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From time to time *The Fibonacci Quarterly* will publish invited papers by well known mathematicians and scientists describing how the authors use Fibonacci numbers, or similar recurrence sequences, in their research and teaching. We are pleased to present here the first paper in the series. The author is the distinguished scholar and teacher George Andrews, who is currently the Evan Pugh Professor of Mathematics at Penn State University. In 2003 Professor Andrews was elected to the National Academy of Sciences, which is one of the highest honors that can be accorded a scientist or engineer.

FIBONACCI NUMBERS AND THE ROGERS-RAMANUJAN IDENTITIES

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ABSTRACT

In my research on the theory of partitions and related questions, I have often used recurrent integer sequences such as the Fibonacci numbers to obtain hints about the behavior of certain partition generating functions. In this article, I will illustrate this process by studying the relationship between the Fibonacci numbers and a sequence of polynomials used by Schur in his second proof of the Rogers-Ramanujan identities. I will try to make clear how generating function proofs of Fibonacci identities lead to analogous results for Schur's polynomials. We shall find that this approach helps explain why the generalization of Fibonacci formulas to Schur's polynomial is sometimes rather intricate.

1. INTRODUCTION

Everyone reading these words knows the Fibonacci sequence:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1. \end{cases} \tag{1.1.1}$$

Less well-known is a sequence of polynomials first considered by I. Schur [12]:

$$S_n(q) = S_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ S_{n-1} + q^{n-2} S_{n-2} & \text{if } n > 1. \end{cases} \tag{1.1.2}$$

Of course, it is immediately obvious that

$$S_n(1) = F_n. \tag{1.1.3}$$

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It is, however, less than obvious that

$$S_{n+1}(q) = \sum_{0 \le 2j \le n} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}$$

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n-5j}{2} \rfloor \end{bmatrix} \tag{1.1.4}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0 & \text{if } B < 0 \text{ or } B > A \\ \frac{(1-q^A)(1-q^{A-1})\cdots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\cdots(1-q)} & \text{otherwise} \end{cases} \tag{1.1.5}$$

and $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Consult [2] for a full proof of (1.4).

The rational functions of q given in (1.5) are in fact polynomials. They are sometimes called Gaussian polynomials and sometimes q-binomial coefficients.

If we allow $n \to \infty$ in formula (1.4) then it is easily shown that what results is

$$1 + \sum_{j=1}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \frac{\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2}}{\prod_{n=1}^{\infty}(1-q^n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \text{ (by [4: p. 113])} \tag{1.1.6}$$

the first of the celebrated Rogers-Ramanujan identities.

Schur also considered a sequence slightly different from (1.2):

$$\mathcal{T}_n(q) = \mathcal{T}_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathcal{T}_{n-1} + q^{n-1}\mathcal{T}_{n-2} & \text{if } n > 1. \end{cases} \tag{1.1.7}$$

As before in (1.3), now

$$\mathcal{T}_n(1) = F_n. \tag{1.1.8}$$

In analogy with (1.5)

$$\mathcal{T}_{n+1}(q) = \sum_{0 \le 2j \le n} q^{j^2+j} \begin{bmatrix} n-j \\ j \end{bmatrix}$$

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-3)/2} \begin{bmatrix} n+j \\ \lfloor \frac{n+1-5j}{2} \rfloor + 1 \end{bmatrix}. \tag{1.1.9}$$

Allowing $n \to \infty$ in (1.9), we obtain the second Rogers-Ramanujan identity

$$
\begin{aligned}
1+\sum_{j=1}^{\infty}\frac{q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)} &= \frac{\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j-3)/2}}{\prod_{n=1}^{\infty}(1-q^n)} \\
&= \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}, \text{ (by [4: p. 113]).}
\end{aligned}
\tag{1.1.10}
$$

Now it is already obvious in light of (1.3) and (1.8) that Schur's polynomial sequences are polynomial- or q-analogs of the Fibonacci numbers. Indeed, we see immediately from (1.4) and (1.9) that

$$
\begin{aligned}
F_{n+1} &= \sum_{0\le 2j\le n}\binom{n-j}{j} \\
&= \sum_{j=-\infty}^{\infty}(-1)^j\binom{n}{\lfloor\frac{n-5j}{2}\rfloor+1} \\
&= \sum_{j=-\infty}^{\infty}(-1)^j\binom{n+1}{\lfloor\frac{n+1-5j}{2}\rfloor+1}.
\end{aligned}
\tag{1.1.11}
$$

The first line of (1.11) is quite familiar [14; p. 155]; however the next two lines are much less well known. In fact, they form the basis of the extensive study of generalized Fibonacci numbers in [1] (cf. [10], [13]).

