Online appendix to:

Debt Dynamics with Fixed Issuance Costs

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A Formulation

A.1 The firm

The firm operates in continous-time and generates cash flows at rate $Y_t \geq 0$. The value of the unlevered firm is defined as

$$V_t = \mathbb{E}_t \int_t^\infty e^{-r(s-t)}(1-\tau)Y_s ds$$

where $r > 0$ denotes the risk-free rate, $\tau \in [0,1)$ denotes the corporate tax rate, and the expectation is with respect to the risk-neutral probability measure $\mathbb{P}$. We assume that the cash flow process Y_t evolves according to

$$dY_t = Y_t \left(\sigma dW_t + \mu dt\right) \tag{1}$$

for some constants $\sigma > 0$ and $0 \leq \mu < r$ where W_t is a risk-neutral Brownian motion. As a result, the value of the unlevered firm is explicitly given by

$$V_t = \left(\frac{1-\tau}{r-\mu}\right) Y_t$$

and thus also evolves as a geometric Brownian motion. This shows that we may equivalently use V_t or Y_t as a state variable and we choose later so stay in line with the existing dynamic capital structure literature .

A.2 Debt contracts

The firm's debt takes the form of a continuum of ex-ante identical, exponentially maturing bonds that have equal seniority and pay coupons at rate $c > 0$. The mass of existing bonds at date $t \geq 0$ defines the face value F_t of the firm's debt and we assume that each bond matures independently of all others with intensity $\xi > 0$ so that debtholders receive payments at rate $(c+\xi)F_t$ as long as the firm operates. The recovery of bonds in default is assumed to be zero.

The firm can at adjust its capital structure all times by retiring or issuing bonds at market value but is subject to a fixed adjustment cost βY_t with $\beta > 0$. As a result, the face value of the firm's debt evolves according to

$$dF_t = -\xi F_{t-} dt + dI_t$$

where I_t is a process with initial value $I_{0-} = 0$ whose increments capture changes in the capital structure of the firm.

A.3 Strategies

As long as $\beta > 0$ any adjustment process I_t whose paths has intervals of continuity leads to an infinite accumulation of adjustment costs. Building on this observation we define a default and adjustment strategy as a pair

$$\mathbf{s} \equiv \{\tau_b(\mathbf{s}), I(\mathbf{s})\}$$

where $\tau_b(\mathbf{s})$ is a stopping time that represents the time of default and $I_t(\mathbf{s})$ is a discrete process of the form

$$I_t(\mathbf{s}) = \sum_{s\in\mathcal{A}(\mathbf{s})} \mathbf{1}_{\{s\le t\}}\Delta I_s(\mathbf{s}) = \sum_{s\in\mathcal{A}(\mathbf{s})} \mathbf{1}_{\{s\le t\}} A_s(\mathbf{s}) F_{s-}$$

where $\mathcal{A}(\mathbf{s})$ is a thin set whose elements represent the moments at which the firm restructures its capital and $A_t(\mathbf{s}) \ge -1$ is a predictable process that represents the relative size of the adjustment conditional on a restructuring at date $t \ge 0$.

We will for the most part focus on Markov equilibria in which the state summarized by the variables F_t and Y_t that determine the cash flows of all stakeholders. Accordingly, a strategy is said to be *Markovian* if

$$\begin{aligned}
\mathcal{A}(\mathbf{s}) &= \{t \ge 0 : (F_{t-}, Y_t) \in \mathcal{R}\}, \\
\tau_b(\mathbf{s}) &= \inf\{t \ge 0 : (F_t, Y_t) \in \mathcal{D}\},
\end{aligned}$$

and

$$A_t(\mathbf{s}) = \mathbf{1}_{\{(F_{t-},Y_t)\in\mathcal{R}\}} A\left(F_{t-}, Y_t\right).$$

for some closed disjoint subsets $\mathcal{D}, \mathcal{R}$ of $\mathbb{R}^2_+$ and some function $A = A(\cdot|\mathbf{s}) : \mathcal{R} \to [-1, \infty)$. If in addition

$$\begin{aligned}
\mathcal{D} &= \left\{(F, Y) \in \mathbb{R}^2_+ : Y/F \in \bar{\mathcal{D}}\right\}, \\
\mathcal{R} &= \left\{(F, Y) \in \mathbb{R}^2_+ : Y/F \in \bar{\mathcal{R}}\right\}, \\
A\left(F, Y\right) &= a\left(Y/F\right)
\end{aligned}$$

for some closed disjoint subsets $\bar{\mathcal{D}}, \bar{\mathcal{R}}$ of $\mathbb{R}_+$ and some function $a = a(\cdot|\mathbf{s}) : \bar{\mathcal{R}} \to [-1, \infty)$ then we say that $\mathbf{s}$ is *reduced* Markovian. Throughout we denote by $\mathcal{S}_0$ the set of all default and adjustment strategies, by $\mathcal{M} \subset \mathcal{S}_0$ the subset of Markovian strategies, and by $\mathcal{M}_r \subset \mathcal{M}$ the subset of reduced Markovian strategies.

A.4 Debt valuation

Denote by $P_t(\mathbf{s}) \geq 0$ the value of an individual bond issued by the firm *when creditors anticipate that management will use the strategy* $\mathbf{s} \in \mathcal{S}_0$. The absence of arbitrage opportunities requires that

$$P_t(\mathbf{s}) = \mathbb{E}_t \left[\int_t^{\tau_m \wedge \tau_b(\mathbf{s})} e^{-r(s-t)} c ds + \mathbf{1}_{\{\tau_m < \tau_b(\mathbf{s})\}} e^{-r(\tau_m - t)} \right]$$

on the set $\{\tau_m \wedge \tau_b(\mathbf{s}) > t\}$ where τ_m is an exponential random variable with mean $m = 1/\xi$ that represents the maturity of the individual bond under consideration. Integrating inside the expectation against the conditional distribution

$$\mathbb{P}_t \left[\tau_m \in ds \,|\tau_m > t \right] = \mathbf{1}_{\{s>t\}} e^{-\xi(s-t)} \xi ds$$

then shows that on the set $\{\tau_b(\mathbf{s}) > t\}$ the market price of an individual bond issued by the firm is given by

$$P_t(\mathbf{s}) = \mathbb{E}_t \left[\int_t^{\tau_b(\mathbf{s})} e^{-\rho(s-t)} (c + \xi) ds \right] \leq \frac{c + \xi}{\rho} \tag{2}$$

with the maturity-adjusted discount rate $\rho \equiv r + \xi$.

If $\mathbf{s}$ is Markovian then the right hand side of the above identity only depends on the path of the pair (F_t, Y_t). As a result, the bond price

$$P_t(\mathbf{s}) = P(F_t, Y_t | \mathbf{s})$$

is a *bounded* function of these variables and, since the adjustment times are predictable, we have that this function satisfies the no-jump condition

$$P(F, Y | \mathbf{s}) = P\left(F\left(1 + A(F, Y)\right), Y | \mathbf{s}\right), \qquad (F, Y) \in \mathcal{R}(\mathbf{s}), \tag{3}$$

which guarantees that the market price of the bond does not react to the occurence of an anticipated restructuring of the firm's capital. By the same token, if $\mathbf{s}$ is reduced Markovian then the right hand side of (2) only depends on the path of the invers leverage process y_t. In that case, the bond price $P_t(\mathbf{s}) = P(y_t|\mathbf{s})$ is a *bounded* function of y_t and the absence of arbitrage requires that this function satisfies

$$P(y|\mathbf{s}) = P\left(\frac{y}{1+a(y)}\middle|\,\mathbf{s}\right), \qquad y \in \bar{\mathcal{R}}(\mathbf{s}), \tag{4}$$

which again guarantees that the market price of the bond does not react to the occurence of an anticipated restructuring.

A.5 Equity valuation without commitment

Because default and capital adjustments are decided upon after debt has been issued management may have incentives to deviate from the policy conjectured by creditors. Absent commitment this implies that creditors will only accept to lend money to the firm if the debt contract is incentive compatible in the sense that management never wants to deviate from the strategy that creditors use to price the bonds at issuance.

If creditors conjecture that the firm will use $\mathbf{a} \in \mathcal{S}_0$ but management instead uses another strategy $\mathbf{s} \in \mathcal{S}_0$ then the value of equity is

$$E_t(\mathbf{s},\mathbf{a}) \equiv \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{s})} e^{-r(s-t)}\left(\delta(F_s,Y_s)ds + P_s(\mathbf{a})dI_s(\mathbf{s}) - \beta Y_s dN_s(\mathbf{s})\right)\right] \tag{5}$$

subject to the cash flow dynamics (1) and

$$dF_s = -\xi F_{s-}dt + dI_s(\mathbf{s}) \tag{6}$$

where

$$\delta(F_s,Y_s) \equiv (1-\tau)Y_s - (\xi + c(1-\tau))\,F_s$$

is the instantaneous cash flow to equity holders and

$$N_t(\mathbf{s}) \equiv \sum_{s\in\mathcal{A}(\mathbf{s})} \mathbf{1}_{\{s\leq t\}}$$

is the counting process induced by the restructuring times of $\mathbf{s}$. To formalize the notion of an equilibrium let $\mathcal{S}$ denote the set of strategies such that

$$\Lambda(\mathbf{s}) \equiv \mathbb{E}\left[\int_0^{\tau_b(\mathbf{s})} e^{-rs}\left(F_s(\mathbf{s}) + Y_s\right)ds + \left(|\Delta F_s(\mathbf{s})| + Y_s\right)dN_s(\mathbf{s})\right] < \infty \tag{7}$$

We then have the following:

Definition 1 *A subgame perfect equilibrium* (SPE) *is a strategy* $\mathbf{a} \in \mathcal{S}$ *such that*

$$E_t(\mathbf{a}, \mathbf{a}) = \sup_{\mathbf{s} \in \mathcal{S}} E_t(\mathbf{s}, \mathbf{a}), \qquad t \geq 0.$$

A Markov perfect equilibrium (MPE) *is a* SPE *in* $\mathcal{M}$ *while a reduced Markov Perfect Equilibrium* (rMPE) *is a* SPE *in* $\mathcal{M}_r$.

