now owners I define the value functions by the system of dynamic programming equations

$$V_{q,t}(m, x) = E_t \left[\int_t^\tau e^{-r(s-t)} q X_s ds + e^{-r(\tau - t)} \left(V_{q,\tau}(m_\tau, X_\tau) + \mathcal{E}_{q,\tau}(m_\tau, X_\tau | G) \right) \right] \tag{5}$$

subject to (1), (2) and (4) where τ is the first time that the agent gets an opportunity to change ownership type by trading, and

$$\mathcal{E}_{q,t}(m, x|G) = \int_{\mathbb{M}} \theta_q((2q-1)(G_t(n, x) - G_t(m, x)))^+ dF_{1-q,t}(n) \tag{6}$$

gives the expected gain to an agent of type (q, m) from a meeting with a randomly selected agent of the complementary ownership type. To pin down a unique equilibrium I further require that the value functions satisfy

$$|V_{q,t}(m, x)| \leq c_v x, \qquad (q, t, m) \in \{0, 1\} \times [0, \infty) \times \mathbb{M}, \tag{7}$$

for some constant $c_v > 0$. This linear growth condition is very natural given the agents' risk neutrality and allows for an intuitive interpretation of the value functions. In particular, I show in Appendix B that under this condition the system of dynamic programming equations (5) is equivalent to

$$V_{q,t}(m,x) = E_t \int_t^{\infty} e^{-r(s-t)} \left(qX_s + \lambda \mathcal{E}_{q,s}(m_s, X_s|G)\right) ds. \tag{8}$$

This formulation shows that the value function of a generic agent is the sum of two components. The first component is given by $qf_t(m,x)$ where

$$f_t(m,x) = E_t \int_t^{\infty} e^{-r(s-t)} X_s ds \tag{9}$$

represents the present value of the dividends from the asset to an agent with perceived growth rate m, i.e. the fundamental value of the asset from the point of view of such an agent. The second component

$$h_{q,t}(m,x) = E_t \int_t^{\infty} e^{-r(s-t)} \lambda \mathcal{E}_{q,s}(m_s, X_s|G) ds \geq 0$$

represents the present value of future trading gains. For asset owners ($q = 1$) this component can be understood as the value of the resale option attached to the asset. For non owners this component gives the value that an agent attaches to being present in the market. Contrary to other models of speculative behavior (e.g. Harrison and Kreps (1978), Morris (1996) and Scheinkman and Xiong (2003)) this value is non zero here because, due to the finite asset supply and the restriction on holdings, the buyer captures a non zero fraction of the trading surplus in equilibrium as long as the seller's bargaining power is not equal to one.

Lemma 1 *The fundamental value of the asset is*

$$f(m,x) = (1+\varphi)\frac{x}{\rho(m)} \tag{10}$$

where $\varphi > 0$ is a constant defined in the appendix and $\rho(m) = r + \eta - m$.

The interpretation of the above formula is intuitive. Indeed, the standard Gordon growth formula implies that $x/\rho(m)$ gives the fundamental value of the asset when the agent has a constant perceived growth rate equal to m and is present in the market for an exponentially distributed period of time. This single period fundamental asset value is then scaled up by the constant factor $1 + \varphi > 1$ to account for the fact that in the model the agent is present in the market for an infinite number of such periods with a randomly selected perceived growth rate in each sub-period.

Remark 3 (Multiplicity) The dynamic programming equation (5) does not uniquely determine the value functions. Indeed, if the functions $V_{q,t}(m,x)$ solve this system then the definition of the gain from becoming an owner implies that the same is true of the functions $V_{q,t}(m,x) + e^{rt}\beta_q$ for any constants β_q. The linear growth condition (7) is meant to rule out such solutions and allows to determine a unique equilibrium. A similar restriction is also necessary in other models of speculative behavior including Harrison and Kreps (1978), Morris (1996) and Scheinkman and Xiong (2003).

Remark 4 (Pessimism and Impatience) Using the independence between Poisson and Brownian shocks I show in Appendix B that $V_{q,t}(m,x) = v_{q,t}(m)x$ for some bounded functions which satisfy the reduced-form system

$$v_{q,t}(m) = E_t \int_t^\infty e^{-\int_t^s (r-m_u)du} \left(q + \lambda \mathcal{E}_{q,s}(m_s, 1|v_1 - v_0)\right) ds. \tag{11}$$

Comparing this system to (8) reveals that the heterogenous beliefs model of this paper is equivalent to a common beliefs model in which agents trade a consol bond and differ through their subjective rates of time preferences. In this alternative interpretation of the model a higher perceived growth rate corresponds to a lower discount rate so that more optimistic agents are less impatient.

3.2 Equilibrium distribution of types

The system of dynamic programming equations (8) characterizes the value functions induced by a pair of distribution functions. To close the model, it remains to determine how these distributions evolve over time as a result of trading.

Assume that a suitable version of the law of large numbers applies (see e.g. Duffie and Sun (2007, 2012)) and consider the ways in which agents enter or exit the group of owners who are more pessimistic than a given growth rate $m \in \mathbb{M}$:

1. An agent may enter because he is a non owner with $m_0 \leq m$ who has just met an owner with perceived growth rate m_1 such that

$$(m_0, m_1) \in \mathbb{T}_{0,t}(m|G) = \left\{(a,b) \in \mathbb{M}^2 : G_t(a,x) \geq G_t(b,x) \text{ and } a \leq m\right\}.$$

   where the notation reflects the fact that, since the gain from becoming an asset owner is homogenous in the cash flow variable by Remark 4, the set on the right hand side does not depend on $x \geq 0$. The contribution of such exits to the instantaneous rate of change of the distribution is given by the integral:

$$\int_{\mathbb{T}_{0,t}(m|G)} \lambda dF_{0,t}(m_0) dF_{1,t}(m_1)$$

where the usual informal notation $dF_{q,t}(m)$ denotes the Borel measure associated with the distribution $F_{q,t}(m)$.

2. An agent may exit because he is an owner with $m_1 \leq m$ who has just met a non owner with perceived growth rate m_0 such that

$$(m_0, m_1) \in \mathbb{T}_{1,t}(m|G) = \left\{(a,b) \in \mathbb{M}^2 : G_t(a,x) \geq G_t(b,x) \text{ and } b \leq m\right\}.$$

   The contribution of such exits to the instantaneous rate of change of the distribution is given by the integral:

$$-\int_{\mathbb{T}_{1,t}(m|G)} \lambda dF_{0,t}(m_0) dF_{1,t}(m_1).$$

3. An agent may enter because he is an owner whose perceived growth rate was just reset to some $m_0 \leq m$. The contribution of such entries to the instantaneous rate of change of the distribution is $\eta s F(m)$.

4. Finally, an agent may exit because his perceived growth rate has been reset. The contribution of such exits is given by: $-\eta F_{1,t}(m)$.

Gathering the contribution of these four channels and using (3) shows that the rates of change in the masses of agents who are more pessimistic than a fixed perceived growth rate m are almost surely given by

$$\dot{F}_{0,t}(m) = -\dot{F}_{1,t}(m) \tag{12}$$

$$\dot{F}_{1,t}(m) = \eta(sF(m) - F_{1,t}(m)) + \int_{\mathbb{T}_{0,t}(m|G)} \lambda dF_{0,t}(m_0) dF_{1,t}(m_1) - \int_{\mathbb{T}_{1,t}(m|G)} \lambda dF_{0,t}(m_0) dF_{1,t}(m_1). \tag{13}$$

Given $F(m)$ and an arbitrary initial condition $F_{1,0}(m)$ these equations fully characterize the distributions of perceived growth rates induced by a pair of value functions and directly leads to the following definition of equilibrium.

Definition 1 *A dynamic search equilibrium is an array* $(V_{q,t}(m,x), F_{q,t}(m))$ *of value functions and distribution functions such that*

1. *The value functions solve* (7) *subject to* (8).

2. *The distribution functions satisfy* (12) *and* (13) *subject to* (3) *and the exogenous initial condition* $F_{1,0}(m)$.

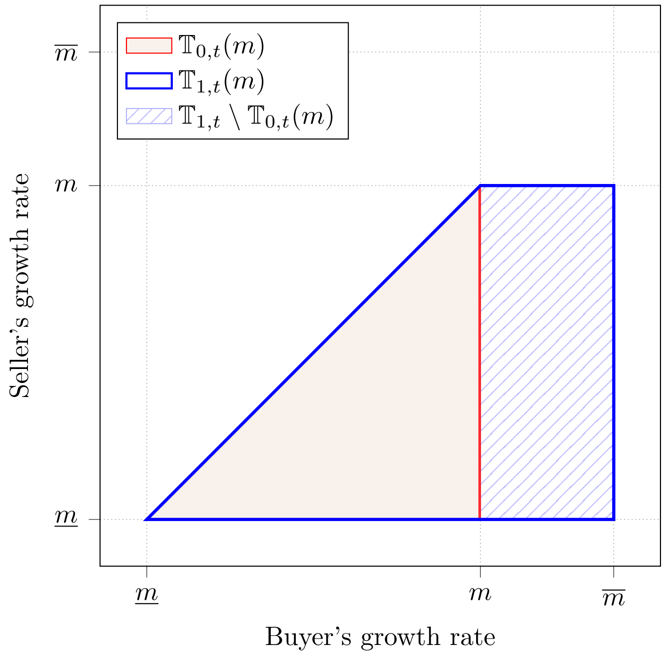
An equilibrium is stationary if both the distribution functions and the value functions are time-independent.

Since intrinsic types are associated with an agent's optimism regarding the cash-flows of the asset it is natural to look for equilibria in which the gain $G_t(m, x)$ from becoming an owner increases with the agent's perceived growth rate. In such a *monotone equilibrium* a trade occurs as soon as an owner meets a more optimistic non owner. This implies that the sets $\mathbb{T}_{q,t}(m|G)$ do not depend on the gain from becoming an owner, and allows to construct an equilibrium in two steps: First, derive the distribution of perceived growth rates among asset owners. Second, compute the induced value functions and verify that the corresponding gain from becoming an owner is an increasing function of an agent's perceived growth rate.

Remark 5 (Initial conditions) Since the distribution of perceived growth rates in the population is stationary it follows from (3) that the initial distributions $F_{q,0}(m)$ are absolutely continuous with respect to the distribution $F(m)$. If the stationarity assumption is relaxed then one can consider initial conditions that need not have this property, i.e. initial conditions that allow for transient types. This extension of the model is pursued in Appendix D and can be used to study the return to the steady state after an aggregate liquidity shock.

Remark 6 (Tie breaking) The definition of the sets $\mathbb{T}_q(m|G)$ implicitly assumes that meetings between an owner and a non owner with the same perceived growth rate result in a trade. This assumption has no impact on the equilibrium distributions because the choice of the tie breaking rule influences the second and third terms on the right of (13) in the same way. While it does not influence the equilibrium, the choice of the tie breaking rule may impact trading volume if the distribution of perceived growth rates includes atoms. From that point of view the assumption that agents trade whenever they are indifferent can be seen as maximizing the trading volume.

FIGURE 1: The sets $\mathbb{T}_{q,t}(m)$ in a monotone equilibrium



Notes. This figure illustrates the sets $\mathbb{T}_{0,t}(m)$, $\mathbb{T}_{1,t}(m)$ and $\mathbb{T}_{1,t} \setminus \mathbb{T}_{0,t}(m)$ (hatched area and thick part of the boundary) under the assumption that the value of becoming an asset owner increases with an agent's perceived growth rate. For the purpose of illustration I assume here that the set $\mathbb{M}$ of potential types has a finite lower bound $\underline{m}$.

3.3 The monotone equilibrium

If the gain from becoming an asset owner increases with an agent's perceived growth rate then the sets of agents between which a trade occurs simplify to

$$\mathbb{T}_{q,t}(m|G) = \mathbb{T}_{q,t}(m) = \left\{(a,b) \in \mathbb{M}^2 : a \geq b \text{ and } a + q(b-a) \leq m\right\},$$

and satisfy

$$\mathbb{T}_{1,t}(m) \setminus \mathbb{T}_{0,t}(m) = \left\{(a,b) \in \mathbb{M}^2 : b \leq m < a\right\}$$

as illustrated by Figure 1. Inserting these sets into the integral equations (12), (13) for the equilibrium distributions and simplifying gives

$$\begin{aligned}\dot{F}_{1,t}(m) &= -\dot{F}_{0,t}(m) = \lambda \mathcal{R}(m, F_{1,t}(m)) \\ &= -\lambda F_{1,t}(m)(1 - s - F(m) + F_{1,t}(m)) + \lambda\phi(sF(m) - F_{1,t}(m)).\end{aligned} \tag{14}$$

with the constant $\phi = \eta/\lambda$. The following proposition provides an explicit expression for the unique solution to this Ricatti equation and shows that it converges to a unique steady state from any initial condition. To state the result let

$$\begin{aligned} F_1^*(m) &= -\tfrac{1}{2}\left(1 - s + \phi - F(m)\right) + \tfrac{1}{2}\Phi(m) \\ &= -\tfrac{1}{2}\left(1 - s + \phi - F(m)\right) + \tfrac{1}{2}\sqrt{(1 - s + \phi - F(m))^2 + 4s\phi F(m)} \end{aligned} \tag{15}$$

denote the strictly positive solution to the characteristic equation $\mathcal{R}(m, x) = 0$ associated with the right hand side of (14).

Proposition 1 *In a monotone equilibrium the distribution of perceived growth rates among asset owners is given by*

$$F_{1,t}(m) = F_1^*(m) + \frac{(F_{1,0}(m) - F_1^*(m))\Phi(m)}{\Phi(m) + (F_{1,0}(m) + \Phi(m) - F_1^*(m))\left(e^{\lambda\Phi(m)t} - 1\right)} \tag{16}$$

and converges strongly to the steady state distribution $F_1^*(m)$ from any initial condition such that (3) *holds.*

To illustrate the convergence of the equilibrium distributions to the steady state Figure 2 plots the equilibrium distributions among owners and non owners at various points in time in a simple environment where perceived growth rates are uniformly distributed among the whole population and $F_{1,0}(m) = sF(m)$. As can be seen from the figure, the distribution of beliefs among owners is initially uniform but as time passes this distribution moves down and to the right, indicating that the market gradually channels the asset towards more optimistic agents. Similarly, the distribution of beliefs among non owners gradually shifts up and to the left, indicating that pessimistic agents are less and less likely to hold the asset as trading unfolds in the market.

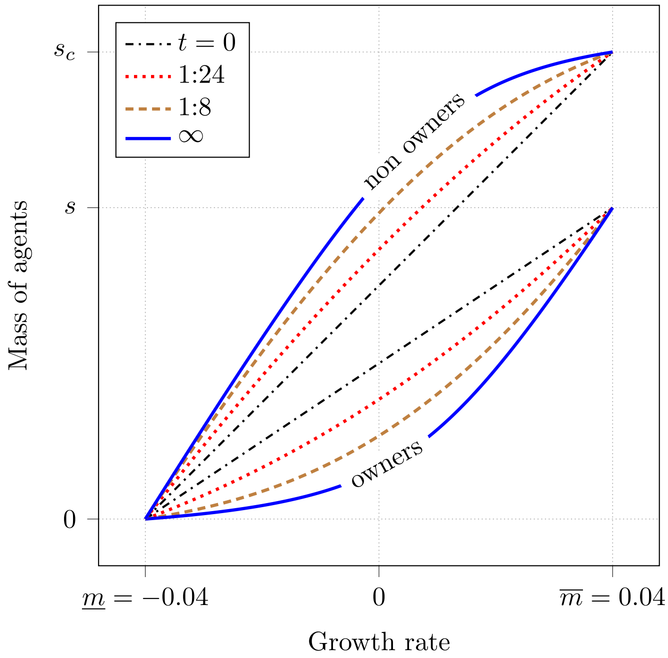
The explicit solution for the steady state distribution of perceived growth rates allows to derive some analytic comparative statics results.

Corollary 1 *The steady state distribution $F_1^*(m)$ is increasing in the asset supply s, and increasing and concave in ϕ with*

$$\begin{aligned} &\lim_{\phi\to\infty} F_1^*(m) = sF(m), \\ &\lim_{\phi\to 0} F_1^*(m) = (1 - s - F(m))^-. \end{aligned}$$

Proposition 1 shows that the steady state distributions only depends on the arrival rates of meetings and changes in perceived growth rates through $\phi = \eta/\lambda$. This constant

Figure 2: Convergence to the steady state distributions

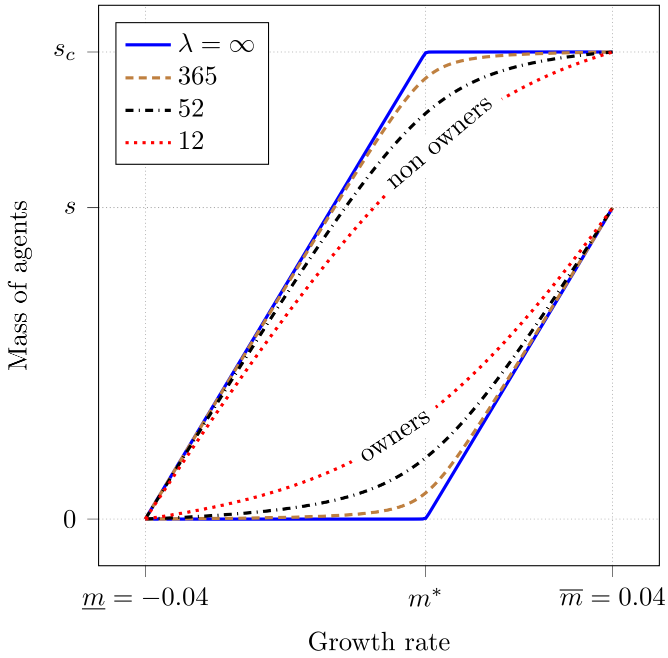


Notes. This figure plots the equilibrium distributions at various point in time in an environment where perceived growth rates are initially uniformly distributed over $[-0.04, 0.04]$, the asset supply is equal to $s = 1 - s_c = 0.4$, and the arrival rates of meetings and changes in perceived growth rates are respectively given by $\lambda = 26$ and $\eta = 2$ so that agents meet once every two weeks and change their mind twice a year on average.

measures the frequency of changes in perceived growth rates relative to the frequency of trading opportunities and therefore indicates the steady state degree of misallocation in the market. In line with this interpretation, the above corollary shows that as ϕ increases, either because meetings become less frequent or because agents change their minds more often, the equilibrium allocation becomes gradually less efficient in that owners tend to be collectively more pessimistic.