Equation (1.11) suggests that interesting information about F_n may be obtained from studying $\mathcal{S}_n$ and $\mathcal{T}_n$. Our object here is the reverse. Can we systematize a study of the generating function for F_n so that we can deduce formulas for $\mathcal{S}_n$ and $\mathcal{T}_n$ from generating function proofs of identities for F_n?

2. GENERATING FUNCTION PROOFS FOR FIBONACCI IDENTITIES

As is well-known, [14; p. 221]

$$\sum_{n=0}^{\infty}F_n x^n = \frac{x}{1-x-x^2}. \tag{2.2.1}$$

Perhaps less widely appreciated is the fact that the vast majority of Fibonacci number identities are easily verified using (2.1). To make this point briefly, we have chosen a few results that are usually proved by other means (cf. [12; Sec 10.14], [14; Chs. 5, 8 and 12]). However, we should note that the generating function method is effectively explored by Koshy ([14; Chs. 18 and 19]).

$$F_0+F_1+\cdots+F_n = F_{n+2}-1 \quad \text{[14; Ch. pp. 228-229];} \tag{2.2.2}$$

$$F_{n+m} = F_{m-1}F_n + F_mF_{n+1} \qquad [14; \text{ p. } 88, \#6; \text{ p. } 363, \text{ eq. } (32.3)]; \tag{2.2.3}$$

$$F_n = 2^{1-n} \sum_{0\le 2j+1\le n} \binom{n}{2j+1} 5^j \qquad [12; \text{ eq. } (10.14.11), \text{ p. } 150]. \tag{2.2.4}$$

Proving these results via generating functions will not be as illuminating as some other proofs nor as elegant. However, the ideas arising from an awareness of the power of this method will serve us well in Section 5.

To treat each identity we consider the generating functions for the sequences defined by each side. If these generating functions are identical the identities are proved.

Proof of (2.2) following [14; pp. 228-229]:

$$\begin{aligned}
\sum_{n=0}^{\infty}(F_0 + F_1 + \cdots + F_n)x^n &= \frac{1}{1-x}\sum_{n=0}^{\infty} F_n x^n \\
&= \frac{x}{(1-x)(1-x-x^2)} \\
&= \frac{1+x}{1-x-x^2} - \frac{1}{1-x} \\
&= x^{-2}\left(\frac{x}{1-x-x^2} - x\right) - \frac{1}{1-x} \\
&= x^{-2}\sum_{n=2}^{\infty} F_n x^n - \frac{1}{1-x} \\
&= \sum_{n=0}^{\infty}(F_{n+2} - 1)x^n,
\end{aligned} \tag{2.2.5}$$

and (2.2) is proved by comparing the coefficients of x^n in the extremes of (2.5).

Proof of (2.3). Here we follow the treatment due to R.T. Hansen [11] (cf. [12; p. 231]). We fix m, then

$$\begin{aligned}
(1-x-x^2)\sum_{n=0}^{\infty} F_{n+m}x^n &= F_m + F_{m+1}x + \sum_{n=2}^{\infty} F_{n+m}x^n \\
&\quad - F_m x - \sum_{n=1}^{\infty} F_{n+m}x^{n+1} - \sum_{n=0}^{\infty} F_{n+m}x^{n+2} \\
&\quad + F_m + xF_{m-1} + \sum_{n=0}^{\infty}(F_{n+m+2} - F_{n+m+1} - F_{n+m})x^{n+2} \\
&= F_m + xF_{m-1}. \quad \text{(where } n \text{ has been replaced by } n+2 \\
&\qquad\qquad \text{in the first sum and } n+1 \text{ in the second)}
\end{aligned} \tag{2.2.6}$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n+m}x^n &= \frac{F_m + xF_{m-1}}{(1-x-x^2)} \\ &= \sum_{n=0}^{\infty} F_m F_{n+1} x^n + \sum_{n=0}^{\infty} F_{m-1}F_n x^n.\end{aligned} \tag{2.2.7}$$

Consequently, (2.3) follows by comparing coefficients of x^n on each side of (2.7).