B Results

B.1 Characterization of SPEs

As a first step towards the construction of equilibria for our default and restructuring game we derive a version of the one shot deviation principle.

Lemma 1 *A strategy* $\mathbf{a} \in \mathcal{S}$ *is a* SPE *if and only if*

$$E_t(\mathbf{a}, \mathbf{a}) = \sup_{\mathbf{s} \in \mathcal{S}} \mathbb{E}_t \left[\int_t^{\theta_t(\mathbf{s}) \wedge \tau_b(\mathbf{s})} e^{-r(s-t)} \delta(F_s, Y_s) ds \right. \\ \left. + \mathbf{1}_{\{\theta_t(\mathbf{s}) < \tau_b(\mathbf{s})\}} e^{-r(\theta_t(\mathbf{s})-t)} \left(E_{\theta_t(\mathbf{s})}(\mathbf{a}, \mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a}) \Delta I_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta Y_{\theta_t(\mathbf{s})} \right) \right] \tag{8}$$

where the stopping time

$$\theta_t(\mathbf{s}) \equiv \inf\{s \geq t : s \in \mathcal{A}(\mathbf{s})\} = \inf\{s \geq t : dI_s(\mathbf{s}) \neq 0\}$$

denotes the time of the first restructuring prescribed by the strategy $\mathbf{s} \in \mathcal{S}$ *on or after an arbitrary date* $t \geq 0$.

Proof. Assume that $\mathbf{a} \in \mathcal{S}$ is a SPE, let $\mathbf{s} \in \mathcal{S}$ and consider for each fixed starting point $t \geq 0$ the one-shot deviation $\mathbf{s}_t$ obtained by following $\mathbf{s}$ until $\tau_b(\mathbf{s}) \wedge \theta_t(\mathbf{s})$ and then reverting to $\mathbf{a}$. Using the equilibrium property of $\mathbf{a}$ together with the law of iterated expectations we deduce that

$$E_t(\mathbf{a},\mathbf{a}) = \mathbb{E}_t\Bigg[\int_t^{\theta_t(\mathbf{a})\wedge\tau_b(\mathbf{a})} e^{-r(s-t)}\delta(F_s,Y_s)ds$$
$$+ \mathbf{1}_{\{\theta_t(\mathbf{a})<\tau_b(\mathbf{a})\}} e^{-r(\theta_t(\mathbf{a})-t)}\left(E_{\theta_t(\mathbf{a})}(\mathbf{a},\mathbf{a}) + P_{\theta_t(\mathbf{a})}(\mathbf{a})\Delta I_{\theta_t(\mathbf{a})}(\mathbf{a}) - \beta Y_{\theta_t(\mathbf{a})}\right)\Bigg]$$
$$\geq E_t(\mathbf{s}_t,\mathbf{a}) = \mathbb{E}_t\Bigg[\int_t^{\theta_t(\mathbf{s})\wedge\tau_b(\mathbf{s})} e^{-r(s-t)}\delta(F_s,Y_s)ds$$
$$+ \mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}} e^{-r(\theta_t(\mathbf{s})-t)}\left(E_{\theta_t(\mathbf{s})}(\mathbf{a},\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a})\Delta I_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta Y_{\theta_t(\mathbf{s})}\right)\Bigg]$$

and the necessity of (8) follows from the arbitrariness of $\mathbf{s} \in \mathcal{S}$. To establish the converse assume that $\mathbf{a} \in \mathcal{S}$ satisfies (8). Since never restructuring and defaulting at the first time that the cash flow becomes negative is feasible we have that $E_t(\mathbf{a},\mathbf{a}) \geq 0$ at all times. Using this property and iterating (8) forward we deduce that

$$E_t(\mathbf{a},\mathbf{a}) \geq \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{s})\wedge\theta_{n,t}(\mathbf{s})} e^{-r(s-t)}\left(\delta(F_s,Y_s)ds + P_s(\mathbf{a})dI_s(\mathbf{s}) - \beta Y_s dN_s(\mathbf{s})\right)\right]$$

where $\theta_{n,t}(\mathbf{s})$ is the time of the nth restructuring after $t \geq 0$. let $Z_{n,t}$ denote the random variable inside the conditional expectation. Since the bond price is bounded and $\delta(F,Y)$ is a linear function we have that

$$\sup_{n\geq 1}|Z_{n,t}| \leq C_0(t)\int_0^{\tau_b(\mathbf{s})} e^{-rs}\left((F_s+Y_s)\,ds + (|\Delta I_s(\mathbf{s})| + Y_s)dN_s(\mathbf{s})\right)$$

for some deterministic function $C_0(t) > 0$ and it follows from (7) that the right hand side has finite expectation. Letting the number of restructuring rounds $n \to \infty$ and appealing to the dominated convergence theorem then gives

$$E_t(\mathbf{a},\mathbf{a}) \geq \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{s})} e^{-r(s-t)}\left(\delta(F_s,Y_s)ds + P_s(\mathbf{a})dI_s(\mathbf{s}) - \beta Y_s dN_s(\mathbf{s})\right)\right]$$

and the desired result now follows from (5). □

Lemma 1 provides a characterization of SPEs in terms of a stochastic control problem in which the controlled process is two dimensional. To further simplify the construction of equilibria we now provide an alternative characterization that only involves a one dimensional controlled process.

Corollary 1 *A default and adjustment strategy* $\mathbf{a} \in \mathcal{S}$ *is a* SPE *if and only if the scaled equity value process* $e_t(\mathbf{a}) = E_t(\mathbf{a}, \mathbf{a})/F_t$ *satisfies*

$$e_t(\mathbf{a}) = \sup_{\mathbf{s} \in \mathcal{S}} \mathbb{E}_t \Bigg[\int_t^{\theta_t(\mathbf{s}) \wedge \tau_b(\mathbf{s})} e^{-\rho(s-t)} \delta(y_s) ds \tag{9}$$
$$+ \mathbf{1}_{\{\theta_t(\mathbf{s}) < \tau_b(\mathbf{s})\}} e^{-\rho(\theta_t(\mathbf{s})-t)} \Big(\big(1 + A_{\theta_t(\mathbf{s})}(\mathbf{s})\big) e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a}) A_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta y_{\theta_t(\mathbf{s})-} \Big) \Bigg]$$

with the discount rate $\rho = r + \xi$ *and the cash flow function* $\delta(y) \equiv \delta(1, y)$.

Proof. The result follows from Lemma 1 by noting that we have

$$F_s = e^{-\xi(s-t)} F_t, \qquad \text{for all } s \in [t, \theta_t(\mathbf{s})),$$

and therefore

$$\begin{aligned} & E_{\theta_t(\mathbf{s})}(\mathbf{a}, \mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a}) \Delta I_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta Y_{\theta_t(\mathbf{s})} \\ & \quad = F_{\theta_t(\mathbf{s})-} \left(\frac{F_{\theta_t(\mathbf{s})}}{F_{\theta_t(\mathbf{s})-}} e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a}) \frac{\Delta I_{\theta_t(\mathbf{s})}(\mathbf{s})}{F_{\theta_t(\mathbf{s})-}} - \beta y_{\theta_t(\mathbf{s})-} \right) \\ & \quad = F_{\theta_t(\mathbf{s})-} \Big(\big(1 + A_{\theta_t(\mathbf{s})}(\mathbf{s})\big) e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a}) A_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta y_{\theta_t(\mathbf{s})-} \Big) \\ & \quad = e^{-\xi(\theta_t(\mathbf{s})-t)} F_t \Big(\big(1 + A_{\theta_t(\mathbf{s})}(\mathbf{s})\big) e_{\theta_t(\mathbf{s})}(\mathbf{a}) + P_{\theta_t(\mathbf{s})}(\mathbf{a}) A_{\theta_t(\mathbf{s})}(\mathbf{s}) - \beta y_{\theta_t(\mathbf{s})-} \Big) \end{aligned}$$

where the second equality follows from the definition of $A_t(\mathbf{s}) \geq -1$ as the relative size of the debt adjustment. □

Proposition 1 (Leverage ratchet effect) *If* $\mathbf{a} \in \mathcal{S}$ *is a* MPE *then* $I_t(\mathbf{a})$ *is a non decreasing process.*

Proof. Assume that $\mathbf{a} \in \mathcal{M}$ is a MPE in which

$$\mathbb{P}\left[\{ s \in \mathcal{A}(\mathbf{a}) : dI_s(\mathbf{a}) < 0 \} \right] \neq 0.$$

To show that this leads to a contradiction consider the deviation $\hat{\mathbf{a}} \in \mathcal{S}_0$ defined by the default time $\tau_b(\hat{\mathbf{a}}) \equiv \tau_b(\mathbf{a})$ and the face value process

$$F_t(\hat{\mathbf{a}}) \equiv \sup_{0 \leq u \leq t} \left\{ e^{\xi(u-t)} F_u(\mathbf{a}) \right\}.$$

Standard results in the theory of Skorokhod reflection problems (see for example **(au-**

thor?) (2) and references therein) show that we have

$$0 \leq \Delta I_t(\hat{\mathbf{a}}) = (F_t(\mathbf{a}) - F_{t-}(\hat{\mathbf{a}}))^+ \leq \Delta I_t(\mathbf{a})^+ \tag{10a}$$

$$0 = (F_t(\mathbf{a}) - F_t(\hat{\mathbf{a}}))\,\Delta I_t(\hat{\mathbf{a}}). \tag{10b}$$

Using these properties we show in Lemma 2 below that $\hat{\mathbf{a}} \in \mathcal{S}$ defines a feasible deviation and it remains to show that this deviation is profitable.