In the limit where $\phi \to \infty$ the steady state equilibrium distributions are proportional to the population wide distribution so that ownership and beliefs are independent. To understand this result observe that in this limit it must be the case that either $\eta \to \infty$ so that beliefs become infinitely volatile, or $\lambda \to 0$ so that trading becomes impossible due to a lack of meetings. In either case, changes of beliefs become too frequent relative to trading opportunities for the decentralized market to channel the asset towards more optimistic agents and, as a result, the equilibrium distributions asymptotically reproduce the population wide distribution albeit on a smaller scale.

FIGURE 3: Steady state distributions of perceived growth rates



Notes. This figure plots the equilibrium distributions for various values of the meeting intensity in an environment where perceived growth rates in the population are uniformly distributed over $[-0.04, 0.04]$, the asset supply is $s = 1 - s_c = 0.4$, and the arrival rate of jumps in perceived growth rates is $\eta = 2$ so that agents change their mind twice a year on average.

On the other hand, the corollary shows that as ϕ decreases to zero, either because meetings become infinitely frequent or because beliefs are fixated, the supports of the steady state distributions become separated. Specifically, there exists a threshold m^* such that all owners are more optimistic than this threshold while all non owners are more pessimistic, see Figure 3 for an illustration. As shown in Section 4.1 below these limiting distributions coincide with the allocation that would prevail in any equilibrium of the corresponding Walrasian market. Therefore, the above result shows that when agents beliefs are fixed or the search friction vanishes the equilibrium of the search market is asymptotically efficient for any distribution of perceived growth rates.

Having derived the distributions of perceived growth rates among owners and non owners I now need to compute the induced value functions and to verify the conjectured monotonicity. Subtracting (8) with $q = 0$ from itself with $q = 1$ gives

$$G_t(m, x) = E_t \int_t^\infty e^{-r(s-t)} \left(X_s + \lambda[\mathcal{E}_{1,s} - \mathcal{E}_{0,s}](m_s, X_s|G)\right) ds. \tag{17}$$

Since the right hand side of (8) only depends on the gain for becoming an asset owner it follows that solving the system of dynamic programming equations (8) subject to (7) is equivalent to solving (17) subject to

$$|G_t(m,x)| \leq c_g x, \qquad (t,m) \in [0,\infty) \times \mathbb{M}. \tag{18}$$

As a first step towards the construction of a function satisfying these requirements I observe that under the conjectured monotonicity the non linear equation (17) is equivalent to the linear equation

$$G_t(m,x) = E_t \int_t^\infty e^{-r(s-t)} \left(X_s + \lambda \mathcal{O}_s(m_s, X_s|G)\right) ds \tag{19}$$

where the linear operator on the right hand side is defined by

$$\begin{aligned}\mathcal{O}_t(m,x|v) &= \int_{\mathbb{M}} (v_t(n,x) - v_t(m,x)) dH_t(n|m) \\ &= \int_{\mathbb{M}} (v_t(n,x) - v_t(m,x)) \left(\mathbf{1}_{\{n>m\}} \theta_1 dF_{0,t}(n) + \mathbf{1}_{\{n \leq m\}} \theta_0 dF_{1,t}(n)\right)\end{aligned} \tag{20}$$

with

$$H_t(n|m) = \theta_0 F_{1,t}(n \wedge m) + \theta_1 (F_{0,t}(n) - F_{0,t}(m))^+$$

and coincides with the nonlinear operator $\mathcal{E}_1 - \mathcal{E}_0$ on nondecreasing functions. To solve this equation consider the equivalent probability measure P^* under which the compensator of an agent's perceived growth rate is

$$Q^*(dt, \mathbb{B}) = Q(dt, \mathbb{B}) + \int_{\mathbb{B}} \lambda dH_t\,(n|m_{t-}) dt, \qquad \mathbb{B} \subseteq \mathcal{B}(\mathbb{R}). \tag{21}$$

Under this probability measure changes in the perceived growth rate are more likely and can be interpreted as occurring not only due to exogenous shocks but also due to trades. Indeed, the compensator is the sum of three terms:

$$Q^*(dt, \mathbb{B}) = Q(dt, \mathbb{B}) + \int_{\mathbb{B} \cap \{n > m_{t-}\}} \lambda \theta_1 dF_{0,t}(n) dt + \int_{\mathbb{B} \cap \{n \leq m_{t-}\}} \lambda \theta_0 dF_{1,t}(n) dt.$$

The first term is the compensator under the original probability measure and reflects exogenous changes in the agent's perceived growth rate. The second term reflects a positive jump that occurs when the generic agent, viewed as an asset owner, meets a randomly selected now owner who is more optimistic. Similarly, the third term reflects a negative jump that occurs when the generic agent, viewed as non asset owner, meets a more pessimistic owner drawn from the equilibrium distribution.

This decomposition of the compensator suggests that the probability P^* should be interpreted as tracking the beliefs of the marginal agent. This interpretation is confirmed by the next result which shows that the equilibrium gain from becoming an owner can be computed as the fundamental value of the asset to an hypothetical agent whose beliefs about the growth rate are represented by this endogenously determined probability measure. To state the result, let $\mathcal{F}_{\mathrm{ns}}$ denote the class of cumulative distribution functions which can be expressed as

$$F(m) = \int_{-\infty}^{m} \delta(n)dn + \sum_{k\in\mathbb{N}} \mathbf{1}_{\{m_k \le m\}}\pi_k$$

for some locally bounded density function $\delta(n)$ and some sequences of points $m_k \in \mathbb{M}$ and point probabilities $\pi_k \in [0,1)$. This class of distributions is general enough to include all the standard distributions used in economics and only excludes pathological cases where the distribution of perceived growth rates in the population contains a singularly continuous part as in the well-known examples derived from the Cantor function (see for example Feller (1968, Chapters I and V)).

Theorem 1 *The unique solution to* (18) *and* (19) *is*

$$G_t(m,x) = E_t^* \int_t^{\infty} e^{-r(s-t)} X_s ds. \tag{22}$$

If the distribution of perceived growth rates is in $\mathcal{F}_{\mathrm{ns}}$ *then this solution is nondecreasing with respect to* $m \in \mathbb{M}$ *and there exists a unique monotone equilibrium.*

The equilibrium gain from becoming an asset owner can be seen as the reservation value of the asset as it gives both the maximal price that a non owner can accept to pay to acquire the asset and the minimal price that an owner can accept to charge to part with the asset. Therefore, the above theorem shows that in a decentralized market the reservation value of the asset to a given agent can be calculated as the expected value of future discounted cash flows under an endogenously determined, agent-specific probability measure. This type of results is standard in frictionless asset pricing, where private values are usually computed under a probability measure constructed from the agent's marginal rates of substitution, but what is unique to the decentralized market setting considered here is that this probability measure entirely summarizes the trading frictions.

A notable feature of the monotone equilibrium derived in Theorem 1 is that, given the opportunity, an agent always trades if the match surplus is positive. In other words, there is no option value of waiting for better terms of trade in a decentralized market. This is due to the fact that the only trading cost is proportional to the match surplus, and is reminiscent of what happens in real options models without fixed costs where the optimal strategy typically consists in exerting action as soon as the surplus becomes different from zero, see Dixit and Pindyck (1994) and Stokey (2009).

The conditional expectation in Theorem 1 does not seem to admit an explicit solution and is also very difficult to compute numerically due to the complex time-dependence induced by the non stationarity of the equilibrium distributions of perceived growth rates. A notable exception is the case where the support of distribution of perceived growth rates in the economy is a finite collection of points. Indeed, the following proposition shows that in that case the necessary computation can be reduced to solving a finite dimensional system of first order linear differential equations.

Proposition 2 *Assume that the distribution of perceived growth rates is supported by a finite collection of N points and set*

$$B_{ij,t} = \delta_{ij}\left(r - m_i + \eta + \lambda H_t(\overline{m}|m_i)\right) - \eta\Delta F(m_j) - \lambda\Delta H_t(m_j|m_i)$$

where δ_{ij} is the Kronecker delta symbol. In the unique monotone equilibrium the gain from becoming an asset owner satisfies

$$G_t(m_i, x) = g_{i,t}x, \qquad 1 \le i \le N,$$

where the function $g_t \in \mathbb{R}^N$ is the unique bounded solution to $\dot{g}_t = B_t g_t - \mathbf{1}$.

To gain more insights into the structure of the monotone equilibrium I consider next the case in which the initial conditions are given by the steady state distributions and derive an explicit solution for the corresponding stationary monotone equilibrium.

3.4 Steady state equilibrium

Assume that the initial condition for (14) is given by the steady state distribution of perceived growth rates among asset owners, and let

$$F_0^*(m) = F(m) - F_1^*(m)$$

denote the corresponding distribution among non owners. Using (22) together with the Markov property of the perceived growth rate process and the law of iterated expectations shows that the bounded function

$$g_t(m) = \frac{G_t(m,x)}{x} = E_t^* \int_t^\infty e^{-\int_t^s (r-m_u)du} ds \tag{23}$$

is time-independent and satisfies

$$g(m) = E^* \int_0^\tau e^{-(r-m)s} ds + E^*[e^{-(r-m)\tau} g(m_\tau)]$$

where the stopping time τ denotes the first time that the agent's perceived growth rate changes. Integrating with respect to the joint distribution of τ and m_τ and simplifying the result then shows that the above equation is equivalent to

$$\begin{aligned} \gamma(m)g(m) &= 1 + \mathcal{D}^*(m|g) \\ &= 1 + \int_{\mathbb{M}} g(n) \big(\eta dF(n) + \mathbf{1}_{\{n>m\}} \lambda\theta_1 dF_0^*(n) + \mathbf{1}_{\{n\le m\}} \lambda\theta_0 dF_1^*(n)\big), \end{aligned} \tag{24}$$

with the discount rate

$$\gamma(m) = \rho(m) + \lambda\theta_1(1 - s - F_0^*(m)) + \lambda\theta_0 F_1^*(m) > 0. \tag{25}$$

The result of Theorem 1 guarantees that this integral equation admits a unique bounded solution and an educated guess suggests that

$$g(m)c = \Gamma(m) = \exp\left(-\int_m^{\overline{m}} \frac{dn}{\gamma(n)}\right) \tag{26}$$

for some free constant. Substituting this conjecture into (24) and using integration by parts to simplify the result shows that this free constant is

$$c = \frac{1}{g(\overline{m})} = \gamma(\overline{m}) - \mathcal{D}^*(\overline{m}|\Gamma) = r - \overline{m} + \mathcal{D}^*(\overline{m}|1-\Gamma) > 0 \tag{27}$$

where the second equality follows from the linearity of $\mathcal{D}^*$ and the inequality follows from Assumption 1 and the definition of the discount rate. This provides an explicit solution for the steady state gain from becoming an asset owner, and combining this expression with arguments similar to those of Lemma 1 allows to derive the unique stationary monotone equilibrium in closed-form.

Theorem 2 *Assume that* $F_{1,0}(m) = F_1^*(m)$. *Then there exists a unique monotone equilibrium that is stationary and given by the distributions*

$$F_{1,t}(m) = F_1^*(m)$$
$$F_{0,t}(m) = F_0^*(m) = F(m) - F_1^*(m)$$

and the time-independent value functions

$$V_q(m,x) = qf(m,x) + \frac{\lambda \mathcal{E}_{q,0}(m,x|G)}{\rho(m)} + \eta f(m,1) \int_{\mathbb{M}} \frac{\lambda \mathcal{E}_{q,0}(n,x|G)}{\rho(n)} dF(n)$$

where the fundamental value of the asset and the steady state gain from becoming an asset owner are defined by (9), (10), (23), (26) *and* (27).

An important question regarding the nature of the steady state equilibrium is whether it is stable, i.e. whether the economy converges to it from any initial conditions. The following result shows that this is indeed the case.

Corollary 2 *Assume that* $F \in \mathcal{F}_{\text{ns}}$ *then the unique monotone equilibrium of Theorem 1 converges to the unique stationary monotone equilibrium of Theorem 2 from any initial condition such that* (3) *holds.*

Remark 7 (Stochastic marginal utility) Assume as in Duffie et al. (2005) and Weill (2008) that an owner receives the stochastic utility flow $U(m_t)X_t dt$ for some bounded increasing function $U(m) \geq 0$. In this case it can be shown that when $F \in \mathcal{F}_{\text{ns}}$ there exists a unique monotone equilibrium that is given by

$$G_t(m,x) = E_t^* \int_t^\infty e^{-r(s-t)} U(m_s) X_s ds$$

and the same distributions as in Proposition 1. The corresponding globally stable steady state equilibrium can also be derived. In particular, the steady state distributions are as in Theorem 2 and the gain from becoming an owner is

$$G(m,x) = xg(m) = x\Gamma(m) \left(c - \int_m^{\overline{m}} \frac{dU(n)}{\gamma(n)\Gamma(n)} \right)$$

where the free constant can be obtained by imposing the boundary condition

$$\gamma(\overline{m})g(\overline{m}) = U(\overline{m}) + \mathcal{D}^*(\overline{m}|g).$$

Since the equilibrium distributions do not depend on $U(m)$ all the predictions of the paper regarding efficiency, volume and trading delays carry over to the case where the marginal utility is a non trivial function. If $U(m)$ is non constant one may even dispense from heterogenous beliefs by assuming that m_t only influences the agent's marginal utility as this is sufficient to generate trade. In this all the formulas of the paper remain valid provided that the function $\rho(m) = \rho(0) = r$ for all $m \in \mathbb{M}$ throughout.

Remark 8 (Singular distributions) As can be seen from the arguments leading to Theorem 2 the assumption that $F \in \mathcal{F}_{\text{ns}}$ is not needed for the existence and uniqueness of the stationary monotone equilibrium. The reason for this simplification is that in the stationary case the gain from becoming an owner is known in closed form and can be directly shown to be increasing for any distribution. By contrast, in the non stationary case the gain from becoming an owner is only known as the unique fixed point of an integral operator and showing the monotonicity of this fixed point requires some assumptions on the structure of the distribution of perceived growth rates.

4 Analysis

4.1 Asymptotic efficiency

In a Walrasian market, equilibrium is characterized by a price process at which agents can trade instantly and such that markets clear. The corresponding allocation is efficient and, assuming that the gain from becoming an owner is increasing in beliefs, it follows that there exists a cutoff $w \in \mathbb{M}$ such that in equilibrium the set of asset owners is contained in the set of agents who are more optimistic than w.

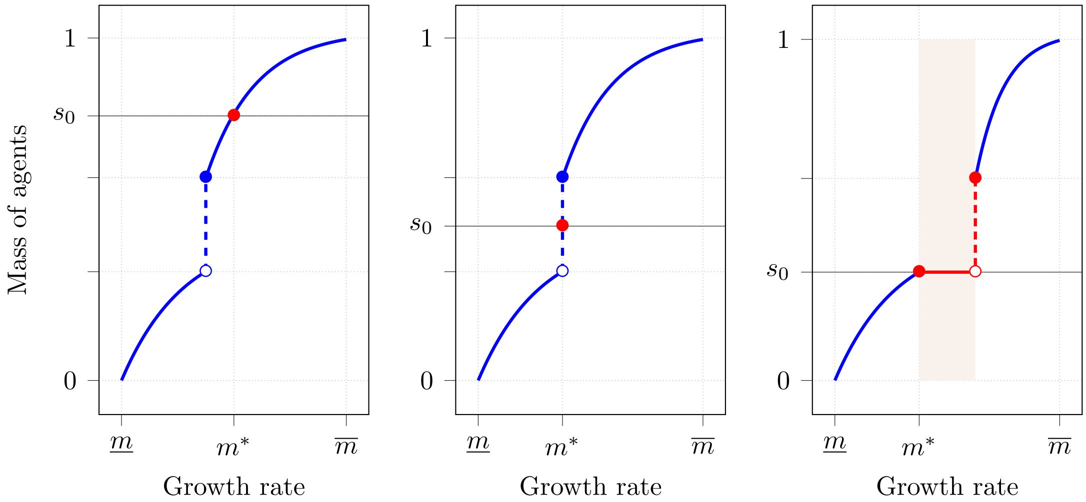
Since the distribution of perceived growth rates can have atoms, the inclusion can be strict in which case some randomization will be required at the margin. Taking this into account shows that the market clearing condition is

$$s = 1 - F(w) + \pi \, \Delta F(w) \tag{28}$$

where $\pi \in [0, 1]$ represents the probability that a given marginal agent is allocated one unit of the asset in equilibrium, and $\Delta F(w)$ gives the mass of agents with perceived growth rate w. It follows that the cutoff is a quantile of order $1 - s$ of the distribution of perceived growth rates and I will denote by

$$m^* = \inf \{m \in \mathbb{M} : F(m) \geq 1 - s\} \tag{29}$$

FIGURE 4: The equilibrium cutoff



Notes. This figure illustrates the determination of the equilibrium cutoff as the smallest quantile of order $s_0 = 1 - s$ of the distribution of perceived growth rates. In the first two cases the set of quantiles is reduced to a singleton while in the third case it is given by the whole shaded interval. For the purpose of illustration I assume here that the set $\mathbb{M}$ of potential types has a finite lower bound $\underline{m}$.

the smallest among such quantiles.

Remark 9 Note that the equilibrium cutoff is generically unique and given by the lowest quantile. Indeed, the only case where the set

$$\mathcal{Q} = \left\{ m \in \mathbb{M} : \lim_{n \uparrow m} F(n) \leq 1 - s \leq F(m) \right\} \tag{30}$$

is not reduced to a single point is when the distribution of perceived growth rates in the population is constant at the level $1 - s$ over a set of positive measure as illustrated by the shaded interval in the right panel of Figure 4.

Fix an arbitrary cutoff $w \in \mathcal{Q}$. If this cutoff is such that $F(w) = 1 - s$ as in the left panel of Figure 4 then the set of owners is exactly the set of agents who are strictly more optimistic than w and no randomization is required. On the contrary, if w is an atom of the distribution such that $F(w) > 1 - s$ as in the middle and right panels of the figure then some randomization is required at the margin, and

$$\pi(w) = \frac{F(w) - (1 - s)}{\Delta F(w)}$$

gives the fraction of marginal agents who hold the asset in equilibrium. In either case, it follows from (28) that the equilibrium distribution of perceived growth rates among owners and non owners are given by

$$F_1^w(m) = (1 - s - F(m))^- \tag{31}$$

$$F_0^w(m) = F(m) - F_1^w(m) = \min\{1 - s, F(m)\} \tag{32}$$

and do not depend on the choice of the cutoff $w \in \mathcal{Q}$. Finally, the equilibrium price must be determined in such a way that an hypothetical agent who is constantly marginal is indifferent between holding or not holding the asset. In the eyes of such an agent the cash flow process has constant drift equal to the cutoff w and it follows that the equilibrium price is given by

$$P_t^w = \Pi(w, X_t) = \frac{X_t}{r - w}. \tag{33}$$

To justify this construction it remains to verify that given this price the distributions of perceived growth rates in (31), (32) are consistent with individual optimality. This verification is carried out in the appendix and delivers the following.