Finally we consider (2.4):

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{1-n} \sum_{0\le 2j+1\le n} \binom{n}{2j+1} 5^j x^n &= \sum_{n,j\ge 0} 2^{1-n-2j-1} \binom{n+2j+1}{2j+1} 5^j x^{n+2j+1} \\ &\qquad \text{(where } n \text{ has been shifted to } n+2j+1\text{)} \\ &= \sum_{j\ge 0} \frac{5^j x^{2j+1}}{4^j} \left(1-\frac{x}{2}\right)^{-2j-2} \\ &= \frac{1}{(1-\frac{x}{2})^2} \frac{x}{1-\frac{5x^2}{4(1-\frac{x}{2})^2}} \\ &= \frac{x}{(1-\frac{x}{2})^2 - \frac{5x^2}{4}} \\ &= \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n\end{aligned} \tag{2.2.8}$$

and (2.4) is proved.

As we stated at the beginning of this section, the object of this section was *not* to provide the most elegant of illuminating proofs of identities (2.2)-(2.4) (indeed, not even novel). Rather we have taken a uniform generating function approach in the hope that in Sections 3-5 we can use the work in this section as guidance to point us to correct analogs of (2.2)-(2.4) for Schur's polynomials. We should note that if we can write down a generating function for both sides of a general 'Fibonacci number identity' then we have just one identity involving power series to verify (rather than infinitely many numerical identities), which in each of these cases turns out to be an identity involving rational functions.

3. SCHUR'S POLYNOMIALS, A BEGINNING

In order to find analogs of (2.1) for $\mathcal{S}_n$ and $\mathcal{T}_n$, we need the q-analog of the binomial series:

$$\sum_{j=0}^{\infty} \begin{bmatrix} n+j \\ j \end{bmatrix} x^j = \frac{1}{(x;q)_{n+1}}, \qquad [4;\text{ p. 36, eq. (3.3.7)}] \tag{3.3.1}$$

where

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{3.3.2}$$

Throughout this section $|x| < 1,\ |q| < 1$.

We may now obtain an analog of (2.1) for $\mathcal{S}_n(q)$.

Theorem 1: $\sum_{n=0}^{\infty} \mathcal{S}_n(q)x^n = \frac{1}{1-x-x^2\eta}x$, *where η is an operator on functions of x defined by*

$$\eta f(x) = f(xq)$$

Remark: *For a full account of the algebra of operators, I recommend [6]. As a readily comprehensible example, we consider*

$$\frac{1}{1-x\eta}f(x) = \sum_{n=0}^{\infty}(x\eta)^n f(x)$$

$$= \sum_{n=0}^{\infty} \underbrace{(x\eta)(x\eta)\dots(x\eta)}_{n \text{ times}} f(x)$$

$$= \sum_{n=0}^{\infty} x^n q^{0+1+2+\cdots+(n-1)} f(xq^n)$$

$$= \sum_{n=0}^{\infty} x^n q^{n(n-1)/2} f(xq^n).$$

The algebra of operators (see [6] again) tells us that the inverse operator is $1-x\eta$, and we can easily convince ourselves of this as follows

$$(1-x\eta)\sum_{n=0}^{\infty} x^n q^{n(n-1)/2} f(xq^n)$$

$$= \sum_{n=0}^{\infty} x^n q^{n(n-1)/2} f(xq^n) - \sum_{n=0}^{\infty} x(xq)^n q^{n(n-1)/2} f(xq^{n+1})$$

$$= \sum_{n=0}^{\infty} x^n q^{n(n-1)/2} f(xq^n) - \sum_{m=1}^{\infty} x^m q^{m(m-1)/2} f(xq^m)$$

(where we have replaced $n+1$ by m in the second sum)

$$= f(x).$$

Proof: Let us denote by $\sigma(x)$ the expression on the left side of the equation in Theorem 1. Hence equivalently, we are to prove

$$(1-x-x^2\eta)\sigma(x) = x.$$

Now,

$$x+\sum_{n=2}^{\infty}\mathcal{S}_n(q)x^n-\sum_{n=0}^{\infty}\mathcal{S}_n(q)x^{n+1}-\sum_{n=0}^{\infty}\mathcal{S}_n(q)x^{n+2}q^n$$

$$=x+\sum_{n=2}^{\infty}(\mathcal{S}_n(q)-\mathcal{S}_{n-1}(q)-q^{n-2}\mathcal{S}_{n-2}(q))x^n$$

$$=x,\qquad \text{(where } n \text{ is replaced by } n-1 \text{ in the second sum and } n-2 \text{ in the third)}$$

by (1.2). □

Lemma 1:

$$(x+x^2\eta)^n x = x^{n+1}\sum_{j=0}^{n} x^j q^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}.$$