Denote by $P(F,Y) = P(F,Y|\mathbf{a})$ the bond price function induced by the assumed Markov perfect equilibrium and by $\bar{c} \equiv c(1-\tau)+\xi$ the after tax cost of debt per unit of face value. A direct calculation using (5) then shows

$$\begin{aligned} E_0\left(\hat{\mathbf{a}},\mathbf{a}\right) - E_0\left(\mathbf{a},\mathbf{a}\right) &= \mathbb{E}\Bigg[\int_0^{\tau_b(\mathbf{a})} e^{-rs}\bar{c}\left(F_s(\mathbf{a}) - F_s(\hat{\mathbf{a}})\right)ds \qquad (11)\\ &+ \int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(P(F_s(\hat{\mathbf{a}}),Y_s)dI_s(\hat{\mathbf{a}}) - \beta Y_s dN_s(\hat{\mathbf{a}})\right)\\ &- \int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(P(F_s(\mathbf{a}),Y_s)dI_s(\mathbf{a}) - \beta Y_s dN_s(\mathbf{a})\right)\Bigg]\\ &\geq \mathbb{E}\left[\int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(\bar{c}G_s ds - P\left(F_{s-}(\mathbf{a}),Y_s\right)\left(dG_s + \xi G_{s-}ds\right)\right)\right]\\ &= \mathbb{E}\left[\int_0^{\tau_b(\mathbf{a})} e^{-rs}(\bar{c} - \xi P\left(F_s(\mathbf{a}),Y_s\right))G_s ds + \int_0^{\tau_b(\mathbf{a})} G_{s-}d\left(e^{-rs}P\left(F_s(\mathbf{a}),Y_s\right)\right)\right] \end{aligned}$$

where $G_t \equiv F_t(\mathbf{a}) - F_t(\hat{\mathbf{a}})$ is the difference between the face value processes associated with the two strategies, the inequality follows from (10b) and (3), and the last equality follows from the fact that

$$P(F_0,Y_0|\mathbf{a})G_0 = G_{\tau_b(\mathbf{a})}P\left(F_{\tau_b(\mathbf{a})}(\mathbf{a}),Y_{\tau_b(\mathbf{a})}\right) = 0.$$

Now, since the process

$$e^{-\rho t}P(F_t(\mathbf{a}),Y_t) + \int_0^t e^{-\rho s}(c+\xi)ds$$

is by construction a martingale on the stochastic interval $[\![0,\tau_b(\mathbf{a})]\!]$ we have that there exists a local martingale M_t such that

$$d(e^{-rt}P\left(F_t(\mathbf{a}),Y_t\right)) = e^{-rt}\left(\xi P\left(F_{t-}(\mathbf{a}),Y_t\right) - (c+\xi)\right)dt + e^{-rt}dM_t.$$

Substituting this evolution into (11) then gives

$$E_0(\hat{\mathbf{a}}, \mathbf{a}) - E_0(\mathbf{a}, \mathbf{a}) \geq \mathbb{E}\left[\int_0^{\tau_b(\mathbf{a})} e^{-rs} G_{s-} dM_s - \int_0^{\tau_b(\mathbf{a})} e^{-rs} c\tau G_s ds\right] \tag{12}$$

and the desired result now follows from Lemma 3 below and the fact that G_t is non positive by construction. □

Lemma 2 *The strategy* $\hat{\mathbf{a}} \in \mathcal{S}$.

Proof. Using (10) we deduce that the deviation $\hat{\mathbf{a}}$ satisfies $dN_t(\hat{\mathbf{a}}) \leq dN_t(\mathbf{a})$ as well as $|\Delta F_t(\hat{\mathbf{a}})| \leq |\Delta F_t(\mathbf{a})|$ and it follows that

$$\mathbb{E}\left[\int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(\left(|\Delta F_s(\hat{\mathbf{a}})| + Y_s\right) dN_s(\hat{\mathbf{a}}) - \left(|\Delta F_s(\mathbf{a})| + Y_s\right) dN_s(\mathbf{a})\right)\right] \leq 0. \tag{13}$$

On the other hand, Itô's formula implies that we have

$$F_s(\hat{\mathbf{a}}) - F_s(\mathbf{a}) = \int_0^s e^{\xi(u-s)}\left(dI_u(\hat{\mathbf{a}}) - dI_u(\mathbf{a})\right)$$

and therefore

$$\begin{aligned}
\int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(F_s(\hat{\mathbf{a}}) - F_s(\mathbf{a})\right) ds &= \int_0^{\tau_b(\mathbf{a})} ds\, e^{-\rho s}\left\{\int_0^s e^{\xi u}\left(dI_u(\hat{\mathbf{a}}) - dI_u(\mathbf{a})\right)\right\} \\
&\leq \int_0^{\tau_b(\mathbf{a})} ds\, e^{-\rho s}\left\{\int_0^s e^{\xi u}|\Delta F_u(\mathbf{a})| dN_u(\mathbf{a})\right\} \\
&= \int_0^{\tau_b(\mathbf{a})} e^{-ru} dN_u(\mathbf{a})|\Delta F_u(\mathbf{a})|\left\{\int_u^{\tau_b(\mathbf{a})} e^{-\rho(s-u)} ds\right\} \\
&\leq \frac{1}{\rho}\int_0^{\tau_b(\mathbf{a})} e^{-ru}|\Delta F_u(\mathbf{a})| dN_u(\mathbf{a}). \qquad (14)
\end{aligned}$$

where the first inequality follows from (10b). Combining (13) and (14) then shows that $\Lambda(\hat{\mathbf{a}}) \leq C_0 \Lambda(\mathbf{a})$ for some $C_0 > 0$ and the desired result follows. □

Lemma 3 *The process*

$$U_t \equiv \int_0^{t\wedge\tau_b(\mathbf{a})} e^{-rs} G_{s-} dM_s$$

that appears in (12) *has expected value zero.*

Proof. Denote by $P_t(\mathbf{a}) = P(F_t(\mathbf{a}), Y_t)$ the bond price along the path of the assumed equilibrium. Itô's formula implies that

$$U_t = e^{-r\theta}G_\theta P_\theta(\mathbf{a}) + \int_0^\theta e^{-rs}\left((c+\xi)G_s ds - P_s(\mathbf{a})\left(dI_s(\mathbf{a}) - dI_s(\hat{\mathbf{a}})\right)\right)\Big|_{\theta\equiv\tau_b(\mathbf{a})\wedge t}$$

and it thus follows from the uniform boundedness of the bond price process, the non positivity of G_t and (10a) that we have

$$\begin{aligned}
\frac{|U_t|}{C_1} &\leq e^{-rt\wedge\tau_b(\mathbf{a})}|G_{t\wedge\tau_b(\mathbf{a})}| + \int_0^{t\wedge\tau_b(\mathbf{a})} e^{-rs}\left((c+\xi)|G_s|ds + |dI_s(\mathbf{a}) - dI_s(\hat{\mathbf{a}})|\right) \\
&= \int_0^{t\wedge\tau_b(\mathbf{a})} e^{-rs}\left((r-c)G_s ds + dI_s(\hat{\mathbf{a}}) - dI_s(\mathbf{a}) + |dI_s(\mathbf{a}) - dI_s(\hat{\mathbf{a}})|\right) \\
&\leq \int_0^{\tau_b(\mathbf{a})} e^{-rs}\left(\left(F_s(\hat{\mathbf{a}}) + F_s(\mathbf{a})\right)|r-c|ds + |\Delta F_s(\mathbf{a})|dN_s(\mathbf{a})\right)
\end{aligned}$$

for some constant $C_1 > 0$. This in turn implies that

$$\mathbb{E}\left\{\sup_{t\geq 0}|U_t|\right\} \leq C_2\left(\Lambda(\mathbf{a}) + \Lambda(\hat{\mathbf{a}})\right) < \infty$$

for some constant $C_2 > 0$ where the second inequality follows from the fact that $\mathbf{a}$ and $\hat{\mathbf{a}}$ are both feasible by Lemma 2. This shows that the local martingale U_t is a uniformly integrable martingale and the desired result follows. □

B.2 Recursive optimal stopping representation

The following lemma shows that the search for Markov equilibria is equivalent to solving a recursive optimal stopping problem.