Proposition 3 *The allocation (31), (32) and the price process (33) form a competitive equilibrium for any $w \in \mathcal{Q}$.*

Comparing the competitive allocation to the distributions in Corollary 1 shows that the steady state distributions of the search market converge to the equilibrium distributions of the Walrasian market as meetings become more frequent. The next proposition shows that the same is true for trading prices. To state the result, let

$$P(m, n, x) = \theta_0 G(m, x) + \theta_1 G(n, x) = (\theta_0 g(m) + \theta_1 g(n))\, x$$

denote the steady state trading price between an asset owner with perceived growth rate m and a non owner with perceived growth rate $n \geq m$.

Proposition 4 *If the bargaining powers $\theta_q \in (0, 1)$ then the equilibrium converges to the Walrasian equilibrium as the meeting intensity increases:*

$$\lim_{\lambda \to \infty} F_q^*(m) = F_q^w(m),$$

$$\lim_{\lambda \to \infty} P(m, n, x) = \lim_{\lambda \to \infty} G(m, x) = \Pi(m^*, x),$$

for all perceived growth rates $m \leq n$ and all $(x, q) \in (0, \infty) \times \{0, 1\}$.

If non asset owners possess all the bargaining power, i.e. if $\theta_1 = 0$, then in every meeting the buyer is able to impose the price that his most favourable to him subject to the seller's participation constraint and all trades occur at the seller's reservation value. Symmetrically, if asset owners possess all the bargaining power then all trades occur at the buyer's reservation value. In either case, it can be shown that the cross-section of realized prices of the search market fails to converge to the Walrasian price despite the fact that the corresponding equilibrium distributions, which do not depend on the bargaining powers, converge to the efficient allocation.

4.2 Trading volume and execution delays

In the model trading volume is naturally defined as the number of meetings that give rise to a trade. Therefore, I have that in the stationary monotone equilibrium trading volume is given by the integral

$$\vartheta = \int_{\mathbb{M}^2} \mathbf{1}_{\{n \leq m\}} \lambda dF_0^*(m) dF_1^*(n) = \int_{\mathbb{M}} \lambda F_1^*(n) dF_0^*(m). \tag{34}$$

Combining the explicit solution for the steady state distributions given in Proposition 1 with a simple change of variable then leads to the following.

Proposition 5 *Assume that the distribution of perceived growth rates is continuous. Then the steady state trading volume is given by*

$$\vartheta = \vartheta_c \equiv \eta s(1-s) \left[(1+\phi) \log\left(1 + \frac{1}{\phi}\right) - 1 \right]$$

The steady state trading volume is increasing in both η and λ and decreases to zero as either or both decrease to zero.

The above proposition offers a striking conclusion: In a decentralized market governed by search and bargaining market trading volume does not depend on the distribution of perceived growth rates among market participants provided that it is continuous. This means in particular that contrary to standard intuition (see for example Banerjee (2011) and the references therein) trading volume depends neither on the support nor on the dispersion of beliefs in the market. The proposition also shows that, under the same continuity assumption, trading volume increases with the intensity of both meetings and changes in beliefs, and is a dome shaped function of the asset supply with a maximum at the point where the masses of potential buyers and sellers coincide. This last result is intuitive. Indeed as supply increases there are simultaneously more assets to trade and less potential buyers. The first effect is initially the strongest, so that trading volume increases for small value of the supply parameter s, but the second effect will eventually dominate as the mass of potential buyers becomes smaller and smaller.

The definition of trading volume given in (34) assumes that meetings involving equally optimistic investors all result in a trade. If the distribution of perceived growth rates is continuous then this assumption is without loss of generality since such meetings occur with zero probability, but it is not so otherwise. Indeed, if the distribution of perceived growth rates has atoms the steady state trading volume depends on the tie breaking rule and is given by

$$\vartheta(p) = \int_{\mathbb{M}} \lambda F_1^*(n) dF_0^*(m) - \sum_{m \in \mathbb{M}} \lambda p \Delta F_0^*(m) \Delta F_1^*(m)$$

where the constant $p \in [0, 1]$ denotes the probability that a meeting between an owner and an equally optimistic non owner fails to result in a trade. Using integration by parts to simplify the right hand side then leads to

$$\vartheta(p) = \vartheta_c + \sum_{m \in \mathbb{M}} \lambda \left[F_1^*(m) \Delta F(m) - \Delta F_1^*(m) \left(1 + p \Delta F_0^*(m) + \frac{\Delta F_1^*(m)}{2} \right) \right]$$

where the constant ϑ_c is defined as in Proposition 5. This confirms that trading volume increases with the probability of trade execution, but whether it is lower or higher than with a continuous distribution is unclear since the sign of the last term depends on the parameters of the model and the distribution of beliefs.

Having calculated the trading volume implied by the model, I now turn to the steady state expected execution delay defined as the expected amount of time $\delta_q(m) = E[\tau_q]$ that an agent has to wait between transactions.

Proposition 6 *The steady state expected execution delay is given by*

$$\begin{aligned} \delta_q(m) &= \left[1 - \eta \int_{\mathbb{M}} \frac{dF(m)}{b_q(m)} \right]^{-1} \frac{1}{b_q(m)} \\ &= \left[\phi \log \left(\frac{\phi}{1 + \phi} \right) + \left(1 - \frac{1 + \phi}{F_q^*(\overline{m})} \right) \log \left(1 - \frac{F_q^*(\overline{m})}{1 + \phi} \right) \right]^{-1} \frac{1}{b_q(m)} \end{aligned} \tag{35}$$

where the second equality holds only if the underlying distribution of perceived growth rates is continuous and

$$b_q(m) = \eta + \lambda q(1 - s - F(m)) + \lambda F_1^*(m) > \eta.$$

The steady state expected execution delay is decreasing in λ, increasing in s for asset owners and decreasing in s for non owners.

4.3 Search frictions and welfare

The results of Section 4.1 show that the monotone equilibrium of the search market converges to the Walrasian equilibrium as the search friction vanishes. However, they do not say anything about the welfare implications of this friction.

As the search friction weakens, the welfare of agents is subject to two opposite effects because the intensity with which agents meet determines both the extensive margin, i.e. the number of trades, and the intensive margin which corresponds here to the expected gains from a meeting. Proposition 6 shows that an increase in the meeting intensity improves the intensive margin. This implies that agents trade more often and leads to an increase in welfare everything else equal. On the other hand, Proposition 1 shows that the steady state mass $F_1^*(m)$ of owners who are more pessimistic than a given $m \in \mathbb{M}$ is decreasing in the meeting intensity, and since the expected difference in beliefs with a compatible seller

$$E[\mathbf{1}_{\{m_1 \le m\}}(m - m_1)] = \int_{\mathbb{M}} \mathbf{1}_{\{n \le m\}} F_1^*(n) dn$$

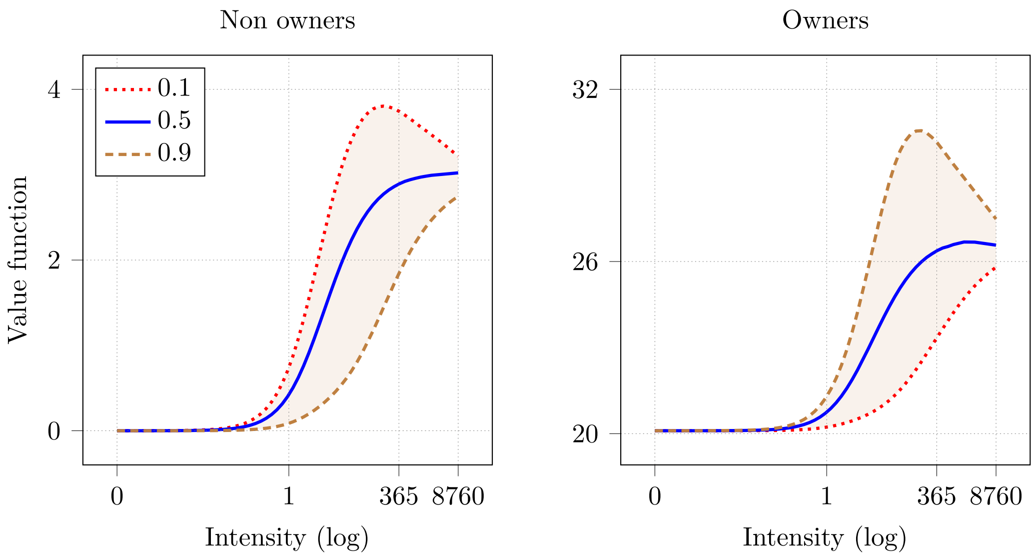
and the expected difference in beliefs with a compatible buyer

$$E[\mathbf{1}_{\{m_0 \ge m\}}(m_0 - m)] = \int_{\mathbb{M}} \mathbf{1}_{\{n > m\}} (1 - s - F(n) + F_1^*(n)) dn$$

both depend positively on this steady state mass, it follows that the counterparties of a given agent tend to be closer to him as the meeting intensity increases. Combining this with the fact that the range of reservation values simultaneously shrinks (see Section 4.4 below) then shows that an increase in the meeting intensity worsens the extensive margin of trade for all agents, and thereby has a detrimental effect on their welfare.

In the absence of trading opportunities, i.e. when the meeting intensity is equal to

FIGURE 5: Welfare and search frictions



Notes. This figure plots the value functions of an owner and a non owner with average beliefs as functions of the meeting intensity for different values of the bargaining power θ_1 of asset owners in an environment where perceived growth rates are initially uniformly distributed over the interval $[-0.04, 0.04]$, the risk free rate is $r = 5\%$, the asset supply is $s = 0.4$ and $\eta = 2$ so that agents change their mind twice a year on average.

zero, the steady state value functions satisfy

$$\lim_{\lambda \to 0} \left(V_q(m, x) - qf(m, x) \right) = 0, \qquad (q, m, x) \in \{0, 1\} \times (0, \infty) \times \mathbb{M}.$$

Combining this with the fact that

$$V_q(m, x) - qf(m, x) = E \int_0^\infty e^{-rt} \lambda \mathcal{E}_{q,s}(m_s, X_s | G) ds > 0$$

for any $\lambda > 0$ when $\theta_q \neq 0$ shows that the first effect dominates in a neighbourhood of the origin and it follows that, starting from a totally illiquid market, both owners and non owners benefit from a higher meeting intensity. Whether their value functions remain monotonic away from the origin is not clear in general, and depends on the bargaining powers, the magnitude of the search friction and the distribution of beliefs. For example, Figure 5 shows that in a market with uniformly distributed beliefs an increase in the meeting intensity may have a very different impact on welfare depending on the bargaining powers, and illustrates the fact that when the bargaining power of a given type is high, so that a decrease in expected trading gains is very costly to that type, an increase in liquidity can lead to a decrease in welfare.

4.4 The cross-section of trading prices

In the stationary monotone equilibrium, the cross-section of trading prices observed by an econometrician who has access to all transaction data is

$$P(m_0, m_1, X_t) = (\theta_0 g(m_1) + \theta_1 g(m_0))\, X_t.$$

In this equation the quantities $m_1 \leq m_0$ are random variables that take value in $\mathbb{M}$ and are distributed according to the cumulative distribution

$$\text{Prob}\left[\{m_0 \leq a\} \cap \{m_1 \leq b\}\right] = \frac{U(a,b)}{\vartheta}$$

where the denominator is the steady state equilibrium trading volume as defined by equation (34), and the function

$$\begin{aligned} U(a,b) &= \int_{\mathbb{M}^2} \mathbf{1}_{\{n \leq a\}} \mathbf{1}_{\{m \leq n \wedge b\}} \lambda dF_0^*(n) dF_1^*(m) \\ &= \lambda F_1^*(b)(F_0^*(a) - F_0^*(b))^+ + \int_{\mathbb{M}} \mathbf{1}_{\{n \leq a \wedge b\}} \lambda F_1^*(n) dF_0^*(n) \end{aligned}$$

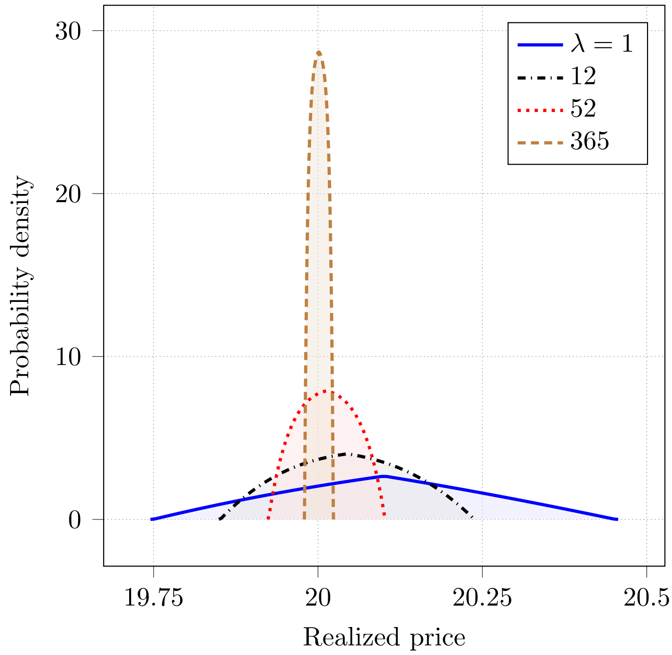
gives the probability that a non owner who is more pessimistic than a meets an owner who is more pessimistic than b with whom a mutually beneficial trade can be agreed upon. When the distribution of perceived growth rates in the population is continuous the above integrals can be computed explicitly using the same change of variable as in the proof of Proposition 5. However, even with this simplification the distribution of realized prices remains too complex to be studied analytically in general.

One exception concerns the impact of liquidity on the support of the distribution. Proposition 4 shows that the interval of realized prices collapses to a single point in the limit of perfect liquidity, and one naturally expects this convergence to be monotonic. This intuition can be confirmed as follows: Theorem 2 shows that bid/ask spread is explicitly given by

$$1 - \frac{\min_{m \in \mathbb{M}} g(m)}{\max_{m \in \mathbb{M}} g(m)} = 1 - \exp\left(-\int_{\mathbb{M}} \frac{dn}{\gamma(n)}\right)$$

and, since the discount rate $\gamma(n)$ is increasing in the meeting intensity (see Appendix B),

Figure 6: Probability density of trading prices

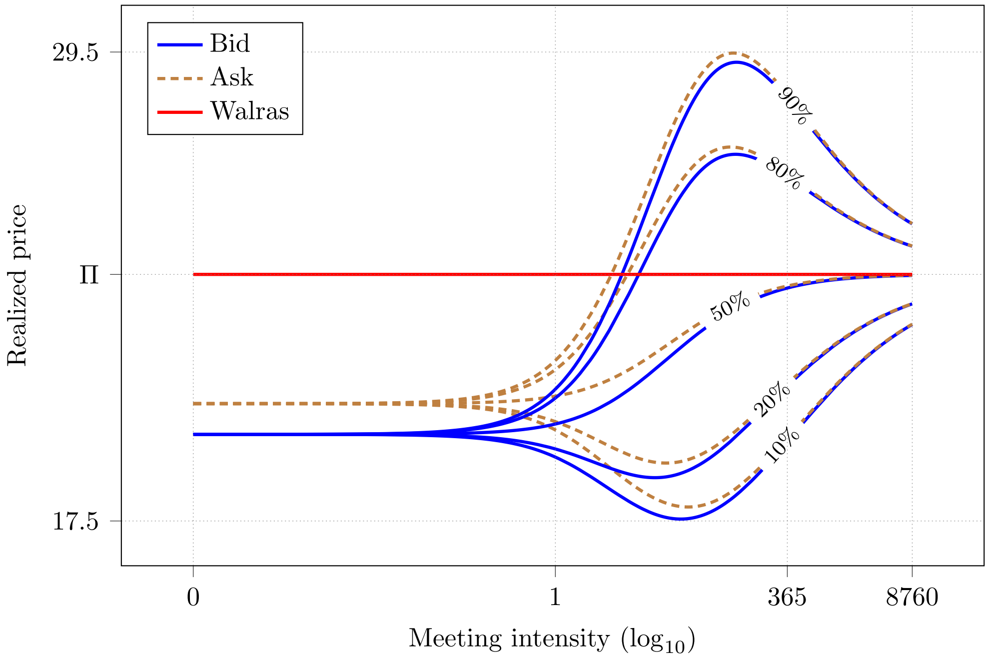


Notes. This figure plots the probability density function of realized prices for various levels of the meeting intensity. To construct this figure I assume that the risk free rate is $r = 5\%$, that the asset supply is $s = 50\%$, that perceived growth rates are uniformly distributed over $[-0.04, 0.04]$ and that agents have equal bargaining powers and change beliefs twice a year on average. Under these assumptions the Walrasian price is given by $\Pi = 20$ independently of the meeting intensity and bargaining powers.

it follows that the relative width of the support is decreasing in the meeting intensity and converges to zero. While it seems difficult to analytically calculate the rate of convergence, numerical experiments suggest that convergence is quite fast. For example, Figure 6 shows that in an environment with uniformly distributed beliefs, homogenous bargaining powers and as many asset owners as non owners the relative support of the distribution is reduced by 50% when the average frequency of meetings increases from monthly to weekly, and by a further 70% when it increases from weekly to daily.

To illustrate the convergence of the cross-section of trading prices to the Walrasian price Figure 7 plots the bid and the ask prices as functions of the meeting intensity for different allocations of the bargaining power in an environment where beliefs are uniformly distributed and there are 40% of asset owners. The figure confirms that the range of prices given by the shaded areas shrinks as the search friction weakens and shows that for a given perceived growth rate the convergence of the gain from becoming an owner is not

Figure 7: Trading prices and search frictions



Notes. This figure plots the bid and ask prices as functions of the meeting intensity for different values of the bargaining power θ_1 of sellers in an economic environment where perceived growth rates are initially uniformly distributed over $[-0.04, 0.04]$, the risk free rate is $r = 5\%$, the asset supply is $s = 0.4$ and $\eta = 2$ so that agents change their mind twice a year on average. Under these assumptions the Walrasian price is given by $\Pi = 23.8095$ independently of the meeting intensity and bargaining powers.

monotonic in general. As explained in Section 4.3 this non monotonicity is due to the conflicting effects of the meeting intensity on the frequency of trades and the expected gains from each individual trade. Interestingly, the figure also shows that while prices include a liquidity discount compared to the Walrasian price for low levels of the meeting intensity they may also include a significant scarcity premium when the meeting intensity and the seller bargaining power are sufficiently high.