Proof: For $n=0$, this asserts $x=x$. Now assume the result for a particular n; then

$$(x+x^2\eta)^{n+1}x=(x+x^2\eta)x^{n+1}\sum_{j=0}^{n}x^jq^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}$$

$$=x^{n+2}\sum_{j=0}^{n}x^jq^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}+x^{n+3}\sum_{j=0}^{n}x^jq^{j^2+n+1+j}\begin{bmatrix} n \\ j \end{bmatrix}$$

$$=x^{n+2}\left(\sum_{j\geq 0}x^jq^{j^2}\left(\begin{bmatrix} n \\ j \end{bmatrix}+q^{n+1-j}\begin{bmatrix} n \\ j-1 \end{bmatrix}\right)\right)$$

$$=x^{n+2}\sum_{j\geq 0}x^jq^{j^2}\begin{bmatrix} n+1 \\ j \end{bmatrix}\qquad \text{(by [4; eq. (3.3.3), p. 35])},$$

and the result follows by mathematical induction. □

Corollary 1:

$$\sum_{n\geq 0}\mathcal{S}_n(q)x^n=\sum_{j\geq 0}\frac{x^{2j+1}q^{j^2}}{(x;q)_{j+1}}.$$

Proof:

$$\sum_{n\geq 0} \mathcal{S}_n(q)x^n = \sum_{j=0}^{\infty} \frac{1}{1-x-x^2\eta}x$$

$$= \sum_{n=0}^{\infty} (x+x^2\eta)^n x$$

$$= \sum_{n=0}^{\infty} x^{n+1} \sum_{j=0}^{n} x^j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}$$

$$= \sum_{n,j\geq 0} x^{n+2j+1} q^{j^2} \begin{bmatrix} n+j \\ j \end{bmatrix}$$

(where n has been replaced by $n+j$)

$$= \sum_{j\geq 0} \frac{x^{2j+1}q^{j^2}}{(x;q)_{j+1}}$$

(by (3.1)). □

Corollary 2:

$$\mathcal{S}_n(q) = \sum_{j\geq 0} q^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}.$$

Proof: Extract the coefficient of x^n in the penultimate line in the proof of Corollary 1.1. □

Using precisely this technique, we can prove the following analog of (2.2).

Theorem 2:

$$\mathcal{S}_{n+2}(q) - 1 = \sum_{j=0}^{n} q^j \mathcal{S}_j(q).$$

Proof:

$$
\begin{aligned}
\sum_{n\geq 0}(\mathcal{S}_{n+2}(q)-1)x^n &= x^{-2}\left(\sum_{n=0}^{\infty}\mathcal{S}_n(q)x^n - x\right) - \frac{1}{1-x} \\
&= x^{-2}\frac{1}{1-x-x^2\eta}x - \frac{x^{-1}}{1-x} \\
&= \frac{1}{1-x}(-x^{-2}(1-x-x^2\eta)+x^{-2}-x^{-1})\frac{1}{1-x-x^2\eta}x \\
&= \frac{1}{1-x}\eta\frac{1}{1-x-x^2\eta}x \\
&= \frac{1}{1-x}\sum_{j\geq 0}\mathcal{S}_j(q)(xq)^j \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}q^j\mathcal{S}_j(q)\right)x^n,
\end{aligned}
\tag{3.3.3}
$$

and extracting the coefficients of x^n in the above extremes we obtain Theorem 3. □

Note that if we let $q \to 1$ in Corollary 2 it reduces directly to the first line of (1.11), and if we let $q \to 1$ in Theorem 2 it reduces to (2.2).