Lemma 4 *A Markovian strategy* $\mathbf{a} \in \mathcal{M}\cap\mathcal{S}$ *is a* MPE *if and only if the induced equity value function satisfies*

$$E(F,Y|\mathbf{a}) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_{F,Y}\left[\int_0^\theta e^{-rt}\delta(\bar{F}_t, Y_t)dt + e^{-r\theta}R\left(\bar{F}_\theta, Y_\theta|\mathbf{a}\right)^+\right] \tag{15}$$

subject to (1) *and the uncontrolled dynamics*

$$d\bar{F}_t = -\xi\bar{F}_t dt$$

where the reward function is defined by

$$R(F,Y|\mathbf{a}) \equiv \sup_{G\in\mathbb{R}_+}\left\{E(G,Y|\mathbf{a}) + (G-F)\,P(G,Y|\mathbf{a}) - \beta Y\right\} \tag{16}$$

and $\mathcal{T}$ denotes the set of stopping times.

Proof of necessity. Assume that $\mathbf{a} \in \mathcal{M} \cap \mathcal{S}$ is a MPE and denote by

$$R(F, Y, G|\mathbf{a}) \equiv E(G, Y|\mathbf{a}) + (G - F)\, P(G, Y|\mathbf{a}) - \beta Y$$

the objective function on the right hand side of (16). Since $(\tau_b(\mathbf{a}), \theta_0(\mathbf{a}))$ are stopping times it follows from (16) and Lemma 1 that

$$\begin{aligned}
E(F, Y|\mathbf{a}) = \mathbb{E}_{F,Y} \Bigg[\int_0^{\tau_b(\mathbf{a}) \wedge \theta_0(\mathbf{a})} & e^{-rt} \delta(\bar{F}_t, Y_t) dt \\
& + e^{-r\theta_0(\mathbf{a})} \mathbf{1}_{\{\theta_0(\mathbf{a}) < \tau_b(\mathbf{a})\}} R\left(\bar{F}_{\theta_0(\mathbf{a})}, Y_{\theta_0(\mathbf{a})}, \bar{F}_{\theta_0(\mathbf{a})}\left(1 + A(\bar{F}_{\theta_0(\mathbf{a})}, Y_{\theta_0(\mathbf{a})})\right) \middle| \, \mathbf{a}\right) \Bigg] \\
& \leq \sup_{(\tau,\theta) \in \mathcal{T}^2} \mathbb{E}_{F,Y} \left[\int_0^{\tau \wedge \theta} e^{-rt} \delta(\bar{F}_t, Y_t) dt + \mathbf{1}_{\{\theta < \tau\}} e^{-r\theta} R\left(\bar{F}_\theta, Y_\theta |\mathbf{a}\right) \right] \\
& \leq \sup_{(\tau,\theta) \in \mathcal{T}^2} \mathbb{E}_{F,Y} \left[\int_0^{\tau \wedge \theta} e^{-rt} \delta(\bar{F}_t, Y_t) dt + \mathbf{1}_{\{\theta < \tau\}} e^{-r\theta} R\left(\bar{F}_\theta, Y_\theta |\mathbf{a}\right)^+ \right] \\
& \leq \sup_{\zeta \in \mathcal{T}} \mathbb{E}_{F,Y} \left[\int_0^{\zeta} e^{-rt} \delta(\bar{F}_t, Y_t) dt + e^{-r\zeta} R\left(\bar{F}_\zeta, Y_\zeta |\mathbf{a}\right)^+ \right]
\end{aligned}$$

To establish the reverse inequality let

$$R_n(F, Y|\mathbf{a}) \equiv \sup_{0 \leq G \leq n} R(F, Y, G|\mathbf{a})$$

and consider the sequence $(\mathbf{s}_n)_{n=1}^\infty$ of one shot deviations defined by

$$\begin{aligned}
\theta_0(\mathbf{s}_n) &\equiv \sigma + \mathbf{1}_{\left\{R_n(\bar{F}_\sigma, Y_\sigma|\mathbf{a}) \leq 0\right\}} \infty, \\
\tau_b(\mathbf{s}_n) &\equiv \mathbf{1}_{\left\{R_n(\bar{F}_\sigma, Y_\sigma|\mathbf{a}) \leq 0\right\}} \sigma + \mathbf{1}_{\left\{R_n(\bar{F}_\sigma, Y_\sigma|\mathbf{a}) > 0\right\}} (\sigma + q_\sigma \circ \tau_b(\mathbf{a})),
\end{aligned}$$

and

$$\bar{F}_{\theta_0(\mathbf{s}_n)}\left(1 + A_{\theta_0(\mathbf{s}_n)}(\mathbf{s}_n)\right) = \underset{0 \leq G \leq n}{\text{argmay}}\, R\left(\bar{F}_{\theta_0(\mathbf{s}_n)}, Y_{\theta_0(\mathbf{s}_n)}, G|\mathbf{a}\right)$$

where σ is an arbitrary but fixed stopping time, and q_σ denotes the Markov shift operator. It is easily seen that $\mathbf{s}_n \in \mathcal{S}$ is a feasible deviation for each $n \geq 1$. Therefore, it follows from Lemma 1 and the specification of $\mathbf{s}_n$ that we have

$$
\begin{aligned}
E(F,Y|\mathbf{a}) \geq \mathbb{E}_{F,Y}\Bigg[\int_0^{\tau_b(\mathbf{s}_n)\wedge\theta_0(\mathbf{s}_n)} & e^{-rt}\delta(\bar{F}_t,Y_t)dt \\
&+ e^{-r\theta_0(\mathbf{s}_n)}\mathbf{1}_{\{\theta_0(\mathbf{s}_n)<\tau_b(\mathbf{s}_n)\}}R_n(\bar{F}_{\theta_0(\mathbf{s}_n)},Y_{\theta_0(\mathbf{s}_n)}|\mathbf{a})\Bigg] \\
&= \mathbb{E}_{F,Y}\left[\int_0^{\sigma} e^{-rt}\delta(\bar{F}_t,Y_t)dt + e^{-r\sigma}R_n(\bar{F}_\sigma,Y_\sigma|\mathbf{a})^+\right].
\end{aligned}
$$

Letting $n\to\infty$ on both sides and invoking the monotone convergence theorem to justify the interchange of limit and expectation then gives

$$
E(F,Y|\mathbf{a}) \geq \mathbb{E}_{F,Y}\left[\int_0^{\sigma} e^{-rt}\delta(\bar{F}_t,Y_t)dt + e^{-r\sigma}R(\bar{F}_\sigma,Y_\sigma|\mathbf{a})^+\right]
$$

and the result follows by taking the supremum over $\sigma\in\mathcal{T}$. □

Proof of sufficiency. Assume that $\mathbf{a}\in\mathcal{M}\cap\mathcal{S}$ satisfies (15) and let $\mathbf{s}\in\mathcal{S}$ be fixed. Because $\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})$ is a stopping time this implies that we have

$$
\begin{aligned}
E(F_t,Y_t|\mathbf{a}) \geq \mathbb{E}_t\Bigg[\int_t^{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})} & e^{-r(s-t)}\delta(\bar{F}_s,Y_s)ds \\
&+ e^{-r(\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})-t)}R\left(\bar{F}_{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})},Y_{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})}\middle|\,\mathbf{a}\right)^+\Bigg] \\
&\geq \mathbb{E}_t\Bigg[\int_t^{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})} e^{-r(s-t)}\delta(\bar{F}_s,Y_s)ds \\
&+ e^{-r(\theta_t(\mathbf{s})-t)}\mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}}R\left(\bar{F}_{\theta_t(\mathbf{s})},Y_{\theta_t(\mathbf{s})}\middle|\,\mathbf{a}\right)\Bigg] \\
&\geq \mathbb{E}_t\Bigg[\int_t^{\tau_b(\mathbf{s})\wedge\theta_t(\mathbf{s})} e^{-r(s-t)}\delta(\bar{F}_s,Y_s)ds \\
&+ e^{-r(\theta_t(\mathbf{s})-t)}\mathbf{1}_{\{\theta_t(\mathbf{s})<\tau_b(\mathbf{s})\}}R\left(\bar{F}_{\theta_t(\mathbf{s})},Y_{\theta_t(\mathbf{s})},\bar{F}_{\theta_t(\mathbf{s})}\left(1+A_{\theta_t(\mathbf{s})}(\mathbf{s})\right)\middle|\,\mathbf{a}\right)\Bigg]
\end{aligned}
$$

and the required result now follows from Lemma 1, the arbitrariness of $\mathbf{s}\in\mathcal{S}$ and the definition of the function $R(F,Y,G|\mathbf{a})$. □

The next result specializes Lemma 4 to the case of rMPEs and will serve as a basis for most of our results on barrier strategies.