4.5 Speculative behavior

In their seminal paper Harrison and Kreps (1978) define speculation as a situation where the right to resell a stock makes investors willing to pay more for it than they would agree to pay if obliged to hold it forever, and show it is a necessary condition for equilibrium if trading is frictionless, agents have constant beliefs over time and holdings are subject to no other restrictions than a ban on short sales.

Following the same definition I say that the equilibrium of the search market exhibits speculative behavior if the cross-section of realized prices is such that

$$\inf_{n\in\mathbb{M}}\left(\theta_1 G(m,x)+\theta_0 G(n\wedge m,x)\right)\geq f(m,x),\qquad m\in\mathbb{M},$$

where the left hand side gives the lowest price that a non asset owner with perceived growth rate m can hope to pay for the asset and the right hand side gives the fundamental value of the asset from the point of view of that same agent. To unveil the conditions under which speculative behavior arises consider first the limiting Walrasian market of Section 4.1 where trading is frictionless subject to the restriction that agents may hold either one or zero unit of the asset. In such a market the equilibrium price is an increasing function of the perceived growth rate

$$m^*(s)=\inf\left\{m\in\mathbb{M}:F(m)\geq 1-s\right\}.$$

of the marginal agent, and since this perceived growth rate is a monotone decreasing function of the asset supply, it is natural to expect that speculative behavior occurs as soon as the asset supply is low enough. The following result confirms this intuition and relies on the continuity of the equilibrium gain from becoming an owner to provide sufficient conditions for speculative behavior to arise in the search market.

Proposition 7 *Speculative behavior occurs in the Walrasian market if and only if*

$$\overline{m}-m^*(s)\leq\eta\left(1-\int_{\mathbb{M}}\frac{\rho(\overline{m})}{\rho(n)}dF(n)\right)\tag{36}$$

and if this condition holds then speculative behavior occurs in the search market for all sufficiently high meeting intensities.

The above result complements the findings of Harrison and Kreps (1978) by showing that with time-varying beliefs and a restriction on asset holdings speculation only occurs if the asset is sufficiently rare. Indeed, the left hand side of (36) is monotone increasing in the asset supply, has value zero at $s=0$ and satisfies

$$\lim_{s\to 1}\left(\overline{m}-m^*(s)\right)=|\mathbb{M}|\geq\int_{\mathbb{M}}(\overline{m}-n)dF(n)\geq\xi\tag{37}$$

where the constant $\xi\geq 0$ denotes the right hand side of condition (36), and the second inequality follows from Assumption 1. Therefore there exists a critical level s^* for the asset supply such that speculation occurs in both the Walrasian market and the search market as soon as the asset is rare enough in the sense that $s \leq s^*$ yet traded at a sufficiently high frequency. For example, in the economic environment of Figure 7 the critical level of the asset supply is given by $s^* \approx 48\%$ and, as can be seen from the figure, the minimal meeting intensity required to induce speculative behavior is monotone decreasing in the bargaining power attributed to asset owners.

An immediate consequence of Proposition 7 is that with restricted asset holdings speculative behavior can only occur if the agents' beliefs are time varying in the sense that $\eta \neq 0$. The intuition for this finding is clear: If beliefs are constant then there is no trading in the stationary equilibrium and, as a result, no agent would ever agree to pay more than his fundamental value. If there are only two types of agents, as in Duffie et al. (2005) and Weill (2008) among many others, then a direct calculation shows that speculative behavior only arises if $s < \Delta F(\overline{m})$ so that not all optimistic agents can hold the asset in equilibrium. In general it follows from (37) that a necessary condition for speculative behavior is that owners be more optimistic than the average.

5 Unrestricted asset holdings

As an extension of the model I now consider the case where agents can hold an arbitrary number of units of the asset subject to a short sale constraint. To avoid the complications induced by tie breaking I assume throughout this section that the distribution of perceived growth rate in the population is *continuous* so that the probability of a meeting between two agents with the same beliefs is zero.

In order to proceed towards the construction of an equilibrium I start by determining the nature and terms of the trades between agents.

5.1 Trading strategies

Consider a meeting between an agent of current type $\sigma_1 = (q_1, m_1)$ and an agent of current type $\sigma_2 = (q_2, m_2)$ where $q_i \in \mathbb{N}$ gives the number of units of the asset held by each agent prior to the meeting. Since agents are subject to a short sale constraint the first agent can buy up to q_2 units, and sell up to q_1 units. Therefore, the maximal trading surplus that can be generated from the meeting is

$$S_t(\sigma_1, \sigma_2, x|V) = \max_{n \in [-q_1, q_2] \cap \mathbb{Z}} \left((V_{q_1+n,t} - V_{q_1,t})(m_1, x) + (V_{q_2-n,t} - V_{q_2,t})(m_2, x) \right).$$

Since agents have linear utility and differ only in their optimism regarding the cash flows of the asset it is natural to look for equilibria in which

$$V_{q,t}(m,x) = V_{0,t}(m,x) + qG_t(m,x) \tag{38}$$

for some non decreasing function $G_t(m,x)$ that gives the gain to the agent from an additional unit of the asset, i.e. the marginal value of the asset. In such a *monotone* equilibrium the maximal surplus from a meeting simplifies to

$$\begin{aligned} S_t(\sigma_1,\sigma_2,x) &= \max_{n\in[-q_1,q_2]\cap\mathbb{Z}} n\left(G_t(m_1,x) - G_t(m_2,x)\right) \\ &= q_2(G_t(m_1,x) - G_t(m_2,x))^+ + q_1(G_t(m_2,x) - G_t(m_1,x))^+ \end{aligned}$$

and can be achieved through Nash bargaining by having the most pessimistic agent sell his whole inventory to the most optimistic agent at the unit price

$$P_t(m_1,m_2) = \theta_1 G_t(m_1 \wedge m_2, x) + \theta_0 G_t(m_1 \vee m_2, x).$$

Taking this trading behavior as given I now determine the induced evolution of the equilibrium distribution of types.

5.2 Equilibrium distribution of asset holdings

Let $F_{q,t}(m)$ denote the mass of agents who hold $q \in \mathbb{N}$ units of the asset and are more pessimistic than a given $m \in \mathbb{M}$. These distributions are initially given and have to be determined endogenously at all subsequent dates subject to

$$0 = F(m) - \sum_{q=0}^{\infty} F_{q,t}(m) = s - \sum_{q=0}^{\infty} qF_{q,t}(\overline{m}). \tag{39}$$

As in the benchmark model, the first constraint requires that the joint distribution of holdings and perceived growth rates be consistent with the distribution of perceived growth rates in the economy. The second constraint is a market clearing condition that requires the average asset holding among the population to equal the asset supply. Note that contrary to the benchmark model there is no reason here to impose an upper bound on the supply parameter so I only assume that $s > 0$.

To determine the evolution of these distributions I proceed as in the restricted holdings case by considering the ways in which agents enter or exit the group of owners who hold exactly q units and are more pessimistic than a given $m \in \mathbb{M}$.

1. An agent may exit because he has perceived growth rate $m_1 \leq m$ and meets an agent with perceived growth rate $m_2 \geq m_1$ to whom he sells his whole inventory. The contribution of such exits is

$$-\int_{\mathbb{M}^2} \mathbf{1}_{\{m_1 \leq m \wedge m_2\}} \lambda dF_{q,t}(m_1) dF(m_2).$$

2. An agent may exit because he has perceived growth rate $m_1 \leq m$ and meets an asset owner with perceived growth rate $m_2 \leq m_1$ whose inventory he buys. The contribution of such exits is

$$-\sum_{n=1}^{\infty} \int_{\mathbb{M}^2} \mathbf{1}_{\{m_2 \leq m_1 \leq m\}} \lambda dF_{q,t}(m_1) dF_{n,t}(m_2).$$

3. An agent may enter because he is of type $(q-n, m_1)$ for some $n \leq q$ and meets an agent of type (n, m_2) with perceived growth rate $m_2 \leq m_1$ whose inventory he buys. The contribution of such entries is

$$\sum_{n=1}^{q} \int_{\mathbb{M}^2} \mathbf{1}_{\{m_2 \leq m_1 \leq m\}} \lambda dF_{n,t}(m_2) dF_{q-n,t}(m_1).$$

4. An agent holding q units may enter or exit because his perceived growth rate has been reset. The contribution of such events is $\eta F_{q,t}(\overline{m})F(m) - \eta F_{q,t}(m)$.

Summing the contributions of these five channels and using integration by parts it can be shown (see Appendix B for details) that for $q \geq 1$

$$\dot{F}_{q,t}(m) = -\lambda F_{q,t}(m) + \nu \sum_{n=0}^{q} F_{n,t}(m) F_{q-n,t}(m) + \eta(F_{q,t}(\overline{m})F(m) - F_{q,t}(m)) \tag{40}$$

where $\nu = (\lambda/2)$ gives the intensity with which an individual agent contacts others. Combining this with the first equality in (39) then shows that the corresponding mass of non asset owners evolves according to

$$\dot{F}_{0,t}(m) = \nu(1 - F_{0,t}(m))^2 - \nu(1 - F(m))^2 + \eta\left(F_{0,t}(\overline{m})F(m) - F_{0,t}(m)\right). \tag{41}$$

and solving that equation delivers the following result.

Proposition 8 *In a monotone equilibrium with unrestricted asset holdings the mass of non asset owner is given by*

$$F_{0,t}(\overline{m}) = 1 - \frac{1 - F_{0,0}(\overline{m})}{1 + \nu t(1 - F_{0,0}(\overline{m}))}$$

and converges to one. In particular, the model with unrestricted asset holdings does not admit a steady state distribution.

The intuition for this finding is clear: with unrestricted asset holdings the more optimistic agent accumulate more and more holdings over time until the whole supply of the asset is held by a vanishingly small group of extremely optimistic agents.

Despite the fact that the model does not converge to a meaningful steady state it is nonetheless interesting to analyze its dynamics over a finite time horizon. In order to do so it is necessary to solve the infinite dimensional system given by (40). As a first step in this direction, consider the discrete-time Fourier transform

$$\Phi_{z,t}(m) = \sum_{q=0}^{\infty} e^{-izq} F_{q,t}(m), \qquad z \in \mathbb{R}.$$

A direct calculation using (40) together with the fact that $\Phi_{0,t}(m) = 1$ shows that this series is absolutely convergent for any $z \in \mathbb{R}$ and satisfies

$$\dot{\Phi}_{z,t}(m) = \nu(1 - \Phi_{z,t}(m))^2 - \nu(1 - F(m))^2 + \eta\left(\Phi_{z,t}(\overline{m})F(m) - \Phi_{z,t}(m)\right). \qquad (42)$$

subject to an exogenously given initial condition $\Phi_{z,0}(m)$ that can be calculated from the initial distribution of holdings and perceived growth rates. In particular, setting $m = \overline{m}$ on both sides of the above equation shows that

$$\dot{\Phi}_{z,t}(\overline{m}) = \nu(1 - \Phi_{z,t}(\overline{m}))^2$$

and solving this Ricatti equation gives

$$1 - \Phi_{z,t}(\overline{m}) = \frac{1 - \Phi_{z,0}(\overline{m})}{1 + \nu t(1 - \Phi_{z,0}(\overline{m}))}. \qquad (43)$$

Using this function as an input then turns (42) into an ordinary Ricatti equation with a time-dependent forcing term whose unique solution can be derived in terms of confluent hypergeometric functions (see e.g. Abramowitz and Stegun (1964)).

Proposition 9 *The unique solution to (42) is given by*

$$\nu(1-\Phi_{z,t}(m)) = \frac{\dot{Y}_{1,z,t}(m) - A_z(m)\dot{Y}_{2,z,t}(m)}{Y_{1,z,t}(m) - A_z(m)Y_{2,z,t}(m)}$$

where the functions $A_z(m) \in \mathbb{C}$ and $Y_{n,z,t}(m) \in \mathbb{R}$ are defined in the appendix.

Given its discrete-time Fourier transforms, the joint equilibrium distribution of holdings and perceived growth rates can be recovered as

$$F_{q,t}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{izq}\Phi_{z,t}(m)dz, \qquad q \in \mathbb{N}.$$

This integral is easily computed numerically and can even be computed in closed form in some cases as illustrated by the following example.

Example 1 The marginal distribution of asset holdings in the population at a given time is defined by

$$q \longmapsto \mu_{q,t} = F_{q,t}(\overline{m}).$$

Setting $m = \overline{m}$ in the differential equation (40) shows that this marginal distribution solves the Smoluchowski coagulations equations

$$\dot{\mu}_{q,t} = -\mu_{q,t}\sum_{j=1}^{\infty} K_{q,j}\mu_{j,t} + \frac{1}{2}\sum_{j=1}^{q-1} K_{q,j}\mu_{j,t}\mu_{q-j,t}$$

with constant coagulation kernel $K_{q,j} = \lambda$. This system of equations was first derived by Smoluchowski (1916) to describe the distribution of cluster sizes resulting from the gradual coagulation of particles according to an additive reaction under which the collision of two clusters of sizes q_1 and q_2 results in a single cluster of size $q_1 + q_2$. See Aldous (1999), Dubovskii (1994) for a survey of the mathematical theory of coagulation.

The general solution to this system can be computed recursively by observing that the right hand side of (43) is the ratio of two power series and then expressing this ratio as a single power series. This gives

$$\mu_{0,t} = 1 - \frac{1-\mu_{0,0}}{1+\nu t(1-\mu_{0,0})}$$

and

$$\mu_{q,t} = \frac{1}{1+\nu t(1-\mu_{0,0})}\left((1-\nu t)\mu_{q,0} + \nu t \sum_{j=1}^{q} \mu_{j,0}\mu_{q-j,t}\right), \qquad q \geq 1.$$

In some cases this recursive computation can even be simplified to obtain an explicit solution for the marginal distribution of asset holdings. For example, if $s \in \mathbb{N}$ and the initial distribution is such that all agents in the market hold s units of the asset, then the distribution at subsequent dates it is given by

$$\mu_{q,t} = \mathbf{1}_{\{q \in s\mathbb{N}\}} \left(\frac{1}{1+\nu t}\right)^2 \left(\frac{\nu t}{1+\nu t}\right)^{q/s-1}$$

for $q \geq 1$ and

$$\mu_{0,t} = 1 - \sum_{q=1}^{\infty} \mu_{q,t} = \frac{\nu t}{1+\nu t}.$$

Similarly, if the marginal distribution of asset holdings in the population is initially geometric with mean $s > 0$ then at subsequent times it is explicitly given by

$$\mu_{q,t} = \frac{s^q}{(1+\nu t)^2}\left(\frac{1+\nu t}{1+s(1+\nu t)}\right)^{1+q}$$

for $q \geq 1$ and

$$\mu_{0,t} = 1 - \sum_{q=1}^{\infty} \mu_{q,t} = 1 - \frac{s}{1+s(1+\nu t)}.$$

In both cases it is easily seen that the marginal distribution converges to a Dirac mass at zero in accordance with Proposition 8. In order to circumvent this behaviour, and thereby obtain a steady state it is necessary to modify the model by introducing the possibility of security issuance and/or default on assets. I leave this extension for further research.

5.3 Equilibrium value functions

Having characterized the distribution of types it now remains to compute the value functions and to verify the conjectured linearity in asset holdings. Taking as given the distribution of beliefs among owners and non owners I define the value functions through the system of dynamic programming equations

$$V_{q,t}(m,x) = E_t\left[\int_t^\tau e^{-r(s-t)} qX_s ds + e^{-r(\tau-t)}(V_{q,\tau}(m_\tau, X_\tau) + \hat{\mathcal{E}}_{q,\tau}(m_\tau, X_\tau|V))\right] \quad (44)$$

where the stopping time τ denotes the first time that the agent gets an opportunity to trade, and the integral operator

$$\hat{\mathcal{E}}_{q,t}(m,x|v) = \sum_{p=0}^{\infty}\int_{\mathbb{M}} R_t((m,q),(n,p),x|v)dF_{p,t}(n)$$

with

$$R_t(\sigma_1,\sigma_2,x|v) = (\theta_0 + \mathbf{1}_{\{n_t^*(\sigma_1,\sigma_2,x|v)\leq 0\}}(\theta_1-\theta_0))S_t(\sigma_1,\sigma_2,x)$$
$$n_t^*(\sigma_1,\sigma_2,x|v) = \underset{n\in[-q_1,q_2]\cap\mathbb{Z}}{\operatorname{argmax}}\ ((v_{q_1+n,t}-v_{q_1,t})(m_1,x)+(v_{q_2-n,t}-v_{q_2,t})(m_2,x))$$

represents the expected payoff to an agent of type (m,q) from a meeting with another randomly selected agent. To pin down a unique equilibrium I further require that the value functions satisfy the linear growth condition

$$|V_{q,t}(m,x)| \leq c_v(1+q)x, \qquad (q,t,m)\in\mathbb{N}\times[0,\infty)\times\mathbb{M}, \quad (45)$$

for some constant $c_v > 0$ and the same arguments as in the model with restricted holdings show that under this condition (44) is equivalent to

$$V_{q,t}(m,x) = E_t\int_t^\infty e^{-r(s-t)}(qX_s + \lambda\hat{\mathcal{E}}_{q,s}(m_s,X_s|V))ds. \quad (46)$$

An immediate calculation using the constraint (39) then shows that solving (46) subject to (45) for a function of the form (38) is equivalent to finding a nondecreasing solution to the single dynamic programming equation

$$G_t(m,x) = E_t\int_t^\infty e^{-r(s-t)}(X_s + \lambda\hat{\mathcal{O}}_s(m_s,X_s|G))ds \quad (47)$$

subject to the linear growth condition (18) where the linear operator on the right hand side is defined by

$$\hat{\mathcal{O}}_t(m,x|v) = \int_{\mathbb{M}} \mathbf{1}_{\{n>m\}}(v_t(n,x)-v_t(m,x))\theta_1 dF(n).$$

In order to construct the unique function satisfying these requirements consider the equivalent probability measure $\hat{P}$ under which the compensator of the changes in an agent's perceived growth rate is given by

$$\hat{Q}(dt, \mathbb{B}) = Q(dt, \mathbb{B}) + \int_{\mathbb{B}\cap\{n>m_{t-}\}} \lambda\theta_1 dF(n)dt, \qquad \mathbb{B} \subseteq \mathcal{B}(\mathbb{M}).$$

As in the model with restricted holdings this auxiliary probability measure can be seen as tracking the beliefs of the marginal agent. Indeed, the first term in the above definition reflects exogenous changes in the agent's perceived growth rate while the second reflects a positive jump that occurs every time that the current holder of the asset meets a more optimistic agent to whom he sells his his whole inventory.