4. EXTENDING (2.3) IS HARDER

If we try to follow the proof of (2.3) directly, we find troublesome powers of q (which vanish when $q = 1$) making a direct generalization of (2.3) impossible. In order for us to overcome this difficulty we must, in fact, generalize $\mathcal{S}_n(q)$:

$$
\mathcal{S}_n(t,q) = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ \mathcal{S}_{n-1}(t,q)+tq^{n-2}\mathcal{S}_{n-2}(t,q) & \text{for } n>1. \end{cases}
\tag{4.4.1}
$$

Now for $m \geq 0$

$$(1 - x - tx^2 q^m \eta) \sum_{n=0}^{\infty} \mathcal{S}_{n+m}(t,q) x^n$$

$$\begin{aligned}
&= \mathcal{S}_m(t,q) + x\mathcal{S}_{m+1}(t,q) + \sum_{n=2}^{\infty} \mathcal{S}_{n+m}(t,q) x^n \\
&\quad - x\mathcal{S}_m(t,q) - \sum_{n=1}^{\infty} \mathcal{S}_{n+m}(t,q) x^{n+1} \\
&\quad - \sum_{n=0}^{\infty} \mathcal{S}_{n+m}(t,q) t q^{m+n} x^{n+2} \qquad (4.4.2) \\
&= \mathcal{S}_m(t,q) + x(\mathcal{S}_{m+1}(t,q) - \mathcal{S}_m(t,q)) \\
&\quad + \sum_{n=2}^{\infty} (\mathcal{S}_{n+m}(t,q) - \mathcal{S}_{n+m-1}(t,q) - tq^{m+n-2}\mathcal{S}_{n+m-2}(t,q)) x^n
\end{aligned}$$

(where n has been replaced by $n-1$ in the second sum and $n-2$ in the third)

$$= \begin{cases} x & \text{if } m = 0 \\ \mathcal{S}_m(t,q) + xtq^{m-1}\mathcal{S}_{m-1}(t,q) & \text{if } m > 0. \end{cases}$$

Thus

$$\sum_{n=0}^{\infty} \mathcal{S}_n(t,q) x^n = \frac{1}{1 - x - tx^2\eta} x, \qquad (4.4.3)$$

and

$$\sum_{n=0}^{\infty} \mathcal{S}_{n+1}(t/q,q) x^n = \frac{1}{1 - x - tx^2\eta}. \qquad (4.4.4)$$

So by (4.2) with t replaced by t/q^m, and $m > 0$,

$$\sum_{n=0}^{\infty} \mathcal{S}_{n+m}(tq^{-m})x^n = \frac{1}{1-x-tx^2\eta}(\mathcal{S}_m(tq^{-m},q) + xtq^{-1}\mathcal{S}_{m-1}(tq^{-m},q))$$

$$= \mathcal{S}_m(tq^{-m},q)\sum_{n=0}^{\infty}\mathcal{S}_{n+1}(t/q,q)x^n \tag{4.4.5}$$

$$+ tq^{-1}\mathcal{S}_{m-1}(tq^{-m},q)\sum_{n=0}^{\infty}\mathcal{S}_n(t,q)x^n.$$

We now replace t by tq^m on both sides of (4.5) and compare coefficients of x^n to obtain:

Theorem 3: *For* $n \geq 0,\ m > 0$,

$$\begin{aligned}\mathcal{S}_{n+m}(t,q) = {} & \mathcal{S}_m(t,q)\mathcal{S}_{n+1}(tq^{m-1},q) \\ & + tq^{m-1}\mathcal{S}_{m-1}(t,q)\mathcal{S}_n(tq^m,q).\end{aligned} \tag{4.4.6}$$

Consequently (4.6) is the desired generalization of (2.3). Note that the new variable t was an essential addition. If it had not been included, it would not have been possible to transform the factor

$$(1 - x - tx^2q^m\eta)$$

into

$$(1 - x - tx^2\eta),$$

which formed the essential step in obtaining (4.6).

Results essentially equivalent to Theorem 3 were first proved by MacMahon [15; pp. 43-46]. Related investigations were given by Andrews, Knopfmacher and Paule [5], Berkovich and Paule [7], and Garrett [9].

5. EXTENDING (2.4) IS BIZARRE

We have chosen to obtain a q-analog of (2.4) precisely because there is currently no q-analog of (2.4) in the literature. By using the operator generating function technique we will see precisely how complications arise as we uncover the q-analog.