Lemma 5 *A strategy $\mathbf{a}\in\mathcal{M}_r\cap\mathcal{S}$ is a rMPE if and only if the induced equity value* *function satisfies*

$$e(y|\mathbf{a}) = \sup_{\theta\in\mathcal{T}} \mathbb{E}_y \left[\int_0^\theta e^{-\rho t}\delta(\bar{y}_t)dt + e^{-\rho\theta}\phi\left(\bar{y}_\theta|\mathbf{a}\right)^+\right] \tag{17}$$

$$= \hat{e}(y) + \sup_{\theta\in\mathcal{T}} \mathbb{E}_y \left[e^{-\rho\theta}\psi\left(\bar{y}_\theta|\mathbf{a}\right)\right] \tag{18}$$

subject to

$$d\bar{y}_t = \bar{y}_t\left(\sigma dW_t + (\xi+\mu)\,dt\right)$$

where

$$\hat{e}(y) \equiv \mathbb{E}_y\left[\int_0^\infty e^{-\rho t}\delta(\bar{y}_t)dt\right] = \frac{\delta(0)}{\rho} + \frac{\delta(y)-\delta(0)}{r-\mu}$$

denotes the equity value associated with never defaulting or restructuring, and the reward functions are defined by

$$\phi(y|\mathbf{a}) \equiv \sup_{z\geq 0}\Phi(y,z|\mathbf{a}) = \sup_{z\in\mathbb{R}_+}\left\{\frac{y}{z}e(z|\mathbf{a}) + \left(\frac{y}{z}-1\right)P(z|\mathbf{a}) - \beta y\right\}$$

$$\psi(y|\mathbf{a}) \equiv \phi(y|\mathbf{a})^+ - \hat{e}(y).$$

In particular, if $\mathbf{a}$ *is a* rMPE *then the induced scaled equity value is nonnegative, convex, and differentiable at all points where* $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+$.

Proof. Equation (17) follows from Lemma 4 by noting that

$$R(\bar{F}_t, Y_t|\mathbf{a})/\bar{F}_t = \sup_{z\geq 0}\left\{\frac{Y_t}{z\bar{F}_t}e\left(z|\mathbf{a}\right) + \left(\frac{Y_t}{z\bar{F}_t}-1\right)P\left(z|\mathbf{a}\right)\right\} - \frac{\beta Y_t}{z\bar{F}_t}$$

$$= \sup_{z\geq 0}\left\{\frac{\bar{y}_t}{z}e\left(z|\mathbf{a}\right) + \left(\frac{\bar{y}_t}{z}-1\right)P\left(z|\mathbf{a}\right)\right\} - \beta\bar{y}_t = \phi(\bar{y}_t|\mathbf{a})$$

and $\bar{F}_t = e^{-\xi t}F_0$. To see that (17) is equivalent to (18) it suffices to observe that the no-action equity value function satisfies the Dynkin identity

$$\hat{e}(y) - \mathbb{E}_y\left[e^{-\rho\zeta}\hat{e}(\bar{y}_\zeta)\right] = \mathbb{E}_y\left[\int_0^\zeta e^{-\rho t}\delta(\bar{y}_t)dt\right]$$

for all stopping times $\zeta\in\mathcal{T}$. Setting $\theta\equiv 0$ in (17) shows that the equity value function is nonnegative. On the other hand, we have that $\psi(y|\mathbf{a})$ is convex as the supremum of a family of affine functions and it thus follows from **(author?)** (1, Theorem 5) and **(author?)** (3, Corollary 7.5) that $v(y)\equiv e(y|\mathbf{a}) - \hat{e}(y)$ is differentiable at all points of the set

$$\{y \geq 0 : v(y|\mathbf{a}) = \psi(y|\mathbf{a})\} = \{y \geq 0 : e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+\}.$$

Since the function $\hat{e}(y)$ is linear this in turn implies that $e(y|\mathbf{a})$ is also convex and differentiable at all points of this set and the proof is complete. □

Corollary 2 *If $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ is a* rMPE *then the induced scaled equity value function is nondecreasing and there exists and constant $0 \leq y_b(\mathbf{a}) < \infty$ such that*

$$e(y|\mathbf{a}) = 0 = \phi(y|\mathbf{a})^+$$

at all points $y \leq y_b(\mathbf{a})$.

Proof. Assume that $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ is a rMPE and observe that since $e(y|\mathbf{a}) \geq \hat{e}(y)$ we have $e(y|\mathbf{a}) > 0$ for all sufficiently large y and thus $\bar{\mathcal{D}}(\mathbf{a}) \neq \mathbb{R}_+$. Let

$$y_b(\mathbf{a}) \equiv \sup\{y \geq 0 : y \in \bar{\mathcal{D}}(\mathbf{a})\}.$$

Since the scaled equity value function is nonnegative and not identically zero we have that $y_b(\mathbf{a}) < \infty$ and that $e'_+(z|\mathbf{a}) > 0$ at some point $z > y_b(\mathbf{a})$. Together with the convexity afforded by Lemma 5 this implies that the scaled equity value is nondecreasing and it follows by continuity that $e(y|\mathbf{a}) = 0 \geq \phi(y|\mathbf{a})^+$ for all $y \leq y_b(\mathbf{a})$. □

Corollary 3 *If $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ is a* rMPE *then*

$$e(y|\mathbf{a}) = \sup_{z \in \mathcal{C}(\mathbf{a})} \Phi\left(y, z|\mathbf{a}\right),$$

$$\{\mathcal{Y}(y) \equiv y/\left(1 + a(y|\mathbf{a})\right)\} = \operatorname*{argmay}_{z \in \mathcal{C}(\mathbf{a})} \Phi\left(y, z|\mathbf{a}\right)$$

for all $y \in \bar{\mathcal{R}}(\mathbf{a})$ where $\mathcal{C}(\mathbf{a}) \equiv \mathbb{R}_+ \backslash (\bar{\mathcal{D}} \cup \bar{\mathcal{R}})(\mathbf{a})$. Furthermore, the scaled equity value is differentiable and satisfies

$$e'(y|\mathbf{a}) = \frac{\partial \Phi}{\partial y}(y, \mathcal{Y}(y)|\mathbf{a})$$

at all points of the restructuring region $\bar{\mathcal{R}}(\mathbf{a})$.

Proof. If $y \in \bar{\mathcal{R}}(\mathbf{a})$ lies then it follows from (17) that

$$e(y|\mathbf{a}) \geq \phi(y|\mathbf{a})^+ = \sup_{z \geq 0} \Phi(y, z|\mathbf{a})^+$$

and from (9) that

$$e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) = \frac{y}{\mathcal{Y}(y)} e(\mathcal{Y}(y)|\mathbf{a}) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)|\mathbf{a}) - \beta y$$

Combining the two shows that we have

$$\begin{aligned} &\bar{\mathcal{R}}(\mathbf{a}) \subseteq \{y \geq 0 : e(y|\mathbf{a}) = \phi(y|\mathbf{a}) \geq 0\} \\ &\mathcal{Y}(y) \in \mathcal{Z} = \operatorname*{argmay}_{z \geq 0} \Phi\left(y, z|\mathbf{a}\right) \end{aligned} \tag{19}$$

and the first part will follow if we can show that the maximizer is unique and lies in $\mathcal{C}(\mathbf{a})$. Suppose to the contrary that $y \in \bar{\mathcal{R}}(\mathbf{a})$ is such that

$$\sup_{z \geq 0} \Phi(y, z|\mathbf{a}) = \Phi(y, z^*|\mathbf{a}).$$

for some $z^* \notin \mathcal{C}(\mathbf{a})$. If $z^* \in \bar{\mathcal{D}}(\mathbf{a})$ then it follows from (19) that we have

$$e(y|\mathbf{a}) = \Phi(y, z^*|\mathbf{a}) = -\beta y < 0$$

which contradicts the nonnegativity of the scaled equity value function. On the other hand, if $z^* \in \bar{\mathcal{R}}(\mathbf{a})$ then

$$\begin{aligned} e(y|\mathbf{a}) &= \Phi(y, z^*|\mathbf{a}) \\ &= \frac{y}{z^*} e(z^*|\mathbf{a}) + \left(\frac{y}{z^*} - 1\right) P(z^*|\mathbf{a}) - \beta y \\ &= \frac{y}{z^*} \Phi(z^*, \mathcal{Y}(z^*)|\mathbf{a}) + \left(\frac{y}{z^*} - 1\right) P(z^*|\mathbf{a}) - \beta y \\ &= \Phi(y, \mathcal{Y}(z^*)) + \left(\frac{y}{z^*} - 1\right) (P(z^*|\mathbf{a}) - P(\mathcal{Y}(z^*)|\mathbf{a})) - \beta y \\ &= \Phi(y, \mathcal{Y}(z^*)) - \beta y < \Phi(y, \mathcal{Y}(z^*)) \end{aligned}$$

where the third equality follows from (9), the fifth equality follows from the no jump condition (4) and the inequality follows from the strict positivity of the fixed cost. This contradicts the fact that $e(y|\mathbf{a}) = \phi(y|\mathbf{a})$ over $\bar{\mathcal{R}}(\mathbf{a})$ and thus establishes that $\mathcal{Z} \subseteq \mathcal{C}(\mathbf{a})$. To complete the proof observe that

$$e(y|\mathbf{a}) = \phi(y|\mathbf{a}) = \sup_{z \in \mathcal{C}(\mathbf{a})} \Phi(y, z|\mathbf{a})$$

is differentiable at all points of $\bar{\mathcal{R}}(\mathbf{a})$ as a result of (19) and Lemma 5, and apply **(author?)** (4, Corollary 4.iii)). □

B.3 The HJB equation

If $v : \mathbb{R}_+ \to \mathbb{R}$ is a convex function then its one sided derivatives $v'_\pm(y)$ are nondecreasing functions of finite variation, and its second distributional derivative is a positive measure that we denote by $v''(dy)$. Consider now the measure

$$\mathcal{O}v(dy) = [(\xi + \mu)yv'_-(y) - \rho v(y)]dy + \frac{1}{2}\sigma^2 y^2 v''(dy).$$

(author?) (3) show that the solution to (18) is intimately related to the set of functions that solve the HJB equation

$$\text{may}\,\{\mathcal{O}v(y), \psi(y|\mathbf{a}) - v(y)\} = 0 \tag{20}$$

in the distributional sense. To make this result precise we start by formally defining the type of weak solutions we are interested in.