Consistent with this interpretation the next result shows that the marginal value of the asset is given by the fundamental value of the asset to an hypothetical agent whose beliefs are represented by the probability measure $\hat{P}$. In order to state the result, introduce the strictly positive discount rate

$$\hat{\gamma}(m) = r - m + \eta + \lambda\theta_1(1 - F(m))$$

and define the bounded function $\hat{\Gamma}(m)$ as in (26) albeit with the function $\hat{\gamma}(m)$ in place of the function $\gamma(m)$.

Theorem 3 *The unique solution to (18) and (47) is*

$$G_t(m, x) = \hat{E}_t \int_t^\infty e^{-r(s-t)} X_s ds = \frac{x\hat{\Gamma}(m)}{r - \overline{m} + \int_{\mathbb{M}} \eta(1 - \hat{\Gamma}(n))dF(n)}$$

and is nondecreasing in the perceived growth rate. In particular, there exists a unique monotone equilibrium with unrestricted asset holdings.

A key difference between the restricted and unrestricted models is that in the later the marginal asset value is stationary and can be computed in closed for any distribution. To understand this result, observe that in the unrestricted model an agent who holds $q + 1$ units has the same buying options as an agent who holds q units but has one additional option to sell. Since the agent can sell the asset to any other agent who is more optimistic, the value of that additional option only depends on the population wide distribution of beliefs and, therefore, is time independent.

By contrast, in the restricted model an asset owner can only sell to a more optimistic non owner and a non owner can only buy from a more pessimistic owner. The equilibrium gain from becoming an asset owner therefore depends on the distribution of beliefs in the two groups and, since these distributions are time dependent prior to reaching the steady state, it follows that the gain from becoming an asset owner is stationary.

A Proofs

Lemma A.1 *Assume that $|Y_t| \leq S_t$ where the process S_t is a geometric Brownian motion with strictly negative drift and arbitrary volatility. Then $\lim_{t\to\infty} Y_t = 0$ and there exists a constant $a > 1$ such that $E_t \sup_{t\geq 0} |Y_t|^a < \infty$.*

Proof. Assume that the process S_t has drift $\mu < 0$ and volatility v and fix an arbitrary constant $1 < a < 1 - 2\mu/v^2$. By application of Itô's lemma I have that the process S_t^a is a geometric Brownian motion with drift $a\mu + a(a-1)v^2/2 < 0$ and the second part now follows from Doob (1949, Eq.(4.2)). To establish the first part I start by observing that the process S_t is a nonnegative supermartingale since $\mu < 0$. This implies that the process converges almost surely to some nonnegative random variable S_∞ and the desired follows since $E[S_\infty] \leq \lim_t E[S_t] = \lim_t e^{\mu t} S_0 = 0$ by Fatou's lemma. Q.E.D.

Proof of Lemma 1. Using the independence between the Brownian motion and the Poisson process I deduce that

$$\psi_t(m) \equiv f_t(m,x)/x = E_t \int_t^\infty e^{-\int_t^s (r-m_u)du} ds$$

and it follows from Assumption 1 that $\psi_t(m)$ is bounded. Let σ denote the next time that the agent's perceived growth rate changes. Using the fact that $m_s = m$ on $[\![t, \sigma[\![$ and applying the law of iterated expectation gives

$$\psi_t(m) = E_t \int_t^\sigma e^{-(r-m)(s-t)} ds + E_t[e^{-(r-m)(\sigma-t)} \psi_\sigma(m_\sigma)].$$

Now integrating inside the expectation with respect to the joint distribution of the stopping time σ and the perceived growth rate m_σ I obtain

$$\psi_t(m) = \int_t^\infty e^{-\rho(m)(s-t)} (1 + \eta \mathcal{B}(\psi_s)) ds \tag{A.1}$$

where the discount rate function $\rho(m)$ is defined as in the statement and

$$\mathcal{B}(v) = \int_{\mathbb{M}} v(n) dF(n).$$

To solve this recursive integral equation I look for a time-independent solution in the form $\psi_t(m) = (1+\varphi)/\rho(m)$. Inserting this conjecture into (A.1) immediately shows that the integration constant is given by

$$1 + \varphi = \frac{1}{1 - \mathcal{B}(\eta/\rho)} > 0 \tag{A.2}$$

where the inequality follows Assumption 1 and the definition of $\rho(m)$. To show that this solution is in fact unique consider the integral operator

$$\mathcal{T}_t(m, v) = \int_t^\infty e^{-\rho(m)(s-t)}(1 + \eta\mathcal{B}(v_s))ds$$

Assumption 1 implies that this operator maps the space $L^\infty(\mathbb{M} \times [0, \infty))$ into itself. Furthermore, a direct calculation shows that

$$|\mathcal{T}_t(m, v_1) - \mathcal{T}_t(m, v_2)| \leq \frac{\eta}{\rho(m)} \sup_{(m,t)\in\mathbb{M}\times[0,\infty)} |v_{1,t}(m) - v_{2,t}(m)| \, .$$

Since $(\eta/\rho(m)) < 1$ as a result of Assumption 1 this shows that the operator $\mathcal{T}$ is a contraction on the space of real valued bounded functions and it thus follows from the Banach fixed point theorem that (A.1) can have at most one bounded solution. Q.E.D.

Proof of Proposition 1. The differential equation (14) is a standard Ricatti equation whose solution can be found in any textbook. In order to show that the solution satisfies the constraint (3) set $m = \overline{m}$ in equation (13) to obtain

$$\dot{F}_{1,t}(\overline{m}) = \lambda F_{1,t}(\overline{m})(s - F_{1,t}(\overline{m})),$$

and observe that the unique solution with initial value s is $F_{1,t}(\overline{m}) = s$. In order to establish the second part I argue as follows. By the first part of the proposition I have that there exists a smooth function such that

$$0 \leq F_{1,t}(m) = B(t, F(m), F_{1,0}(m)) \leq F(m)$$

Since the initial distribution is absolutely continuous with respect to $F(m)$ this shows that $F_{1,t}(m)$ is also absolutely continuous with respect to $F(m)$ and it therefore follows from the Radon-Nikodym theorem that

$$F_{1,t}(\mathbb{A}) = \int_{\mathbb{A}} \left(\alpha^c_{1,t}(m)dF^c(m) + \alpha^d_{1,t}(m)dF^d(m)\right) \tag{A.3}$$

for any set $\mathbb{A} \subseteq \mathbb{M}$ where the finite measures $dF^c(m)$ and $dF^d(m)$ denote respectively the continuous part and the purely atomic part of the population wide distribution, and the nonnegative functions $\alpha^i_{1,t}(m)$ are defined by

$$\alpha^c_{1,t}(m) = \frac{\partial B}{\partial x}(t, F(m), F_{1,0}(m)) + \alpha^c_{1,0}(m)\frac{\partial B}{\partial y}(t, F(m), F_{1,0}(m)),$$

and

$$\alpha^d_{1,t}(m) = \sum_{k=1}^{\infty} \mathbf{1}_{\{m=m_k\}} \frac{B(t, F(m_k), F_{1,0}(m_k)) - B(t, F(m_{k-1}), F_{1,0}(m_{k-1}))}{p(m_k)}.$$

Furthermore, since the distribution $F_{1,t}(m)$ is nonnegative and bounded from above by the population wide distribution I have that the functions $\alpha^i_{1,t}(m)$ are uniformly bounded. Similarly, I know from the first part of the proposition that

$$0 \leq F^*_1(m) = B^*(F(m)) \leq F(m)$$

for some smooth function and it follows that

$$F^*_1(\mathbb{A}) = \int_{\mathbb{A}} (\alpha^{c,*}_1(m)dF^c(m) + \alpha^{d,*}_1(m)dF^d(m))$$

for any set $\mathbb{A} \subseteq \mathbb{M}$ where the nonnegative functions $\alpha^{i,*}_1(m)$ are uniformly bounded and explicitly defined by

$$\alpha^{c,*}_1(m) = \frac{\partial B^*}{\partial x}(F(m)),$$

and

$$\alpha^{d,*}_1(m) = \sum_{k=1}^{\infty} \mathbf{1}_{\{m=m_k\}} \frac{B^*(F(m_k)) - B^*(F(m_{k-1}))}{p(m_k)}.$$

A direct calculation based on (15), (16) then shows that $\lim_t \alpha^i_{1,t}(m) = \alpha^{i,*}_1(m)$ and, since the integrands in (A.3) are uniformly bounded, it follows from the dominated convergence theorem that

$$\begin{aligned}\lim_{t\to\infty} F_{1,t}(\mathbb{A}) &= \lim_{t\to\infty} \int_{\mathbb{A}} \left(\alpha^c_{1,t}(m)dF^c(m) + \alpha^d_{1,t}(m)dF^d(m)\right) \\ &= \int_{\mathbb{A}} \lim_{t\to\infty} \left(\alpha^c_{1,t}(m)dF^c(m) + \alpha^d_{1,t}(m)dF^d(m)\right) = F^*_1(\mathbb{A}).\end{aligned}$$

Since $\mathbb{A} \subseteq \mathbb{M}$ is arbitrary this shows that the equilibrium distribution $F_{1,t}(m)$ converges strongly to its steady state counterpart $F_1^*(m)$ and completes the proof. Q.E.D.

Proof of Corollary 1. A direct calculation shows that

$$-\frac{\partial F_0^*(m)}{\partial s} = \frac{\partial F_1^*(m)}{\partial s} = \frac{F_1^*(m) + \phi F(m)}{\Phi(m)} \geq 0$$

which implies the desired monotonicity in s. Using the definition of the steady state distribution it can be shown that

$$\frac{\partial F_1^*(m)}{\partial \phi} = \frac{sF(m) - F_1^*(m)}{\Phi(m)} = \frac{s(1-s)F(m)(1-F(m))}{(\phi + F_1^*(m) + (1-s)(1-F(m)))\Phi(m)}$$

and the desired monotonicity follows by observing that all the terms on the right hand side are nonnegative. Knowing that $F_1^*(m)$ is increasing in ϕ a further calculation shows that $\Phi(m)$ is also increasing in ϕ and it now follows from the above equation that the steady state distribution is concave in ϕ. The expression for the limiting values follows by sending ϕ to 0 and ∞ in (15), I omit the details. Q.E.D.

Lemma A.2 *Assume that $F \in \mathcal{F}_{\text{ns}}$. Then the function*

$$Y_t(m, x) = E_t^* \int_t^\infty e^{-r(s-t)} X_s ds$$

is nondecreasing in the agent's perceived growth rate and satisfies (18)*.*

Proof. The same arguments as in the proof of equation (11) show that

$$y_t(m) = \frac{Y_t(m, x)}{x} = E_t^* \int_t^\infty e^{-\int_t^s (r - m_u) du} ds \tag{A.4}$$

and the second part of the statement now follows from Assumption 1 which implies that the function $y_t(m)$ satisfies

$$\|y\|_\infty = \sup_{(t,m) \in [0,\infty) \times \mathbb{M}} |y_t(m)| \leq (r - \overline{m})^{-1}. \tag{A.5}$$

In order to prove that the first part of the statement and thereby complete the proof I will proceed in several steps.

Step 1: *Continuity and differentiability in time.* Let $\tau \geq t$ denote the first time that the agent's growth rate changes. Using (A.4) in conjunction with the law of iterated expectations and the definition of P^* gives

$$y_t(m) = E_t^*\left[e^{-(r-m)(\tau-t)}y_\tau(m_\tau) + \int_t^\tau e^{-(r-m)(s-t)}ds\right]$$
$$= \int_t^\infty e^{-\int_t^s \gamma_u(m)du}\left(1 + \int_{\mathbb{M}} y_s(k)(\eta dF(k) + \lambda dH_s(k|m))\right) ds$$

where the discount rate

$$\gamma_t(m) = \rho(m) + \lambda\theta_1(1 - s - F_{0,t}(m)) + \lambda\theta_0 F_{1,t}(m) \tag{A.6}$$

is right-continuous, strictly positive and bounded. Since all the terms inside the integral are uniformly bounded this implies that the function $y_t(m)$ is continuously differentiable in time for each fixed $m \in \mathbb{M}$ and satisfies

$$\dot{y}_t(m) = \gamma_t(m)y_t(m) - 1 - \int_{\mathbb{M}} y_t(k)(\eta dF(k) + \lambda dH_t(k|m))$$

Step 2: *Continuity in space.* Let $n > m$ be elements of $\mathbb{M}$ and define

$$c_t(n,m) = y_t(n) - y_t(m).$$

Using the above equation in conjunction with (20) shows that

$$\dot{c}_t(n,m) = \gamma_t(n)y_t(n) - \gamma_t(m)y_t(m) + \int_{\mathbb{M}} y_t(k)\lambda(dH_t(k|m) - dH_t(k|n))$$
$$= \gamma_t(n)c_t(n,m) - y_t(m)(n-m) - \int_m^n c_t(m,k)d\nu_t(k)$$

where $\nu_t(m)$ is the right continuous and bounded function defined by

$$\nu_t(m) = \lambda\left(\theta_1 F_{0,t}(m) - \theta_0 F_{1,t}(m)\right).$$

Using the facts that the function $c_t(n,m)$ is uniformly bounded and that the discount rate is strictly positive shows that

$$c_t(n,m) = \int_t^\infty e^{-\int_t^s \gamma_u(n)du}\left(y_s(m)(n-m) + \int_m^n c_s(m,k)d\nu_s(k)\right) ds \tag{A.7}$$

and, since all the terms below the integral are uniformly bounded it now follows from the dominated convergence theorem and the right-continuity of the functions $\gamma_t(m)$ and

$\nu_t(m)$ with respect to m that

$$\lim_{n\to m} c_t(n,m) = \int_t^\infty e^{-\int_t^s \gamma_u(m)du} \left(\lim_{n\to m} \int_m^n c_s(m,k)d\nu_s(k) \right) ds = 0$$

which establishes the right-continuity of the function $y_t(m)$ with respect to m. Similarly, if $n < m$ are arbitrary elements of $\mathbb{M}$ then

$$c_t(n,m) = \int_t^\infty e^{-\int_t^s \gamma_u(n))du} \left(y_s(m)(n-m) - \int_n^m c_s(m,k)d\nu_s(k) \right) ds$$

and, since all the terms below the integral are uniformly bounded by construction, it follows from the dominated convergence theorem and the integration by parts formula for Lebesgue Stieljes integrals that

$$\begin{aligned}\lim_{n\to m} c_t(n,m) &= \int_t^\infty e^{-\int_t^s \gamma_u(m-)du} \left(-\lim_{n\to m} \int_n^m c_s(m,k)d\nu_s(k) \right) ds \\ &= \int_t^\infty e^{-\int_t^s \gamma_u(m-)du} \lim_{n\to m} \left(-y_s(m)\Delta\nu_s(k) + \int_n^m y_s(k)d\nu_s(k) \right) ds = 0.\end{aligned}$$

This shows that the function $y_t(m)$ is left continuous with respect to m and combining this property with the first part finally shows that it is continuous.

Step 3: *Monotonicity in space.* Fix $k \in \mathbb{N}$, let $m_k \leq m < m + 1/n < m_{k+1}$ for some $n \in \mathbb{N}$ and consider the difference quotient

$$\begin{aligned}nc_t(m+1/n, m) &= \frac{y_t(m+1/n) - y_t(m)}{1/n} \\ &= \int_t^\infty e^{-\int_t^s \gamma_u(m+1/n)du} \left(y_s(m) + n \int_m^{m+1/n} c_s(m,k)d\nu_s(k) \right) ds\end{aligned}$$

where the second equality follows from (A.7). Combining the decomposition in (A.3) with the assumption $F \in \mathcal{F}_{\text{ns}}$ shows that the quantity inside the bracket on the right hand side of the above expression can be written as

$$\begin{aligned}A_{n,s}(m) &= y_s(m) + n \int_m^{m+1/n} c_s(m,k)d\nu_s(k) \\ &= y_s(m) + n \int_m^{m+1/n} c_s(m,k)\lambda(\theta_1\alpha^c_{0,s}(k) - \theta_0\alpha^c_{1,s}(k))f(k)dk\end{aligned}$$

for some nonnegative functions such that

$$\alpha_{0,t}^c(m) + \alpha_{1,t}^c(m) = 1.$$

Using this decomposition together with the uniform boundedness of $y_t(m)$, the definition of the constants θ_q and the local boundedness of the population wide density function then shows that

$$\begin{aligned}|A_{n,t}(m)| &\leq |y_t(m)| + \lambda n \int_m^{m+1/n} |c_t(m,k)|(\theta_1\alpha_{0,t}^c(k) + \theta_0\alpha_{1,t}^c(k))f(k)dk \\ &\leq |y_t(m)| + \lambda n \int_m^{m+1/n} |c_t(m,k)|f(k)dk \\ &\leq \|y\|_\infty \left(1 + 2\lambda n \int_m^{m+1/n} f(k)dk\right) \leq (1 + 2\lambda K(m))\,\|y\|_\infty\end{aligned}$$

where the finite constant $K(m) > 0$ is an upper bound on the population wide density function in a right neighbourhood of the point m. Since

$$\lim_{n\to\infty}(A_{n,t}(m) - y_t(m)) = \lim_{n\to\infty} n \int_m^{m+1/n} c_s(m,k)d\nu_s(k) = 0,$$

due to the continuity of the function $y_t(m)$ it follows from the dominated convergence theorem and the right continuity of the discount rate that the function $y_t(m)$ is right differentiable with

$$D^+y_t(m) = \lim_{n\to\infty} nc_t(m + 1/n, m) = \int_t^\infty e^{-\int_t^s \gamma_u(m)du} y_s(m)ds \geq 0$$

and the required monotonicity now follows from the result of Step 2 and Titchmarsh (1975, Example IV.11.3), see Hagood and Thomson (2006). Q.E.D.