To this end we require some definitions and another lemma. In the definitions we employ the idea of a partition and its conjugate. A partition is simply a representation of an integer as a sum of positive integers. For example, the partitions of 4 are 4, $3+1,\ 2+2,\ 2+1+1$ and $1+1+1+1$. Each partition has a Ferrers graph which is left justified array of rows of dots where in the j^{th} row there are λ_j dots where λ_j is the j^{th} part of the partition. Thus the Ferrers graph of $5+5+3+2+1+1$ is

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \\ \bullet & & & & \\ \bullet & & & & \end{array}$$

The conjugate partition is obtained by reading columns instead of rows. So in this instance the conjugate to $5+5+3+2+2+1$ is $6+4+3+2+2$.

Definition 5.5.1: By $C_+(n)$ we denote the set of the conjugates of all the partitions with distinct parts each $\leq n$.

So $C_+(3) = \{3+2+1, 2+2+1, 2+1+1, 2+1, 1+1+1, 1+1, 1, \phi\}$ because these are the conjugates of $3+2+1, 3+2, 3+1, 2+1, 3, 2, 1, \phi$, the partitions with distinct parts each ≤ 3.

A moment's reflection reveals that $C_+(n)$ consists of all partitions with at most n parts in which each integer not exceeding the largest part appears at least once.

Definition 5.5.2: By $C_0(n)$ we denote the set of partitions in $C_+(n)$ to which have been added 0's so that there are exactly $n+1$ parts in each partition.

Thus $C_0(3) = \{3+2+1+0, 2+2+1+0, 2+1+1+0, 2+1+0+0, 1+1+1+0, 1+1+0+0, 1+0+0+0, 0+0+0+0\}$.

Lemma 2:

$$\eta^r(f(x)(A+\eta))^m g(x)$$

$$= \sum_{\substack{\pi \in C_0(m) \\ \pi=\{\pi_1,\pi_2,\cdots,\pi_{m+1}\} \\ \pi_0 \geq \pi_1 \geq \cdots \geq \pi_m = 0}} A^{m-\pi_0} g(xq^{\pi_0+r}) f(xq^{\pi_1+r}) \cdots f(xq^{\pi_m+r})$$

Proof: When $m=0$, the Lemma asserts $g(xq^r) = q(xq^r)$. Now assume that the result is true for a particular m. Then

$$\begin{aligned}
&\eta^r(f(x)(A+\eta))^{m+1} g(x) \\
&\quad = \eta^r f(x)(A+\eta)(f(x)(A+\eta))^m g(x) \\
&\quad = f(xq^r)(A\eta^r + \eta^{r+1})(f(x)(A+\eta))^m g(x) \\
&\quad = f(xq^r) \sum_{\substack{\pi \in C_0(m) \\ \pi=\{\pi_0,\pi_1,\cdots,\pi_m\} \\ \pi_0 \geq \cdots \geq \pi_m = 0}} A^{m+1-\pi_0} g(xq^{\pi_0+r}) f(xq^{\pi_1+r}) \cdots f(xq^{\pi_m+r}) \\
&\quad + f(xq^r) \sum_{\substack{\pi \in C_0(m) \\ \pi=\{\pi_0,\pi_1,\cdots,\pi_m\} \\ \pi_0 \geq \pi_1 \geq \cdots \geq \pi_m = 0}} A^{(m+1)-(\pi_0+1)} g(xq^{\pi_0+1+r}) f(xq^{\pi_1+1+r}) \cdots f(xq^{\pi_m+1+r}) \\
&\quad = \sum_{\substack{\pi \in C_0(m+1) \\ \pi=\{\pi_0,\pi_1,\cdots,\pi_{m+1}\} \\ \pi_0 \geq \pi_1 \geq \cdots \geq \pi_{m+1} = 0}} A^{m+1-\pi_0} g(xq^{\pi_0+r}) f(xq^{\pi_1+r}) \cdots f(xq^{\pi_{m+1}+r}),
\end{aligned}$$

where the first sum in the penultimate line covers those elements of $C_0(m+1)$ where $\pi_m = 0$ and the second sum treats those elements where $\pi_m > 0$ in which case $\pi_m = 1$ (see comment following Definition 5.1).