Definition 2 *A function $v : (0, \infty) \to \mathbb{R}$ is a solution to (20) in the sense of distributions if it is convex and such that*

i) $v(y) \geq \psi(y|\mathbf{a})$ *for all* $y \geq 0$

ii) $\mathcal{O}v(dy)$ *is a non positive measure on* $\mathbb{R}_+$

iii) $\mathcal{O}v(dy)$ *does not charge the set* $\{y \geq 0 : v(y) > \psi(y|\mathbf{a})\}$

Proposition 2 $\mathbf{a} \in \mathcal{M}_r \cap \mathcal{S}$ *is a* rMPE *if and only if*

$$v(y) \equiv e(y|\mathbf{a}) - \hat{e}(y)$$

solves (20) in the sense of distributions subject to the boundary conditions

$$\limsup_{y\downarrow 0} y^{-\Pi} v(y) = \limsup_{y\downarrow 0} y^{-\Pi} \psi(y|\mathbf{a}) < \infty, \tag{21}$$

$$\limsup_{y\uparrow \infty} y^{-\Theta} v(y) = \limsup_{y\uparrow \infty} y^{-\Theta} \psi(y|\mathbf{a}) < \infty, \tag{22}$$

where $\Pi < 0$ and $\Theta > 1$ denote the two solutions to (28).

Proof. This follows from Lemma 5 and **(author?)** (3, Theorems 6.3|4) using the fact that in our case the state space is the positive real line with inaccessible boundaries and the reward function is convex and thus continuous. □

B.4 Barrier strategies

In view of Proposition 1 and Corollary 2 we can essentially restrict the search for rMPEs (but not the set of possible deviations) to the subset of default and adjustment strategies such that $dI_t(\mathbf{a}) \geq 0$ and

$$\tau_b(\mathbf{a}) = \inf\{t \geq 0 : y_t \leq y_b(\mathbf{a})\}$$

for some constant default threshold $y_b(\mathbf{a}) > 0$. A class of strategies of particular interest within that subset is the class of *barrier* strategies which is illustrated in Figure 1 and formally defined as follows.

Definition 3 *A strategy* $\mathbf{a} \in \mathcal{M}_r$ *is a barrier strategy if*

$$\mathcal{D}(\mathbf{a}) = \left\{(F, Y) \in \mathbb{R}^2_+ : Y \leq y_b F\right\}$$
$$\mathcal{R}(\mathbf{a}) = \left\{(F, Y) \in \mathbb{R}^2_+ : Y \geq y_u F\right\}$$

and

$$A(F, Y|\mathbf{a}) = \frac{y}{\mathcal{Y}(y)} - 1 \Big|_{y=\frac{Y}{F}}$$

for some $0 < y_b \leq y_u$ *and some function* $\mathcal{Y} : [y_u, \infty) \to (y_b, y_u)$ *that determines the target level of inverse leverage after the adjustment. In what follows we denote the set of barrier strategies by* $\mathcal{B}$.

Remark 1 The requirement that $\mathcal{Y}(y)$ takes values in $(y_b(\mathbf{a}), y_u(\mathbf{a}))$ rather than in $[0, y]$ is without loss of generality for equilibrium purposes since adjustments that move the state to a point inside $(\mathcal{D} \cup \mathcal{R})(\mathbf{a})$ are strictly suboptimal as long as the fixed cost of adjustment is strictly positive.

Assume that the firm follows a barrier strategy $\mathbf{a} \in \mathcal{B}$. Then (6) implies that the face value of debt satisfies

$$dF_t = -\xi F_t dt + \mathbf{1}_{\{y_{t-} \geq y_u(\mathbf{a})\}} \left(\frac{Y_t}{\mathcal{Y}(y_{t-})} - F_{t-}\right), \qquad \text{on } \{t < \tau_b(\mathbf{a})\}$$

and it follows that the induced inverse leverage process is an autonomous Markov process that evolves according to

$$dy_t = y_{t-}\sigma dW_t + y_{t-}(\xi + \mu)\, dt + \mathbf{1}_{\{y_{t-} \geq y_u(\mathbf{a})\}} (\mathcal{Y}(y_{t-}) - y_{t-}) \tag{23}$$

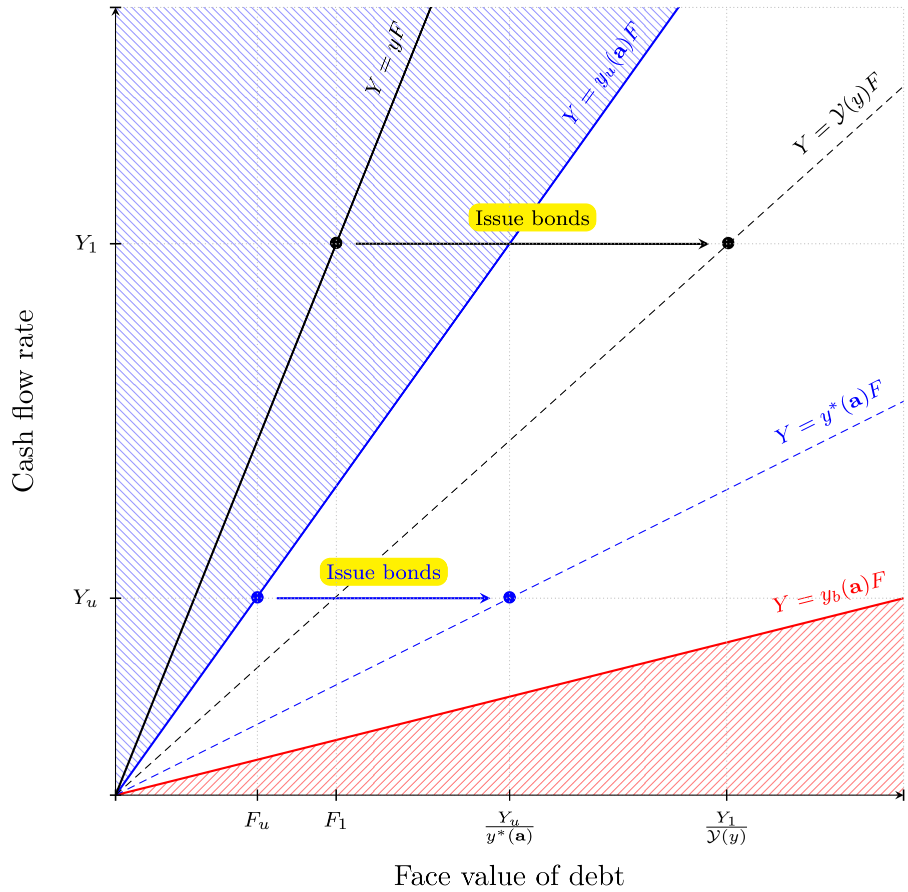

Figure 1: Illustration of a barrier strategy. In the figure ▨ represents the default region $\mathcal{D}(\mathbf{a})$, ▨ represents the restructuring region $\mathcal{R}(\mathbf{a})$, the complement ▭ represents the continuation region, and the arrows indicate increases in the face value of debt that move the state from the restructuring region to the continuation region.

until the first time that it reaches the barrier level $y_b(\mathbf{a})$ where the strategy requires shareholders to file for bankruptcy.

Before proceeding with the computation of the security values induced by a barrier strategy we first prove that all barrier strategies are feasible.

Lemma 6 $\mathcal{B} \subseteq \mathcal{S}$.