Remark A.1 Using the bound (A.5) in conjunction with the fact that the discount rate is uniformly bounded away from zero it is easily shown that

$$\sup_{(t,m)\in[0,\infty)\times\mathbb{M}} \|D^+y_t(m)\|_\infty \leq (r - \overline{m})^{-2}$$

and it thus follows from Assumption 1, Hagood and Thomson (2006, Theorem 9) and Fubini's theorem that $y_t(m)$ is absolutely continuous and satisfies

$$y_t(\overline{m}) - y_t(m) = \int_m^{\overline{m}} D^+y_t(n)dn = \int_t^\infty ds \int_m^{\overline{m}} e^{-\int_t^s \gamma_u(n)du} y_s(n)dn.$$

This in turn shows that $y_t(m)$ is almost everywhere jointly differentiable in (t, m) and implies that it solves the hyperbolic partial differential equation

$$\frac{d}{dt}(D^+ y_t(m)) = \gamma_t(m) D^+ y_t(m) - y_t(m).$$

Note however that unless $F(m)$ is continuous this solution should be understood in the weak sense since both the discount rate and the space derivative on the right hand side are discontinuous at each of the countably many atoms of the population wide distribution of perceived growth rates.

Proof of Theorem 1. Assume that the function $G_t(m, x)$ solves equation (19) subject to (18) so that the process

$$M_t = e^{-rt} G_t(m_t, X_t) + \int_0^t e^{-rs} \left(X_s + \lambda \mathcal{O}_s(m_s, X_s | G) \right) ds$$

is a martingale over any finite horizon under the probability measure P. Using this property together with Girsanov theorem and the fact that

$$\Delta M_t = e^{-rt} \Delta G_t(m_t, X_t)$$

I obtain that the process

$$M_t^* = M_t - \int_0^t e^{-rs} \lambda \mathcal{O}_s(m_s, X_s | G) ds = e^{-rt} G_t(m_t, X_t) + \int_0^t e^{-rs} X_s ds$$

is a local martingale under the equivalent probability measure P^*. This in turn implies that there exists an sequence of stopping times $(\sigma_n)_{n=1}^{\infty}$ that is almost surely increasing with $t \leq \sigma_n \to \infty$ and such that

$$\begin{aligned} G_t(m, x) &= \lim_{n \to \infty} E_t^* \left[e^{-r(\sigma_n - t)} G_{\sigma_n}(m_{\sigma_n}, X_{\sigma_n}) + \int_t^{\sigma_n} e^{-r(s-t)} X_s ds \right] \\ &= \lim_{n \to \infty} E_t^* \left[e^{-r(\sigma_n - t)} G_{\sigma_n}(m_{\sigma_n}, X_{\sigma_n}) + \int_t^{\infty} e^{-r(s-t)} X_s ds \right] \end{aligned}$$

where the second equality follows from the nonnegativity of cash flows and the monotone convergence theorem. Using Assumption 1 together with (18) then shows that

$$e^{-r(\sigma_n - t)} |G_{\sigma_n}(m_{\sigma_n}, X_{\sigma_n})| \leq c_g \left(\frac{X_t}{S_t} \right) S_{\sigma_n}$$

where the process S_t is a geometric Brownian motion with drift $\overline{m} - r < 0$ under the probability measure P^* and it now follows from Lemma A.1 and the dominated convergence theorem that

$$G_t(m, x) = E_t^* \int_t^\infty e^{-r(s-t)} X_s ds = Y_t(m, x).$$

This shows that $Y_t(m, x)$ is the unique solution to (19) such that (18) holds and the proof is complete since it follows from Lemma A.2 that this solution is nondecreasing with respect to the agent's perceived growth rate for any $F \in \mathcal{F}_{\text{ns}}$. Q.E.D.

Proof of Proposition 2. The result follows from Theorem 1 and the definition of the equivalent probability measure P^*. I omit the details. Q.E.D.

Proof of Theorem 2. The equilibrium distributions and the gain from becoming an owner have already been derived so the only thing that requires a proof is the expression for the value function but this follows by direct calculation. Q.E.D.

Proof of Corollary 2. From Proposition 1 I know that $F_{q,t}(m)$ converges to the steady state distribution $F_q^*(m)$ so all there is to prove is that the gain from becoming an owner also converges to its steady state counterpart. Let the discount rate $\gamma_t(m)$ be as in (A.6) and consider the bounded function defined by

$$\begin{aligned} g_t(m) = x^{-1} G_t(m, x) = E_t^* \int_t^\infty e^{-\int_t^s (r - m_u) du} ds \\ = \int_t^\infty e^{-\int_t^s \gamma_u(m) du} \left(1 + \int_{\mathbb{M}} g_s(n) \left(\eta dF(n) + \lambda dH_s(n|m)\right)\right) ds \end{aligned}$$

where the second equality follows from the definition of the probability measure P^* and the law of iterated expectations. Combining this identity with an application of l'Hôpital rule I obtain that the limit exists and satisfies

$$g^*(m) := \lim_{t \to \infty} g_t(m) = \lim_{t \to \infty} \frac{1}{\gamma_t(m)} \left(1 + \int_{\mathbb{M}} g_t(n) \left(\eta dF(n) + \lambda dH_t(n|m)\right)\right).$$

By Proposition 1 I have that the equilibrium distributions converge strongly to their steady state counterparts and it follows that $\lim_t \gamma_t(m) = \gamma(m)$. On the other hand, since the function $g_t(m)$ is uniformly bounded the same arguments as in the proof of Proposition 1 show that

$$\lim_{t \to \infty} \int_{\mathbb{M}} g_s(n) \left(\eta dF(n) + \lambda dH_s(n|m)\right) = \int_{\mathbb{M}} g^*(n) \left(\eta dF(n) + \lambda dH^*(n|m)\right)$$

where the bounded function

$$H^*(n|m) = \theta_0 F_1^*(n \wedge m) + \theta_1 (F_0^*(n) - F_0^*(m))^+$$

is the steady state counterpart of $H_t(n|m)$. This shows that $g^*(m)$ solves the integral equation (24) and the desired result follows since the unique bounded solution to that equation is given by the function $g(m)$ of Theorem 2. Q.E.D.

Proof of Proposition 3. Fix an arbitrary cutoff $w \in \mathcal{Q}$. In order to establish that the price process $P_t = \Pi(w, X_t)$ is an equilibrium I need to prove that the conjectured trading behavior is optimal given the possibility of continuous trading at this price.

As a first step in this direction I start by calculating the value functions associated with the conjectured trading behavior. Define a pair of stopping times by setting

$$\tau_q = \inf\{t \geq 0 : (2q-1)(w - m_t) \geq 0\}$$

and assume that an agent of ownership type q trades upon the occurrence of the stopping time τ_q. The value functions associated with this trading strategy are defined by the system of dynamic programming equations

$$V_q(m, x) = E\left[\int_0^{\tau_q} e^{-rs} q X_s ds + e^{-r\tau_q}((2q-1)P_{\tau_q} + V_{1-q}(m_{\tau_q}, X_{\tau_q}))\right]. \tag{A.8}$$

The next lemma provides an explicit formula for the unique pair of functions that solve this equation subject to the linear growth condition (7).

Lemma A.3 *The unique solution to* (A.8) *such that* (7) *holds is*

$$V_q(m, x) = \Pi(w, x)\left(q + \frac{(w-m)^-}{\rho(m)} + \frac{\eta(1+\varphi)}{\rho(m)} \int_{\mathbb{M}} \frac{(w-n)^-}{\rho(n)} dF(n)\right) \tag{A.9}$$

Proof. Using the same arguments as in the proof of equation (11) it can be shown that solving the system (A.8) subject to (7) is equivalent to finding a pair of bounded functions that solve the reduced-form system

$$v_q(m) = E\left[\int_0^{\tau_q} e^{-\int_0^s (r-m_u)du} q ds + e^{-\int_0^{\tau_q} (r-m_u)du}((2q-1)p + v_{1-q}(m_{\tau_q}))\right] \tag{A.10}$$

with the constant $p = \Pi(w; 1)$. The definition of the stopping times τ_q immediately implies that any solution to this system must satisfy

$$v_q(m) = (2q-1)p + v_{1-q}(m) \tag{A.11}$$

for all $m \in \mathbb{M}$. Let the stopping time τ denote the first time that the agent's perceived growth rate changes. Combining (A.10) and (A.11) with the law of iterated expectations I obtain that

$$\begin{aligned} v_q(m) &= E\int_0^\tau e^{-(r-m)s} q ds + E\left[e^{-(r-m)\tau}((2q-1)p + v_{1-q}(m_\tau))\right] \\ &= E\int_0^\tau e^{-(r-m)s} q ds + E\left[e^{-(r-m)\tau} v_q(m_\tau)\right] = \frac{1}{\rho(m)}\left(q + \eta\int_{\mathbb{M}} v_q(n) dF(n)\right) \end{aligned}$$

for all $m \in \mathbb{M}$ such that $(2q-1)(w-m) < 0$. Using these restrictions together with the definition of the candidate price process I obtain

$$\begin{aligned} \lim_{m\uparrow w} v_1(m) &= \lim_{m\uparrow w}(p + v_0(m)) \\ &= p + \lim_{m\uparrow w}\left(\frac{\eta}{\rho(m)}\int_{\mathbb{M}} v_0(n) dF(n)\right) \\ &= \frac{1}{\rho(w)}\left(1 + \eta\int_{\mathbb{M}} v_1(n) dF(n)\right) = \lim_{m\downarrow w} v_1(m) \end{aligned}$$

This shows that any solution to (A.10) must be continuous at w and combining this property with the above identities gives

$$v_0(m) = \frac{1}{\rho(m)}\left(p(w-m)^- + \eta\int_{\mathbb{M}} v_0(n) dF(n)\right). \tag{A.12}$$

Using the same arguments as in the proof of Lemma 1 it can be shown that the unique bounded solution to this integral equation is

$$v_0(m) = p\left(\frac{(w-m)^-}{\rho(m)} + \frac{\eta(1+\varphi)}{\rho(m)}\int_{\mathbb{M}} \frac{(w-n)^-}{\rho(n)} dF(n)\right)$$

where $\varphi > 0$ is defined as in (A.2). The corresponding formula for $v_1(m)$ now follows from identity (A.11) and the proof is complete. Q.E.D.

To establish that the stopping times τ_q are optimal I need to verify that the agent never has an incentive to deviate from this strategy in the sense that

$$V_q(m,x) = \sup_{\tau\in\mathbb{S}} E\left[\int_0^\tau e^{-rs}qX_s ds + e^{-r\tau}((2q-1)P_\tau + V_{1-q}(m_\tau, X_\tau))\right] \quad \text{(A.13)}$$

where $\mathbb{S}$ denotes the set of all stopping times. This verification is carried out in the next lemma and concludes the proof. Q.E.D.

Lemma A.4 *The pair of functions defined by* (A.9) *satisfies* (A.13).

Proof. Combining (A.11) with (A.12) shows that the nonnegative function $v_q(m)$ solves the integral equation

$$v_q(m) = \frac{1}{\rho(m)}\left(q + p((2q-1)(w-m))^+ + \eta\int_{\mathbb{M}} v_q(n)dF(n)\right).$$

Applying Itô's lemma to the discounted value function and using the above integral equations it is easily deduced that

$$\begin{aligned} d\left(e^{-rt}V_q(m_t,X_t)\right) &= d\left(e^{-rt}v_q(m_t)X_t\right) \\ &= e^{-rt}dM_t - e^{-rt}\left(qX_t + P_t((2q-1)(w-m_t))^+\right)dt \end{aligned}$$

for some local martingale M_t, and it follows that the process

$$\begin{aligned} Y_{q,t} &= e^{-rt}V_q(m_t,X_t) + \int_0^t e^{-rs}qX_s ds \\ &= V_{q,0}(m_0,X_0) + \int_0^t e^{-rs}dM_s - \int_0^t e^{-rs}P_s((2q-1)(w-m_t))^+ ds \end{aligned}$$

is a nonnegative supermartingale. Combining this with Doob's optional sampling theorem for supermartingales (see Dellacherie and Meyer (1980, Theorem 2.16)) gives

$$\begin{aligned} V_q(m,x) &\geq \sup_{\tau\in\mathbb{S}} E\left[\int_0^\tau e^{-rs}qX_s ds + e^{-r\tau}V_q(m_\tau,X_\tau)\right] \\ &= \sup_{\tau\in\mathbb{S}} E\left[\int_0^\tau e^{-rs}qX_s ds + e^{-r\tau}((2q-1)P_\tau + V_{1-q}(m_\tau,X_\tau))\right] \end{aligned}$$

where the equality follows from (A.11) and the definition of $v_q(m)$. Using (A.8) then shows that the reverse inequality also holds and completes the proof. Q.E.D.

Proof of Proposition 4. The convergence of the distributions follows directly from (31), (32) and Corollary 1 so it suffices to establish the convergence of the trading prices.

Consider the bounded function

$$\beta(m) = \frac{x}{G(m,x)} - r + \overline{m} = \frac{\mathcal{D}^*(\overline{m}|1-\Gamma)}{\Gamma(m)} + (r-\overline{m})\left(\frac{1}{\Gamma(m)} - 1\right).$$

Using the definition of the nonnegative functions $\gamma(m)$ and $\Gamma(m)$ together with the assumption that $\theta_q \in (0,1)$ it is easily deduced that

$$\lim_{\lambda\to\infty}(1/\gamma(m)) = \lim_{\lambda\to\infty}(1-1/\Gamma(m)) = 0. \tag{A.14}$$

and therefore

$$\begin{aligned}
\lim_{\lambda\to\infty}\beta(m) &= \lim_{\lambda\to\infty}\mathcal{D}^*(\overline{m}|1-\Gamma) \\
&= \lim_{\lambda\to\infty}\int_{\mathbb{M}}(1-\Gamma(n))(\eta dF(n)+\lambda\theta_0 dF_1^*(n)) \\
&= \lim_{\lambda\to\infty}\int_{\mathbb{M}}\lambda\theta_0(1-\Gamma(n))dF_1^*(n) \\
&= \lim_{\lambda\to\infty}\int_{\mathbb{M}}\lambda\theta_0 F_1^*(n)\frac{\Gamma(n)}{\gamma(n)}dn \\
&= \lim_{\lambda\to\infty}\int_{\mathbb{M}}\frac{\theta_0 F_1^*(n)dn}{\theta_0 F_1^*(n)+\theta_1(1-s-F_0^*(n))} = \overline{m}-m^*
\end{aligned}$$

where the fourth equality follows from the integration by parts formula, the fifth equality follows from (A.14) and the definition of $\gamma(m)$, and the last equality follows from the convergence of the distributions and the definition of the quantile in (29). Combining this with the definition of $\beta(m)$ then shows that

$$\lim_{\lambda\to\infty}G(m,x) = \lim_{\lambda\to\infty}\left(\frac{x}{r-\overline{m}+\beta(m)}\right) = \frac{x}{r-m^*} = \Pi(m^*,x)$$

and completes the proof. Q.E.D.

Proof of Proposition 5. Assume that the distribution function $F(m)$ is continuous. Using the definition of ϑ and integration by parts I obtain

$$\begin{aligned}
\vartheta(s,\eta,\lambda) &= \int_{\mathbb{M}^2}\mathbf{1}_{\{m\geq n\}}\lambda dF_1^*(n)dF_0^*(m) \\
&= \int_{\mathbb{M}}\lambda F_1^*(m)dF_0^*(m) \\
&= \int_{\mathbb{M}}\lambda F_1^*(m)dF(m) - \frac{\lambda}{2}F_1^*(\overline{m})^2 = \int_0^1\lambda\ell(x)dx - \frac{\lambda s^2}{2}
\end{aligned}$$

where I have set

$$\ell(x) = -\frac{1}{2}(1-s+\phi-x) + \frac{1}{2}\sqrt{4s\phi x + (1-s+\phi-x)^2}, \tag{A.15}$$

the third equality follows from (3) and the last equality follows from (15), the market clearing condition and the change of variable formula for Stieltjes integrals. Computing the integral leads to the formula in the statement and the comparative statics follow by differentiating the result with respect to λ and η. Q.E.D.

Remark A.2 If the cumulative distribution $F(m)$ fails to be continuous then the first two equalities remain valid but the third and fourth transform to

$$\begin{aligned}\vartheta(s,\eta,\lambda) &= \int_{\mathbb{M}} \lambda F_1^*(m) dF(m) - \int_{\mathbb{M}} \lambda F_1^*(m) dF_1^*(m) \\ &= \int_0^1 \lambda \ell(F \circ I(x)) dx - \frac{\lambda}{2}\left(s^2 + \sum_{m\in\mathbb{M}} \Delta F_1^*(m)^2\right)\end{aligned}$$

where the function

$$I(x) = \inf\{m \in \mathbb{M} : F(m) \geq x\}$$

denotes the quantile function. In particular, if the support of the distribution is a countable collection of isolated points then

$$\vartheta(s,\eta,\lambda) = \sum_{k=1}^{\infty} \lambda\left(p(k)\ell(F(m_k)) - \frac{1}{2}\Delta\ell(F(m_k))^2\right) - \frac{\lambda s^2}{2}$$

where the constant $p(k) = \lim_{x\uparrow m_k}(F(m_k) - F(x))$ gives the mass of the k^{th} point. In this case the equilibrium trading rate actually depends on the exogenously fixed distribution of perceived growth rates in the population.