The Lemma follows by mathematical induction. $\square$

Before we state Theorem 4, we require some further observations about $C_0(n)$. We note that $C_0(n)$ consists of sequences

$$\pi_0 \geq \pi_1 \geq \cdots \geq \pi_n = 0$$

where $\pi_{i-1} = \pi_i$ or $\pi_i + 1$. So if we write

$$\pi_{i-1} = \pi_i + b_i \quad (b_i = 0 \text{ or } 1, \text{ for } 1 \leq i \leq n),$$

then clearly

$$\pi_{i-1} = b_i + b_{i+1} + \cdots + b_n.$$

We are now prepared to establish a very unlikely q-analog of (2.4).

Theorem 4: *For $n > 0$*

$$\begin{aligned}\mathcal{S}_n(q) = {} & \frac{n}{2^{n-1}} \\ & + \frac{1}{2^{n-1}} \sum_{\substack{1 \leq R_1 < \cdots < R_m \leq n-1 \\ R_{j+1} \geq R_j + 2 \text{ for } 1 \leq j \leq m-1}} R_1(R_2 - R_1 - 1)(R_3 - R_2 - 1) \cdots (R_m - R_{m-1} - 1) \\ & \times \prod_{1 \leq i \leq m} (1 + 4q^{R_i}).\end{aligned}$$

Remark: *It is not difficult to see that as $q \to 1$ this result converges to (2.4). Clearly the inner product becomes 5^m, and for $m > 1$ it is an exercise in mathematical induction to show that for $m > 0$*

$$\sum_{\substack{1 \leq R_1 < \cdots < R_m \leq n-1 \\ R_{j+1} \geq R_j + 2 \text{ for } 1 \leq j \leq m-1}} R_1(R_2 - R_1 - 1)(R_3 - R_2 - 1) \cdots (R_m - R_{m-1} - 1)(n - 1 - R_m) = \binom{n}{2m+1}.$$

Indeed, to prove this by mathematical induction on m, it is easiest to prove it simultaneously with

$$\sum_{\substack{1 \leq R_1 < \cdots < R_m \leq n-1 \\ R_{j+1} \geq R_j + 2 \text{ for } 1 \leq j \leq m-1}} R_1(R_2 - R_1 - 1)(R_3 - R_2 - 1) \cdots (R_m - R_{m-1} - 1) = \binom{n}{2m}.$$

Proof: Returning to the proof of Theorem 1, we recall

$$(1 - x - x^2\eta)\sigma(x) = x.$$

Hence

$$\left(\left(1-\frac{x}{2}\right)^2 - x^2\left(\frac{1}{4}+\eta\right)\right)\sigma(x) = x$$

or

$$\left(1-\left(\frac{x}{1-\frac{x}{2}}\right)^2\left(\frac{1}{4}+\eta\right)\right)\sigma(x) = \frac{x}{(1-\frac{x}{2})^2}.$$

Consequently

$$\sum_{n\geq 0}\mathcal{S}_n(q)x^n = \frac{1}{\left(1-\left(\frac{x}{1-\frac{x}{2}}\right)^2\left(\frac{1}{4}+\eta\right)\right)}\frac{x}{(1-\frac{x}{2})^2},$$

which is precisely the q-analog of what was done in the proof of (2.4).

Hence

$$\begin{aligned}
&\sum_{n\geq 0}\mathcal{S}_n(q)x^n \\
&= \sum_{m=0}^{\infty}\left(\frac{x^2}{(1-\frac{x}{2})^2}\left(\frac{1}{4}+\eta\right)\right)^m \frac{x}{(1-\frac{x}{2})^2} \\
&= \frac{x}{(1-\frac{x}{2})^2} + \sum_{m=1}^{\infty}\sum_{\substack{\pi\in C_0(m)\\ \pi=\{\pi_1,\cdots,\pi_{m+1}\}\\ \pi_1\geq\cdots\geq\pi_{m+1}=0}}\left(\frac{1}{4}\right)^{m-\pi_1}\frac{xq^{\pi_1}}{(1-\frac{xq^{\pi_1}}{2})^2}\frac{x^{2m}q^{2(\pi_2+\cdots+\pi_{m+1})}}{\left(1-\frac{xq^{\pi_1}}{2}\right)^2\cdots\left(1-\frac{xq^{\pi_{m+1}}}{2}\right)^2} \\
&= \sum_{n=0}^{\infty}\frac{nx^n}{2^{n-1}} + \sum_{m=1}^{\infty}\sum_{\substack{\pi\in C_0(m)\\ \pi=\{\pi_1,\cdots,\pi_{m+1}\}\\ \pi_1\geq\cdots\geq\pi_{m+1}=0}}\left(\frac{1}{4}\right)^{m-\pi_1} x^{2m+1}q^{\pi_1+2(\pi_2+\cdots+\pi_{m+1})} \\
&\quad\times\sum_{r_1,r_2,\cdots,r_{m+1}\geq 0}(r_1+1)\cdots(r_{m+1}+1)\left(\frac{x}{2}\right)^{r_1+r_2+\cdots+r_{m+1}} q^{r_1\pi_1+\cdots+r_{m+1}\pi_{m+1}}.
\end{aligned}$$