Proof. Fix a barrier strategy $\mathbf{a} \in \mathcal{B}$. Since the pair (F_t, Y_t) forms a Markov process we have that $\Lambda(\mathbf{a}) = \Lambda(F_0, Y_0)$ for some (possibly infinite) function $\Lambda : \mathbb{R}_+^2 \to \mathbb{R} \cup \{\infty\}$ that satisfies the boundary conditions

$$\Lambda(F, Y) = 0, \qquad (F, Y) \in \mathcal{D}(\mathbf{a}), \tag{24}$$

$$\Lambda(F, Y) = Y\left(1 + \frac{1}{\mathcal{Y}(y)}\right) + \Lambda\left(\frac{Y}{\mathcal{Y}(y)}, Y\right), \qquad (F, Y) \in \mathcal{R}(\mathbf{a}). \tag{25}$$

On the other hand, a standard calculation using Girsanov's theorem and the law of iterated expectations shows that

$$\Lambda(F, Y) = \lambda(y)Y, \qquad (F, Y) \in \mathbb{R}_+ \backslash (\mathcal{D} \cup \mathcal{R})(\mathbf{a})$$

for some function $\lambda : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ that satisfies

$$\lambda(y) = G(y) + H(y) \left(1 + \frac{1}{\mathcal{Y}(y_u(\mathbf{a}))} - \frac{1}{y} + \lambda(\mathcal{Y}(y_u(\mathbf{a})))\right) \tag{26}$$

with $H, G : \mathbb{R}_+ \to \mathbb{R}_+$ uniformly bounded and such that

$$\min\{G(y), 1 - H(y)\} > 0, \quad y \in \mathcal{C}(\mathbf{a}) \equiv (y_b(\mathbf{a}), y_u(\mathbf{a})). \tag{27}$$

Combining (24), (25), and (26) we deduce that the strategy is feasible if and only if the constant

$$\lambda(\mathcal{Y}(y_u(\mathbf{a}))) = \frac{G(\mathcal{Y}(y_u(\mathbf{a}))) + H(\mathcal{Y}(y_u(\mathbf{a})))}{1 - H(\mathcal{Y}(y_u(\mathbf{a})))}$$

is finite and the desired result now follows from (27) since the point $\mathcal{Y}(y_u(\mathbf{a}))$ lies by assumption in the set $\mathcal{C}(\mathbf{a})$. □

Let now $\mathbf{a} \in \mathcal{B}$ be a barrier strategy and denote by

$$\mathcal{L}f(y) \equiv y\,(\xi + \mu)\, f'(y) + \frac{1}{2}\sigma^2 y^2 f''(y)$$

the differential operator associated with the continuous part of this stochastic differential equation. Standard arguments relying on Itô's lemma and the continuity of the bond price at the issuance point show that the function

$$P(y_t|\mathbf{a}) = P_t(\mathbf{a}) = \mathbb{E}_t \left[\int_t^{\tau_b(\mathbf{a})} e^{-\rho(s-t)}\,(c + \xi)\, ds \right]$$

is the unique solution to

$$\begin{aligned} \rho P(y) &= \mathcal{L}P(y) + c + \xi, && y \in (y_b(\mathbf{a}), y_u(\mathbf{a})), \\ P(y) &= P(\mathcal{Y}(y)), && y \geq y_u(\mathbf{a}), \\ P(y) &= 0, && y \leq y_b(\mathbf{a}). \end{aligned}$$

in the space of functions that are bounded on $\mathbb{R}_+$ and twice continuously differentiable on the continuation region $\mathcal{C}(\mathbf{a}) \equiv (y_b(\mathbf{a}), y_u(\mathbf{a}))$.

To describe the solution to this differential problem denote by $\Theta > 1$ and $\Pi < 0$ the solutions to the quadratic equation

$$-\rho + (\xi + \mu)\, y + \frac{1}{2}\sigma^2 y(y-1) = 0 \tag{28}$$

induced by the continuous part of (23); and by $y^*(\mathbf{a}) \equiv \mathcal{Y}(y_u(\mathbf{a}))$ the level of inverse leverage to which the firm moves upon reaching from the inside the right boundary of the continuation region.

Lemma 7 *Assume that* $\mathbf{a} \in \mathcal{B}$ *is a barrier strategy. Then*

$$P_t(\mathbf{a}) = P(y_t|\mathbf{a}) \equiv \mathbf{1}_{\{y_t \in \mathcal{C}(\mathbf{a})\}} \pi(y|\mathbf{a}) + \mathbf{1}_{\{y_t \geq y_u(\mathbf{a})\}} \pi\left(\mathcal{Y}(y_t)|\mathbf{a}\right)$$

with the function

$$\pi(y|\mathbf{a}) \equiv \frac{1}{\rho}\,(c+\xi)\left(1 + A_\pi(\mathbf{a}) y^\Theta + B_\pi(\mathbf{a}) y^\Pi\right),$$

and the constants

$$A_\pi(\mathbf{a}) \equiv \frac{y_u(\mathbf{a})^\Pi - y^*(\mathbf{a})^\Pi}{y_b(\mathbf{a})^\Pi\left(y_u(\mathbf{a})^\Theta - y^*(\mathbf{a})^\Theta\right) + y_b(\mathbf{a})^\Theta\left(y^*(\mathbf{a})^\Pi - y_u(\mathbf{a})^\Pi\right)},$$

$$B_\pi(\mathbf{a}) \equiv \frac{y^*(\mathbf{a})^\Theta - y_u(\mathbf{a})^\Theta}{y_b(\mathbf{a})^\Pi\left(y_u(\mathbf{a})^\Theta - y^*(\mathbf{a})^\Theta\right) + y_b(\mathbf{a})^\Theta\left(y^*(\mathbf{a})^\Pi - y_u(\mathbf{a})^\Pi\right)}.$$

In particular, the bond price function is strictly concave on $\mathcal{C}(\mathbf{a})$ *with* $P'(y_b(\mathbf{a})|\mathbf{a}) > 0$, $P'(y^*(\mathbf{a})|\mathbf{a}) \geq 0$, *and* $P'(y_u(\mathbf{a})|\mathbf{a}) \leq 0$.

Proof. The first part follows by direct calculation and the second by noting that since $\Theta > 1$, $\Pi < 0$, and $y_b(\mathbf{a}) \leq y^*(\mathbf{a}) \leq y_u(\mathbf{a})$ we have $A_\pi(\mathbf{a}), B_\pi(\mathbf{a}) \leq 0$. □

Remark 2 The derivatives of the bond price at the points $y^*(\mathbf{a})$ and $y_u(\mathbf{a})$ are either both non zero or both equal to zero depending on whether the length of the continuation region $|\mathcal{C}(\mathbf{a})|$ is strictly positive or zero. In the latter case, the bond price function coincides with the solution that obtains when imposing a reflecting boundary condition at the upper threshold.

Consider now the scaled equity value that prevails when creditors correctly anticipate that management will use the barrier strategy $\mathbf{a}$:

$$e_t(\mathbf{a}) = \mathbb{E}_t\left[\int_t^{\tau_b(\mathbf{a})} e^{-\rho(s-t)}\left(\delta(y_s)ds + \left(A_s(\mathbf{a})\left(e_s(\mathbf{a}) + P_s(\mathbf{a})\right) - \beta y_{s-}\right) dN_s(\mathbf{a})\right)\right].$$

When $\mathbf{a}$ is a barrier strategy (or more generally a reduced Markov strategy) all the terms in the conditional expectation only depend on the path of the Markov process described by (23). As a result, $e_t(\mathbf{a}) = e(y_t|\mathbf{a})$ for some deterministic function and standard results show that this function is the unique solution to

$$\rho e(y) = \mathcal{L}e(y) + \delta(y), \qquad y \in \mathcal{C}(\mathbf{a}), \tag{29}$$

$$e(y) = 0, \qquad y \leq y_b(\mathbf{a}),$$

$$e(y) = \frac{y}{\mathcal{Y}(y)}e(\mathcal{Y}(y)) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)|\mathbf{a}) - \beta y, \qquad y \geq y_u(\mathbf{a}), \tag{30}$$

in the space of functions that are finite on $\mathbb{R}_+$ and twice continuously differentiable on the continuation region $\mathcal{C}(\mathbf{a})$.

Lemma 8 *Assume that $\mathbf{a} \in \mathcal{B}$ is a barrier strategy. Then*

$$e_t(\mathbf{a}) = e(y_t|\mathbf{a}) \equiv \mathbf{1}_{\{y_t \in \mathcal{C}(\mathbf{a})\}}\varepsilon(y_t|\mathbf{a}) + \mathbf{1}_{\{y_t \geq y_u(\mathbf{a})\}}\overline{\varepsilon}(y_t|\mathbf{a})$$

where

$$\varepsilon(y|\mathbf{a}) \equiv \hat{e}(y) + A_\varepsilon(\mathbf{a})y^\Theta + B_\varepsilon(\mathbf{a})y^\Pi$$

$$\overline{\varepsilon}(y|\mathbf{a}) \equiv \frac{y}{\mathcal{Y}(y)}\varepsilon(\mathcal{Y}(y)|\mathbf{a}) + \left(\frac{y}{\mathcal{Y}(y)} - 1\right) P(\mathcal{Y}(y)|\mathbf{a}) - \beta y$$

and $(A_\varepsilon(\mathbf{a}), B_\varepsilon(\mathbf{a}))$ are the unique solutions to the value matching conditions

$$\varepsilon(y_b(\mathbf{a})|\mathbf{a}) = 0,$$

$$\varepsilon(y_u(\mathbf{a})|\mathbf{a}) = \overline{\varepsilon}(y_u(\mathbf{a})|\mathbf{a}),$$

at the endpoints of the continuation region.

Proof. Follows by direct calculation. □

B.5 rMPEs in barrier strategies

We start with a result that specializes the differential characterization of Proposition 2 to the case of barrier strategies.