Proof of Proposition 6. Consider an agent of ownership type q and denote his perceived growth rate process by m_t. The next time that this agent trades is the first time τ_q at which he meets an agent of the complementary ownership type whose perceived growth rate satisfies

$$(2q-1)(n-m_\tau) \geq 0.$$

In the steady state monotone equilibrium the arrival rate of this event is

$$\lambda_q(m_t) = \lambda q(1 - s - F_0^*(m_t)) + \lambda(1-q)F_1^*(m) = b_q(m_t) - \eta$$

and it follows that

$$\delta_q(m) = E[\tau_q] = E\int_0^\infty t d\left(1 - e^{-\int_0^t \lambda_q(m_s)ds}\right) = E\int_0^\infty e^{-\int_0^t \lambda_q(m_s)ds}dt.$$

Let σ denote the first time that the agent's perceived growth rate changes. Combining the above expression with the law of iterated expectations gives

$$\begin{aligned}\delta_q(m) &= E\int_0^\sigma e^{-\int_0^t \lambda_q(m_s)ds}dt + E\left[e^{-\int_0^\sigma \lambda_q(m_s)ds}\delta_q(m_\sigma)\right] \qquad \text{(A.16)}\\ &= E\int_0^\sigma e^{-\lambda_q(m)t}dt + E\left[e^{-\lambda_q(m)\sigma}\delta_q(m_\sigma)\right] = \frac{1}{b_q(m)}\left(1 + \eta\int_{\mathbb{M}}\delta_q(n)dF(n)\right)\end{aligned}$$

where the second equality follows from the fact that the agent's perceived growth rate is constant over $[\![0,\sigma]\!]$ and the third equality follows from the fact that

$$P\left(\{\sigma \in dt\} \cup \{m_\sigma \leq n\}\right) = \eta e^{-\eta t}F(n)dt.$$

Integrating both sides of (A.16) with respect to the cumulative distribution $F(m)$ and solving the resulting equation gives

$$1 + \eta\int_{\mathbb{M}}\delta_q(m)dF(m) = \left(1 - \eta\int_{\mathbb{M}}\frac{dF(n)}{b_q(n)}\right)^{-1}$$

and substituting back into (A.16) produces the desired formula. Now assume that the function $F(m)$ is continuous. Combining Proposition 1 with the change of variable formula for Stieljes integrals gives

$$\eta\int_{\mathbb{M}}\frac{dF(n)}{b_q(n)} = \int_0^1 \phi(q(1-s-x) + \phi + \ell(x))^{-1}dx = \kappa(\phi, F_q^*(\overline{m}))$$

where the function $\ell(x)$ is defined as in (A.15) and

$$\kappa(\phi, x) = 1 + \phi\log\left(\frac{1+\phi}{\phi}\right) + \left(1 - \frac{1+\phi}{x}\right)\log\left(\frac{1+\phi}{1+\phi-x}\right).$$

To complete the proof it remains to establish the comparative statics of $\delta_q(m)$ with respect to the meeting intensity. A direct calculation shows that

$$\lambda \frac{\partial^2 b_q(m)}{\partial \lambda^2} = \phi^2 \frac{\partial^2 F_1^*(m)}{\partial \phi^2}.$$

Since the distribution function $F_1^*(m)$ is concave in ϕ this shows that $b_q(m)$ is concave in the meeting intensity and it follows that

$$\begin{aligned}\frac{\partial b_q(m)}{\partial \lambda} \geq \lim_{\lambda\to\infty} \frac{\partial b_q(m)}{\partial \lambda} &= \lim_{\lambda\to\infty} q(1-s-F_0^*(m)) + (1-q)F_1^*(m) \\ &= q(1-s-F(m))^+ + (1-q)(1-s-F(m))^- \geq 0.\end{aligned}$$

This shows that $b_q(m)$ is an increasing function of the meeting intensity and the desired result now follows from equation (35) by noting that the distribution function $F(m)$ does not depend on the meeting intensity.

To complete the proof it remains to establish the comparative statics of $\delta_q(m)$ with respect to the asset supply. For non owners the result follows from (35) by noting that $b_0(m)$ is increasing in s as a result of Corollary 1. To obtain the result for asset owners I start by observing that $b_1(m)$ is a convex in s since

$$\frac{\partial^2 b_1(m)}{\partial s^2} = \frac{2\eta F(m)(1+\phi)(1-F(m))}{\Phi(m)^3} \geq 0.$$

This implies that

$$\frac{\partial^2 b_1(m)}{\partial s^2} \leq \left.\frac{\partial^2 b_1(m)}{\partial s^2}\right|_{s=1} = \frac{\eta(F(m)-1)}{\phi + F(m)} \leq 0$$

and the desired conclusion now follows from (35) by noting that the function $F(m)$ does not depend on the asset supply. Q.E.D.

Proof of Proposition 8. Evaluating (41) at the point $m = \overline{m}$ shows that the mass of non asset owners solves the Ricatti differential equation

$$\dot{F}_{0,t}(\overline{m}) = \nu(1 - F_{0,t}(\overline{m}))^2$$

subject to a fixed initial condition $F_{0,0}(\overline{m})$. The formula of the statement as well the convergence result now follow by direct calculation. Q.E.D.

Proof of Proposition 9. Consider a candidate solution of the form

$$1 - \Phi_{z,t}(m) = \nu^{-1} \frac{\dot{Y}_{z,t}(m)}{Y_{z,t}(m)}$$

for some function $Y_{z,t}(m)$. Inserting this guess into (42) and simplifying the resulting expression shows that this function solves

$$\ddot{Y}_{z,t}(m) + \eta \dot{Y}_{z,t}(m) = R_{z,t}(m) Y_{z,t}(m)$$

with the discount rate

$$R_{z,t}(m) = \nu\eta(1 - \Phi_{z,t}(\overline{m})F(m)) + \nu^2(1 - F(m))^2$$

Using the fact that the function $\Phi_{z,t}(\overline{m})$ is given by (43) it can be shown that two linearly independent solutions to this equation are given by

$$Y_{n,z,t}(m) = e^{-\frac{1}{2}(\eta + B_{z,t}(m))t} B_{z,t}(m) M_n \left(1 + \frac{\phi F(m)}{1 + \phi - F(m)}, 2, B_{z,t}(m)\right)$$

where $M_n(a, b, c)$ denotes the confluent hypergeometric of the $n-$th kind (see for example Abramowitz and Stegun (1964)) and I have set

$$B_{z,t}(m) = \frac{2(1 + \phi - F(m))}{1 - \Phi_{z,t}(\overline{m})}.$$

It follows that the general solution to (42) is explicitly given by

$$\nu(1 - \Phi_{z,t}(m)) = \frac{\dot{Y}_{1,z,t}(m) - A\dot{Y}_{2,z,t}(m)}{Y_{1,z,t}(m) - AY_{2,z,t}(m)}$$

for some function $A = A_z(m) \in \mathbb{C}$ and imposing the initial condition $\Phi_{z,0}(m)$ then leads to the formula in the statement. Q.E.D.

Proof of Theorem 3. Assume that the function $G_t(m, x)$ solves equation (47) subject to (18) so that the process

$$M_t = e^{-rt} G_t(m_t, X_t) + \int_0^t e^{-rs} \left(X_s + \lambda \hat{\mathcal{O}}_s(m_s, X_s|G)\right) ds$$

is a martingale over any finite horizon under the probability measure P. Using this property together with Girsanov theorem and the fact that

$$\Delta M_t = e^{-rt}\Delta G_t(m_t, X_t)$$

I obtain that the process

$$\hat{M}_t = M_t - \int_0^t e^{-rs}\lambda\hat{\mathcal{O}}_s(m_s, X_s|G)ds = e^{-rt}G_t(m_t, X_t) + \int_0^t e^{-rs}X_s ds$$

is a local martingale under the equivalent probability measure $\hat{P}$. This in turn implies that there exists an sequence of stopping times $(\sigma_n)_{n=1}^{\infty}$ that is almost surely increasing with $t \leq \sigma_n \to \infty$ and such that

$$\begin{aligned} G_t(m, x) &= \lim_{n\to\infty} \hat{E}_t\left[e^{-r(\sigma_n - t)}G_{\sigma_n}(m_{\sigma_n}, X_{\sigma_n}) + \int_t^{\sigma_n} e^{-r(s-t)}X_s ds\right] \\ &= \lim_{n\to\infty} \hat{E}_t\left[e^{-r(\sigma_n - t)}G_{\sigma_n}(m_{\sigma_n}, X_{\sigma_n}) + \int_t^{\infty} e^{-r(s-t)}X_s ds\right] \end{aligned}$$

where the second equality follows from the nonnegativity of cash flows and the monotone convergence theorem. Using Assumption 1 together with (18) then shows that

$$e^{-r(\sigma_n - t)}|G_{\sigma_n}(m_{\sigma_n}, X_{\sigma_n})| \leq c_g\left(\frac{X_t}{S_t}\right)S_{\sigma_n}$$

where the process S_t is a geometric Brownian motion with drift $\overline{m} - r < 0$ under the probability measure $\hat{P}$ and it now follows from Lemma A.1 and the dominated convergence theorem that

$$G_t(m, x) = \hat{E}_t\int_t^{\infty} e^{-r(s-t)}X_s ds.$$

Using this identity together with the same arguments as in Section 3.4 then shows that the uniformly bounded function defined by $g_t(m) = G_t(m, x)/x$ is time independent and satisfies the integral equation

$$g(m)\hat{\gamma}(m) - 1 = \hat{\mathcal{D}}(m|g) \equiv \int_{\mathbb{M}} g(n)\left(\eta dF(n) + \mathbf{1}_{\{n>m\}}\lambda\theta_1 dF(n)\right).$$

This in turn implies that I must have $g(m) = c\hat{\Gamma}(m)$ for some free constant and imposing the boundary condition

$$g(\overline{m}) = 1 + \mathcal{D}^{\mathrm{u}}(\overline{m}|g) = \int_{\mathbb{M}}(1 + \eta g(n))dF(n)$$

gives the formula of the statement. The monotonicity of the solution follows from that of the function $\hat{\Gamma}(m)$ and the proof is complete. Q.E.D.

B Arguments omitted from the text

Proof of Equation (8). Assume that the functions $V_{q,t}(m,x)$ solve (5) subject to (7). Integrating in (5) with respect to the exponential distribution of the first time at which the agent gets an opportunity to trade gives

$$V_{q,t}(m,x) = E_t \int_t^\infty e^{-(r+\lambda)(s-t)} \left(qX_s + \lambda V_{q,s}(m_s, X_s) + \lambda \mathcal{E}_{q,s}(m_s, X_s|G)\right) ds$$

and it follows that

$$e^{-(r+\lambda)t} V_{q,t}(m_t, X_t) + \int_0^t e^{-(r+\lambda)s} \left(qX_s + \lambda V_{q,s}(m_s, X_s) + \lambda \mathcal{E}_{q,s}(m_s, X_s|G)\right) ds$$

is a martingale over any finite horizon. Combining this with Emery's inequality (see for example Protter (2004, Theorem 3 p. 246)) and Itô's lemma I deduce that

$$e^{-rt} V_{q,t}(m_t, X_t) + \int_0^t e^{-rs} \left(qX_s + \lambda \mathcal{E}_{q,s}(m_s, X_s|G)\right) ds$$

is also a martingale over any finite horizon and, since the integral term is nonnegative, it follows from the monotone convergence theorem that

$$V_{q,t}(m,x) = \lim_{n\to\infty} E_t \left[e^{-r(n-t)} V_{q,n}(m_n, X_n) + \int_t^\infty e^{-r(s-t)} (qX_s + \lambda \mathcal{E}_{q,s}(m_s, X_s|G)) ds \right].$$

The growth condition (7) and Assumption 1 imply that

$$e^{-r(n-t)} |V_{q,n}(m_n, X_n)| \le c_q e^{-r(n-t)} X_n \le S_n$$

where the process S_t is a geometric Brownian motion with drift $\overline{m} - r < 0$. Combining this bound with Lemma A.1 and the dominated convergence theorem then gives

$$\lim_{n\to\infty} E_t \left[e^{-r(n-t)} V_{q,n}(m_n, X_n) \right] = 0$$

and it follows that the functions $V_{q,t}(m,x)$ solve (8). Conversely, if the functions $V_{q,t}(m,x)$ solve (8) subject to (7) then the law of iterated expectations gives

$$V_{q,t}(m,x) = E_t\left[e^{-r(\tau-t)}V_{q,\tau}(m_\tau, X_\tau) + \int_t^\tau e^{-r(s-t)}(qX_s + \lambda\mathcal{E}_{q,s}(m_s, X_s|G))ds\right]$$

where the stopping time $\tau > t$ denotes the first time that the agent gets an opportunity to trade and the desired result now follows by noting that

$$E_t\left[e^{-r(\tau-t)}\mathcal{E}_{q,\tau}(m_\tau, X_\tau|G)\right] = E_t\int_t^\tau e^{-r(s-t)}\lambda\mathcal{E}_{q,s}(m_s, X_s|G)ds$$

due to standard properties of exponential random variables. Q.E.D.

Proof of Equation (11). Consider the equivalent probability measure defined by

$$\overline{P}(A) = E\left[\mathbf{1}_{\{A\}}e^{-\int_0^t m_{u-}du}\left(\frac{X_t}{X_0}\right)\right], \qquad A \in \mathcal{F}_t.$$

Using the definition of the operators $\mathcal{E}_0$ and $\mathcal{E}_1$ together with Bayes' rule for conditional expectations I obtain

$$\begin{aligned} v_{q,t}(m_t) = V_{q,t}(m_t, X_t)/X_t &= E_t\int_t^\infty e^{-r(s-t)}\left(\frac{X_s}{X_t}\right)(q + \lambda\mathcal{E}_{q,s}(m_s, 1, v_1 - v_0))\,ds \\ &= \overline{E}_t\int_t^\infty e^{-\int_t^s (r-m_{u-})du}(q + \lambda\mathcal{E}_{q,s}(m_s, 1, v_1 - v_0))\,ds \end{aligned}$$

and the desired conclusion now follows from the fact that, since its density only depends on the Brownian motion, the new probability measure leaves the distribution of the agent's perceived growth rate unchanged. Q.E.D.

Comparative statics of $\gamma(m)$ and $\Gamma(m)$. A direct calculation shows that

$$\frac{\partial\gamma(m)}{\partial\lambda} = \theta_1(1 - s - F(m)) + F_1^*(m) - \phi\frac{\partial F_1^*(m)}{\partial\phi} \tag{B.17}$$

Using the fact that the function $F_1^*(m)$ is increasing and concave in ϕ I obtain

$$\frac{\partial}{\partial\phi}\left[F_1^*(m) - \phi\frac{\partial F_1^*(m)}{\partial\phi}\right] = -\phi\frac{\partial^2 F_1^*(m)}{\partial\phi^2} \geq 0$$

This shows that the right hand side of (B.17) is increasing in ϕ and since

$$\begin{aligned}\lim_{\phi\to 0}\frac{\partial\gamma(m)}{\partial\lambda} &= \theta_1(1-s-F(m)) + \lim_{\phi\to 0}F_1^*(m)\\ &= \theta_1(1-s-F(m)) + (1-s-F(m))^-\\ &= \theta_1(1-s-F(m))^+ + \theta_0(1-s-F(m))^- \geq 0\end{aligned}$$

it follows that the functions $\gamma(m)$ and $\Gamma(m)$ are both increasing in the intensity of meetings between agents. Q.E.D.

Proof of Equation (40). Summing up the contributions of the four types of entries and exits I obtain that the rate of change in the distribution is given by

$$\dot{F}_{q,t}(m) = \eta(F_{q,t}(\overline{m})F(m) - F_{q,t}(m)) + \mathbb{E}_{q,t}(m) - \mathbb{X}_{q,t}(m) \tag{B.18}$$

where

$$\begin{aligned}\mathbb{E}_{q,t}(m) &= \sum_{n\geq 1}\int_{\mathbb{M}^2}\mathbf{1}_{\{m_2\leq m_1\leq m\}}\lambda dF_{n,t}(m_2)dF_{q-n,t}(m_1)\\ \mathbb{X}_{q,t}(m) &= \int_{\mathbb{M}^2}\lambda\left(\mathbf{1}_{\{m_1\leq m\wedge m_2\}}dF(m_2) + \sum_{n=1}^{\infty}\mathbf{1}_{\{m_2\leq m_1\leq m\}}dF_{n,t}(m_2)\right)dF_{q,t}(m_1)\end{aligned}$$

give respectively the total entry rate and the total exit rate. To simplify this expression consider first the entry rate and observe that

$$\begin{aligned}\mathbb{E}_{q,t}(m) &= \sum_{n=1}^{q}\int_{\mathbb{M}^2}\mathbf{1}_{\{m_2\leq m_1\leq m\}}\lambda dF_{n,t}(m_2)dF_{q-n,t}(m_1)\\ &= \sum_{n=0}^{q}\int_{\mathbb{M}}\mathbf{1}_{\{m_1\leq m\}}\lambda F_{n,t}(m_1)dF_{q-n,t}(m_1) - \int_{\mathbb{M}}\mathbf{1}_{\{m_1\leq m\}}\lambda F_{0,t}(m_1)dF_{q,t}(m_1)\\ &= \sum_{n=0}^{q}\nu F_{n,t}(m)F_{q-n,t}(m) - \int_{\mathbb{M}}\mathbf{1}_{\{m_1\leq m\}}\lambda F_{0,t}(m_1)dF_{q,t}(m_1)\end{aligned}$$

where the last equality follows from the assumed continuity of the distribution of perceived growth rates and the fact that

$$d\left(\sum_{n=0}^{q}F_{n,t}(m)F_{q-n,t}(m)\right) = 2\sum_{n=0}^{q}F_{n,t}(m)dF_{q-n,t}(m).$$

Now consider the exit rate and observe that

$$\begin{aligned}
\mathbb{X}_{q,t}(m) &= \int_{\mathbb{M}^2} \lambda \left(\mathbf{1}_{\{m_1 \le m \wedge m_2\}} dF(m_2) + \mathbf{1}_{\{m_2 \le m_1 \le m\}} (dF - dF_{0,t})(m_2) \right) dF_{q,t}(m_1) \\
&= \int_{\mathbb{M}^2} \lambda \left(\mathbf{1}_{\{m_1 \le m\}} dF(m_2) - \mathbf{1}_{\{m_2 \le m_1 \le m\}} dF_{0,t}(m_2) \right) dF_{q,t}(m_1) \\
&= \lambda F_{q,t}(m) - \lambda \int_{\mathbb{M}} \mathbf{1}_{\{m_1 \le m\}} F_{0,t}(m_1) dF_{q,t}(m_1)
\end{aligned}$$

where the first equality follows from (39) and the last from the fact that the total mass of the population is equal to one. Substituting the expressions for $\mathbb{E}_{q,t}(m)$ and $\mathbb{X}_{q,t}(m)$ into (B.18) and simplifying the result gives (40). Q.E.D.

Proof of Equation (41). Combining (39) with (40) shows that

$$\begin{aligned}
-\dot{F}_{0,t} = \sum_{q=1}^{\infty} \dot{F}_{q,t} &= \lambda(F_{0,t} - F) + \nu \sum_{q=1}^{\infty} \sum_{n=0}^{q} F_{n,t} F_{q-n,t} + \eta(F_{0,t} - F_{0,t}(\overline{m}) F(m)) \\
&= \lambda(1 - F_{0,t})(F_{0,t} - F) + \nu \sum_{q=1}^{\infty} \sum_{n=1}^{q-1} F_{n,t} F_{q-n,t} + \eta(F_{0,t} - F_{0,t}(\overline{m}) F(m)) \\
&= \lambda(1 - F_{0,t})(F_{0,t} - F) + \nu \sum_{n=1}^{\infty} F_{n,t} \sum_{q=n+1}^{\infty} F_{q-n,t} + \eta(F_{0,t} - F_{0,t}(\overline{m}) F(m)) \\
&= \lambda(1 - F_{0,t})(F_{0,t} - F) + \nu(F - F_{0,t})^2 + \eta(F_{0,t} - F_{0,t}(\overline{m}) F(m)) \\
&= \nu(1 - F)^2 - \nu(1 - F_{0,t}) + \eta(F_{0,t} - F_{0,t}(\overline{m}) F(m))
\end{aligned}$$

Q.E.D.