Proposition 3 *A barrier strategy is a rMPE if and only if the induced equity value function $e(y|\mathbf{a})$ is a solution to (20) in the sense of distributions.*

Proof. Combining Corollary 3, Lemma 7, and Lemma 8 we deduce that there exists a constant $k > 0$ such that

$$|e(y|\mathbf{a})| \vee |\phi(y|\mathbf{a})| \leq k\,(1 + |y|)\,, \qquad y \geq 0.$$

Since $\Pi < 0$ and $\Theta > 1$ this implies that we have

$$\lim_{y \downarrow 0} y^{-\Pi} f(y) = \lim_{y \uparrow \infty} y^{-\Theta} f(y) = 0, \qquad \text{for } f \in \{e(\cdot|\mathbf{a}), \phi(\cdot|\mathbf{a})^+\}.$$

This shows that the boundary conditions (21) and (22) hold for any barrier strategy and the desired result now follows from Proposition 2. □

The next result provides a set of *necessary* conditions for a barrier strategy to form an equilibrium. To state the result let

$$s(y|\mathbf{a}) \equiv \frac{1}{y}\left(e(y|\mathbf{a}) + P(y|\mathbf{a})\right)$$

denote the value of the firm per unit of cash flow.

Lemma 9 *Assume that the barrier strategy* $\mathbf{a} \in \mathcal{B}$ *is a* rMPE. *Then the following conditions are satisfied:*

i) $y_b(\mathbf{a}) \leq y_0$,

ii) $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+ = 0$ *on* $(0, y_b(\mathbf{a})]$,

iii) $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+ > 0$ *on* $[y_u(\mathbf{a}), \infty)$,

iv) Smooth pasting and value matching at the default boundary:

$$e'(y_b(\mathbf{a})|\mathbf{a}) = e(y_b(\mathbf{a})|\mathbf{a}) = 0. \tag{31}$$

v) Smooth pasting and value matching at restructuring points:

$$e'(y|\mathbf{a}) = s(\mathcal{Y}(y)|\mathbf{a}) - \beta = s(y|\mathbf{a}), \qquad y \geq y_u(\mathbf{a}). \tag{32}$$

vi) Optimality of restructuring:

$$\{\mathcal{Y}(y)\} = \operatorname*{argmay}_{z \geq 0} \Phi(y, z|\mathbf{a}) = \operatorname*{argmay}_{z \in \mathcal{C}(\mathbf{a})} \Phi(y, z|\mathbf{a}), \qquad y \geq y_u(\mathbf{a}).$$

Proof of i). This follows by observing that if $\mathbf{a}$ is an rMPE with $y_b(\mathbf{a}) > y_0$ then $e(y|\mathbf{a}) = 0 < e_0(y)$ for all $y \in (y_0, y_b(\mathbf{a}))$ which contradicts (9). □

Proof of ii). If $\mathbf{a} \in \mathcal{B}$ is a rMPE then it follows from Lemma 5 and the definition of the strategy that we have $e(y|\mathbf{a}) = 0 \geq \phi(y|\mathbf{a})^+$ for all $y \leq y_b(\mathbf{a})$. □

Proof of iv). Since by definition $e(y|\mathbf{a}) = 0$ for all $y \leq y_b(\mathbf{a})$ it follows from Lemma 5 that $e(y|\mathbf{a}) = \phi(y|\mathbf{a})^+ = 0$ over that region. This in turn implies that the scaled equity value function is differentiable at all points $y \leq y_b(\mathbf{a})$ and the desired result follows by noting that $e'_-(y|\mathbf{a}) = 0$ at any such point. □

Proof of iii). Since by definition $e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) \leq \phi(y|\mathbf{a})$ for $y \geq y_u(\mathbf{a})$ it follows from Lemma 5 that we have

$$0 \leq e(y|\mathbf{a}) = \Phi(y, \mathcal{Y}(y)|\mathbf{a}) = \phi(y|\mathbf{a}), \qquad y \geq y_u(\mathbf{a}).$$

To see that the inequality is strict note that due to Item iv) the scaled equity value function solves (29) subject to (31). In particular,

$$\lim_{y \downarrow y_b(\mathbf{a})} \frac{1}{2}\sigma^2 y^2 e''(y|\mathbf{a}) = -\delta(y_b(\mathbf{a})) > 0$$

where the strict inequality follows from Item i) and the definition of the no-issuance default threshold. The above inequality implies that we have $e(y|\mathbf{a}) > 0$ in a right neighborhood of $y_b(\mathbf{a})$ and thus for all $y > y_b(\mathbf{a})$ by convexity. □

Proof of vi). This follows directly from Corollary 3. □

Proof of v). Since $e(y|\mathbf{a}) = \phi(y|\mathbf{a}) > 0$ for all $y \geq y_u(\mathbf{a})$ by Item iii) it follows from Lemma 5 that the scaled equity value function, and thus also $\phi(y|\mathbf{a})$, is differentiable at all points $y \geq y_u(\mathbf{a})$. On the other hand, by Item vi) we have that $\mathcal{Y}(y)$ is the unique maximizer of the function $z \mapsto \Phi(y, z|\mathbf{a})$ over the compact set $\mathcal{C}(a)$ and the validity of (32) now follows from **(author?)** (4, Corollary 4) and (30). □

Lemma 10 *Assume that the barrier strategy* $\mathbf{a} \in \mathcal{B}$ *satisfies Conditions i) and iv) of Lemma 9. Then*

vi) $e(y|\mathbf{a})$ *is nonnegative, nondecreasing, convex on the interval* $[0, y_u(\mathbf{a}))$ *and strictly positive on the interval* $(y_b(\mathbf{a}), y_u(\mathbf{a}))$

vii) $e(y|\mathbf{a}) \geq e_0(y)$ *for all* $y \leq y_u(\mathbf{a})$ *if and only if* $y_b(\mathbf{a}) \leq y_0$.

Proof of vi). Since $e(y|\mathbf{a})$ solves (29) subject to value matching and smooth pasting at the default boundary the uniqueness of the solution to second order differential equations implies that the constants in Lemma 8 can be expressed as

$$A_\varepsilon(\mathbf{a}) = \frac{y_b(\mathbf{a})^{-\Theta}(1-\tau)\,(y_b(\mathbf{a})-y_0)\,(\Pi-1)}{(r-\mu)(\Theta-\Pi)} \geq 0,$$

$$B_\varepsilon(\mathbf{a}) = \frac{y_b(\mathbf{a})^{-\Pi}(1-\tau)(y_b(\mathbf{a})(\Theta-1)\Pi + y_0\Theta(1-\Pi))}{(r-\mu)\Pi(\Pi-\Theta)} \geq 0.$$

Therefore, $e(y|\mathbf{a})$ is convex on the interval $[0, y_u(\mathbf{a}))$ and remaining claims in the statement follow by observing that because

$$\lim_{y\downarrow y_b(\mathbf{a})} \frac{1}{2}\sigma^2 y^2 e''(y) = -\delta(y_b(\mathbf{a})) > 0$$

we must have $\min\{e, e'\}(y|\mathbf{a}) > 0$ in a right neighbourhood of $y_b(\mathbf{a})$ and thus over the whole interval since the scaled equity value is convex. □

Proof of vii). The necessity of the condition is clear since in its absence $e(y|\mathbf{a}) = 0 < e_0(y)$ for all $y \leq (y_0, y_b(\mathbf{a}))$. Now assume that $y_b(\mathbf{a}) \leq y_0$. If $y_u(\mathbf{a}) \leq y_0$ then the result follows from Item vi)) since $e_0(y) = 0$ on $[0, y_0]$. Assume from now on that $y_u(\mathbf{a}) > y_0$. Proceeding as in the first part of the proof shows that

$$w(y) = e(y|\mathbf{a}) - e_0(y) = A_\varepsilon(\mathbf{a})y^\Theta + \mathbf{1}_{\{y>y_0\}}\bar{B}(\mathbf{a})y^\Pi, \qquad y \in [y_0, y_u(\mathbf{a}))$$

where $A_\varepsilon(\mathbf{a}) \geq 0$ and

$$\bar{B}(\mathbf{a}) \equiv \frac{y_0^{1-\Pi}(1-\tau)}{(r-\mu)\Pi} + \frac{y_b(\mathbf{a})^{-\Pi}(1-\tau)(y_b(\mathbf{a})(\Theta-1)\Pi + y_0\Theta(1-\Pi))}{(r-\mu)(\Pi-\Theta)\Pi}$$

Noting that $0 = \bar{B}(\mathbf{a})\big|\, y_{b(\mathbf{a})=y_0}$ and

$$\frac{d\bar{B}(\mathbf{a})}{dy_b(\mathbf{a})} = \frac{(1-\tau)(\Pi-1)(y_b(\mathbf{a})(\Theta-1) - y_0\Theta)}{y_b(\mathbf{a})^{1+\Pi}(r-\mu)(\Theta-\Pi)} \geq 0, \qquad y_b(\mathbf{a}) \leq y_0$$

we deduce that $\bar{B}(\mathbf{a}) \leq 0$. This implies that $w(y)$ is non decreasing on $[y_0, y_u(\mathbf{a})]$ and the thesis follows by observing that $w(y_0) = e(y_0|\mathbf{a}) \geq 0$. □

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