C Market makers

Assume now that in addition to a continuum of agents the market also includes a unit mass of competitive market makers who have access to a frictionless interdealer market and keep no inventory. An agent contacts market makers with intensity $\alpha \ge 0$. When an agent contacts a market maker, they bargain over the terms of a potential trade and I assume that the result of this negotiation is given by the Nash bargaining solution with bargaining power $z \in [0, 1]$ for the market maker.

C.1 Pricing in the interdealer market

Let Π_t denote the asset price on the interdealer market and consider a meeting between a market maker and an agent with perceived growth rate m who holds $q \in \{0, 1\}$ units of the asset. The assumption of Nash bargaining implies that such a meeting results in a trade if and only if the trade surplus

$$\begin{aligned} S_{q,t}(m_t, X_t|G) &= (2q-1)(\Pi_t - G_t(m_t, X_t)) \\ &= (2q-1)(\Pi_t - V_{1,t}(m_t, X_t) + V_{0,t}(m_t, X_t)) \end{aligned}$$

is nonnegative in which case the realized price is

$$\hat{P}_t = zG_t(m_t, X_t) + (1-z)\Pi_t.$$

If the gain from becoming an owner is increasing in the perceived growth rate then there must exist a cutoff $w_t \in \mathbb{M}$ such that only those owners with perceived growth rate $m \le w_t$ are willing to sell to market makers, while only those non owners with $m \ge w_t$ are willing to buy. Since market makers must be indifferent to trading with marginal agents this in turn implies that the price on the interdealer market is

$$\Pi_t = G_t(w_t, X_t) = V_{1,t}(w_t, X_t) - V_{0,t}(w_t, X_t).$$

To guarantee that the cutoff is unique and constant over time I will from now impose the following assumption. As explained in Remark 9 this assumption only rules out the non generic cases in which the distribution of perceived growth rates in the economy is constant at the level $1 - s$ over an open interval.

Assumption 2 *$F \in \mathcal{F}_{\text{ns}}$ and the quantile set $\mathcal{Q}$ defined in (30) is a singleton.*

To determine the cutoff and thereby complete the description of the interactions between agents and market makers, I use the fact that since market makers keep no inventory their positions must net out to zero. The total mass of owners who contact market makers to sell is $\alpha F_{1,t}(w_t)$. On the other hand, the total mass of non owners who contact market makers to buy the asset is

$$\alpha(1 - s - F_{0,t}(w_t) + \Delta F_{0,t}(w_t)).$$

Since the distribution of perceived growth rates can have atoms, some randomization may be required at the margin. Taking this into account shows that the interdealer market clearing condition is

$$F_{1,t}(w_t) - (1-\pi_{1,t})\Delta F_{1,t}(w_t) = 1 - s - F_{0,t}(w_t) + \pi_{0,t}\Delta F_{0,t}(w_t) \tag{C.19}$$

where $\pi_{q,t} \in [0,1]$ denotes the probability with which market makers execute orders from marginal agents of ownership type q. Combining this condition with Assumption 2 shows that the cutoff is uniquely given by the lowest quantile

$$w_t = m^* = \inf\{m \in \mathbb{M} : F(m) \geq 1 - s\},$$

and it now remains to determine the execution probabilities. Two cases may arise depending on the properties of the distribution. If $F(m^*) = 1 - s$ as illustrated in the left panel of Figure 4 then the execution probabilities are uniquely defined by $\pi_{q,t} = q$ and only marginal buyers get rationed in equilibrium. On the contrary, if the cutoff is an atom such that $F(m^*) > 1 - s$ as in the middle panel of the figure then the execution probabilities are not uniquely defined. In this case, one may for example take

$$\pi_{0,t} = 1 - \pi_{1,t} = \frac{F(m^*) - (1-s)}{\Delta F(m^*)}$$

so that a fraction of both marginal buyers and marginal sellers get rationed in equilibrium, but many other choices are also compatible with market clearing. This choice has by construction no influence on the welfare of agents, and I verify below but it also does not have any impact on the evolution of the equilibrium distribution of types.

C.2 Equilibrium distribution of types

Since agents can now trade both among themselves and with market makers, the evolution of the equilibrium distributions of perceived growth rates must include additional entry and exit terms to reflect the new trading opportunities.

Let $\pi_{q,t}$ be execution probabilities such that (C.19) holds and consider the group of asset owners who are more pessimistic than a fixed $m \in \mathbb{M}$. In addition to the entry channels of the benchmark model, an agent may enter this group because he is a non owner with $n \leq m$ who buys the asset from a market maker. In a monotone equilibrium, the contribution of such entries is

$$\mathbb{E}_t(m) = \int_{\mathbb{M}} \alpha(\mathbf{1}_{\{n>m^*\}} + \pi_{0,t}\mathbf{1}_{\{n=m^*\}})\mathbf{1}_{\{n\leq m\}} dF_{0,t}(n)$$
$$= \alpha(F_{0,t}(m) - F_{0,t}(m^*))^+ + \mathbf{1}_{\{m^*\leq m\}}\alpha\pi_{0,t}\Delta F_{0,t}(m^*)$$

where the last term takes into account the fact that not all meetings with marginal buyers result in a trade. On the other hand, an agent may exit this group because he is an asset owner with $n \leq m$ who sells to a market maker. In a monotone equilibrium, the contribution of such exits is

$$\mathbb{X}_t(m) = \int_{\mathbb{M}} \alpha(\mathbf{1}_{\{n<m^*\}} + \pi_{1,t}\mathbf{1}_{\{n=m^*\}})\mathbf{1}_{\{n\leq m\}} dF_{1,t}(n)$$
$$= \alpha F_{1,t}(m \wedge m^*) - \mathbf{1}_{\{m^*\leq m\}}\alpha(1-\pi_{1,t})\Delta F_{1,t}(m^*).$$

Gathering these contributions and using the market clearing conditions (3) and (C.19) shows that the total contribution of intermediated trades is independent from the choice of the choice of the execution probabilities and given by

$$\mathbb{E}_t(m) - \mathbb{X}_t(m) = -\alpha F_{1,t}(m) + \alpha(1 - s - F(m))^-.$$

Finally, combining this with (14) shows that the equilibrium rate of change in the mass of asset owners who are more pessimistic than a fixed $m \in \mathbb{M}$ is

$$\dot{F}_{1,t}(m) = \lambda\mathcal{R}(m, F_{1,t}(m)) + \alpha(1 - s - F(m))^- - \alpha F_{1,t}(m) \tag{C.20}$$

and does not depends on the choice of the execution probabilities. In order to solve this Ricatti differential equation set $\psi = \alpha/\lambda$ and let

$$F_1^*(m) = -\frac{1}{2}(1 - s + \phi + \psi - F(m)) + \frac{1}{2}\Psi(m)$$

where

$$\Psi(m) = \sqrt{(1 - s + \phi + \psi - F(m))^2 + 4s\phi F(m) + 4\psi(1 - s - F(m))^-}$$

denote the strictly positive solution to the characteristic equation associated with the differential equation (C.20). The following result is a the direct counterpart of Proposition 1 for the model with market makers.

Proposition C.1 *In a monotone equilibrium with market makers the distribution of growth rates among asset owners is given by*

$$F_{1,t}(m) = F_1^*(m) + \frac{(F_{1,0}(m) - F_1^*(m))\Psi(m)}{\Psi(m) + (F_{1,0}(m) + \Psi(m) - F_1^*(m))\left(e^{\lambda\Psi(m)t} - 1\right)}.$$

and converges strongly to the steady state distribution $F_1^*(m)$ from any initial condition such that (3) holds.

Proof. The proof is analogous to that of Proposition 1. Q.E.D.

Corollary C.1 *The steady state distribution $F_1^*(m)$ is increasing in the asset supply s, increasing and concave in ϕ, and decreasing and convex in ψ with*

$$\lim_{\phi\to 0} F_1^*(m) = \lim_{\psi\to\infty} F_1^*(m) = (1 - s - F(m))^-$$
$$\lim_{\phi\to\infty} F_1^*(m) = \lim_{\psi\to 0} F_1^*(m) = sF(m).$$

Proof. The proof is analogous to that of Corollary 1. Q.E.D.

C.3 Equilibrium value functions

Having characterized the equilibrium distribution of beliefs among owners and non owners it now remains to compute the individual value functions and to verify the conjectured monotonicity of the gain from becoming an owner.

Taking as given the distribution of perceived growth rates among the populations of owners and non owners I define the value functions by the system

$$V_{q,t}(m,x) = E_t\Bigg[\int_t^\tau e^{-r(s-t)}qX_s ds + e^{-r(\tau-t)}V_{q,\tau}(m_\tau, X_\tau) \\ + e^{-r(\tau-t)}\left(\mathbf{1}_{\{\tau=\tau_a\}}\mathcal{E}_{q,\tau} + \mathbf{1}_{\{\tau=\tau_m\}}\mathcal{M}_{q,\tau}\right)(m_\tau, X_\tau|G)\Bigg] \tag{C.21}$$

where the stopping time $\tau = \tau_m \wedge \tau_a$ denotes the first time that the agent gets an opportunity to trade with either a market maker or another agent, the operator $\mathcal{E}_q$ defined as in (6) represents the expected gain from a meeting with an agent of the complementary ownership type, and the operator

$$\mathcal{M}_{q,t}(m,x|G) = (1-z)S_{q,t}(m,x|G)^+ \\ = (1-z)((2q-1)(G_t(m^*,x) - G_t(m,x)))^+$$

represents the payoff from a meeting with a market maker. To pin down a unique monotone equilibrium I further require that the value functions satisfy (7) and the same arguments as in the model without market makers show that under this growth condition the system (C.21) is equivalent to

$$V_{q,t}(m,x) = E_t \int_t^{\infty} e^{-r(s-t)} \left(qX_s + \lambda \mathcal{E}_{q,s}(m_s, X_s|G) + \alpha \mathcal{M}_{q,s}(m_s, X_s|G)\right) ds.$$

Using the conjectured monotonicity of the gain from becoming an asset owner it can be shown that solving this system is in turn equivalent to solving the single dynamic programming equation

$$G_t(m,x) = E_t \int_t^{\infty} e^{-r(s-t)}(X_s + \lambda \mathcal{O}_s(m_s, X_s|G) + \alpha(1-z)(G_t(m^*, X_s) - G_t(m_s, X_s)))ds. \tag{C.22}$$

subject to (18) where the linear operator $\mathcal{O}$ is defined as in (20) but with the equilibrium distributions of Proposition C.1. To construct the solution to this equation consider the probability measure $\hat{P}$ under which the compensator of an agent's perceived growth rate is given by

$$\hat{Q}(dt, \mathbb{B}) = Q^*(dt, \mathbb{B}) + \mathbf{1}_{\{m^* \in \mathbb{B}\}} \alpha(1-z)dt.$$

Compared to the benchmark model with only bilateral meetings the compensator includes an additional term that reflects the new trading opportunities available in the market: market makers can be simply interpreted as additional agents whose perceived growth is constantly equal to m^*. As in the model without intermediated trades this auxiliary probability measure should be interpreted as tracking the beliefs of the marginal agent and can be used to compute the equilibrium. To state the result set

$$\Gamma_\alpha(m) = \exp\left(-\int_m^{\overline{m}} \frac{dn}{\gamma(n) + \alpha(1-z)}\right)$$

where the discount rate $\gamma(m) > 0$ is defined as in (25) albeit with the steady state distributions of Proposition C.1 instead of those of Proposition 1.

Theorem C.1

1. *The unique solution to* (C.22) *such that* (18) *holds is*

$$G_t(m,x) = \hat{E}_t \int_t^\infty e^{-r(s-t)} X_s ds$$

and increases with the agent's perceived growth rate. In particular, there exists a unique monotone equilibrium with market makers.

2. *Assume that* $F_{1,0}(m) = F_1^*(m)$. *Then the unique monotone equilibrium with market makers is stationary and given by*

$$G(m,x) = \frac{x\Gamma_\alpha(m)}{r - \overline{m} + \alpha(1-z)(1-\Gamma_\alpha(m^*)) + \mathcal{D}^*(\overline{m}|1-\Gamma_\alpha)}$$

and the steady state distributions of Proposition C.1.

Proof. The proof is analogous to those of Theorems 1 and 2. Q.E.D.

As in the model without intermediated trades it is possible to show that the steady state equilibrium is globally stable and converges to the competitive equilibrium as trading becomes instantaneous.

Corollary C.2

1. *The unique monotone equilibrium converges to the stationary monotone equilibrium from any initial condition as* $t \to \infty$.

2. *The stationary monotone equilibrium with market makers converges to the Walrasian equilibrium as* $\lambda \to \infty$ *if the bargaining powers* $\theta_q \in (0,1)$ *and as* $\alpha \to \infty$ *if the dealer bargaining power* $z < 1$.

Proof. The proof is analogous to those of Corollary 2 and Proposition 4. Q.E.D.

In the model with market makers trading volume is the sum of two terms: The number of bilateral meetings that give rise to a trade and the number of trades intermediated by market makers. As in the benchmark model, the first term

$$\vartheta_b(s,\eta,\lambda,\alpha) = \int_{\mathbb{M}^2} \mathbf{1}_{\{n \le m\}} \lambda dF_1^*(n) dF_0^*(m).$$

represents the number of trades that occur due to bilateral meetings among agents. Since market makers do not keep inventory, each trade that they execute is backed by an offsetting trade. In accordance with the conventions of the NYSE (see Atkins and Dyl

(1997) for a discussion) I therefore define the number of intermediated trades as half the number of trades executed by market makers (or equivalently as the total number of sell orders they execute) that is

$$\vartheta_m(s,\eta,\lambda,\alpha) = \alpha F_1^*(m^*) - \alpha(1-\pi_1)\Delta F_1^*(m^*)$$

where the constant $\pi_1 \in [0,1]$ is the probability with which marginal sell orders are executed. The total trading volume is then defined by $\vartheta = \vartheta_b + \vartheta_m$. The following result provides an explicit solution for the equilibrium trading volume and constitutes the direct counterpart of Proposition 5.

Proposition C.2 *Assume that the distribution $F(m)$ is continuous. Then the steady state trading volume is explicitly given by*

$$\begin{aligned}\vartheta(s,\eta,\lambda,\alpha) &= \eta(1-s)(\psi + s(1+\phi))\log\left(1 + \frac{F_1^*(m^*)}{(1-s)\phi}\right) \\ &- \eta s\left[1 - s + (\psi + (1-s)(1+\phi))\log\left(1 + \frac{F_1^*(m^*)}{\psi + (1-s)(1+\phi)}\right)\right]\end{aligned}$$

and satisfies

$$\lim_{\lambda\to 0}\vartheta(s,\eta,\lambda,\alpha) = \lim_{\lambda\to 0}\vartheta_m(s,\eta,\lambda,\alpha) = \frac{\alpha\eta s(1-s)}{\alpha+\eta}$$

where the constant

$$F_1^*(m^*) = -\frac{1}{2}(\phi+\psi) + \frac{1}{2}\sqrt{4s(1-s)\phi + (\phi+\psi)^2}$$

gives the steady state mass of owners who trade with market makers. In particular, trading volume is increasing in α, λ and η.

Proof. The proof is analogous to that of Proposition 5. Q.E.D.

The above proposition shows that the conclusion of Proposition 5 remains valid in the presence of market makers: The steady state trading volume does not depend on the distribution of beliefs in the economy as long as it is continuous and is fully determined by the meeting intensities, the frequency of changes in beliefs and the asset supply.

D Non stationary initial distribution

Assume that the initial distribution of perceived growth rates in the population is given by an arbitrary cumulative distribution function $F_0(m)$ which need not even be absolutely continuous with respect to $F(m)$. In this case the distribution of perceived growth rates in the economy is given by

$$F_t(m) = e^{-\eta t} F_0(m) + \left(1 - e^{-\eta t}\right) F(m)$$

and converges to $F(m)$. On the other hand, the same arguments as in Section 3.3 show that in a monotone equilibrium the distributions of perceived growth rate among the population of asset owners solves the Ricatti equation

$$\dot{F}_{1,t}(m) = -\lambda F_{1,t}(m)^2 - \lambda(Q_0(m) + e^{-\eta t} Q_1(m))F_{1,t}(m) + \lambda Q_2(m)$$

with

$$\begin{aligned}
Q_0(m) &= 1 - s + \phi - F(m) \\
Q_1(m) &= F(m) - F_0(m) \\
Q_2(m) &= s\phi F(m).
\end{aligned}$$

Given an initial condition $F_{1,0}(m)$ this Ricatti equation admits a unique solution that can be expressed in terms of the confluent hypergeometric function of the first kind $M_1(a, b; x)$ (see Abramowitz and Stegun (1964)) as

$$\lambda F_{1,t}(m) = \lambda(F_t(m) - F_{0,t}(m)) = \frac{\dot{Y}_{+,t}(m) - A(m)\dot{Y}_{-,t}(m)}{Y_{+,t}(m) - A(m)Y_{-,t}(m)}$$

with

$$Y_{\pm,t}(m) = e^{-\frac{\lambda}{2}(Q_0(m) \pm \Phi(m))t} M_1 \left(\frac{Q_0(m) \pm \Phi(m)}{2\phi}, 1 \pm \frac{\Phi(m)}{\phi}; e^{-\eta t} \frac{Q_1(m)}{\phi} \right)$$

and

$$\begin{aligned}
A(m) &= \frac{\dot{Y}_{+,0}(m) - \lambda F_{1,0}(m) Y_{+,0}(m)}{\dot{Y}_{-,0}(m) - \lambda F_{1,0}(m) Y_{-,0}(m)} \\
\Phi(m) &= \sqrt{Q_0(m)^2 + 4Q_2(m)} = \sqrt{(1 - s + \phi - F(m))^2 + 4s\phi F(m)}.
\end{aligned}$$

Furthermore, it follows from basic properties of the confluent hypergeometric function that this solution converges to a unique steady state that is the same as in the model with stationary initial distribution:

$$\lim_{t\to\infty} F_{1,t}(m) = \frac{1}{2}\left(\Phi(m) - Q_0(m)\right) = F_1^*(m).$$

The unique monotone equilibrium can then be computed as in Theorem 1 by substituting the above time-dependent distributions into (21). On the other hand, the same arguments as in the proof of Corollary 2 show that this monotone equilibrium converges to the same stationary monotone equilibrium as in Theorem 2.

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