Therefore

$$\mathcal{S}_n(q) = \frac{n}{2^{n-1}} + \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\substack{r_1,\cdots,r_{m+1}\geq 0 \\ r_1+r_2+\cdots+r_{m+1}\leq n-2m-1}} 2^{1-n}(r_1+1)(r_2+1)\cdots(r_{m+1}+1)$$

$$\sum_{\substack{\pi\in C_0(m)\pi=\{\pi_1,\cdots,\pi_{m+1}\} \\ \pi_1\geq\cdots\geq\pi_{m+1}=0}} 4^{\pi_1} q^{(r_1+1)\pi_1+(r_2+2)\pi_2+\cdots+(r_{m+1}+2)\pi_{m+1}}$$

$$= \frac{n}{2^{n-1}} + \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\substack{r_1,\cdots,r_m\geq 0 \\ r_1+\cdots+r_m\leq n-2m}} 2^{1-n}(r_1+1)(r_2+1)\cdots(r_m+1)$$

$$\times (n-2m-r_1-\cdots-r_m)$$

$$\times \sum_{\substack{\pi\in C_0(m) \\ \pi=\{\pi_1,\cdots,\pi_{m+1}\} \\ \pi_1\geq\cdots\geq\pi_{m+1}=0}} 4^{\pi_1} q^{(r_1+1)\pi_1+(r_2+2)\pi_2+(r_3+2)\pi_3+\cdots+(r_m+1)\pi_m}.$$

Now given the alternative way of describing the partitions in $C_0(n)$ presented just prior to the statement of Theorem 4, we see that the interior sum above may be rewritten as

$$\sum_{0\leq b_1,\cdots,b_m\leq 1} 4^{b_1+b_2+\cdots+b_m} q^{(r_1+1)b_1+(r_1+r_2+3)b_2+\cdots+(r_1+r_2+\cdots+r_m+2m-1)b_m}.$$

So we now change variables by

$$R_1 = r_1 + 1$$

$$R_{j+1} = R_j + r_{j+1} + 2$$

and the interior sums now become

$$\sum_{\substack{1\leq R_1<\cdots<R_m\leq n-1 \\ R_{j+1}\geq R_j+2 \text{ for } 1\leq j\leq m-1}} R_1(R_2-R_1-1)(R_3-R_2-1)\cdots(R_m-R_{m-1}-1)(n-1-R_m),$$

and so we conclude

$$\sum_{n\geq 0} \mathcal{S}_n(q)x^n = \frac{n}{2^{n-1}}$$

$$+ \sum_{\substack{1\leq R_1<\cdots<R_m\leq n-1 \\ R_{j+1}\geq R_j+2 \text{ for } 1\leq j\leq m-1}} R_1(R_2-R_1-1)\cdots(R_m-R_{m-1}-1)(n-1-R_m)$$

$$\times \prod_{i=1}^{m}(1+q^{4R_i}),$$

which is the desired result. □

6. CONCLUSION

I hope that Theorem 1 (the extension of (2.1)), Corollary 1 (the extension of the first line of (1.11)), Theorem 2 (the extension of (2.2)), Theorem 3 (the extension of (2.3)), and Theorem 4 (the very unlikely extension of (2.4)) have shown the richness and the power of using operator methods to study Schur's polynomials as extensions (or q-analogs) of Fibonacci numbers.

There are other identities which do not seem to lend themselves to our operator methods. For example, it is not too difficult to generalize

$$\sum_{j=0}^{n} \binom{n}{j} F_{n-j} = F_{2n} \tag{6.6.1}$$

to

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} t^j q^{j(n-1)} \mathcal{S}_{n-j}(t,q) = \mathcal{S}_{2n}(t,q). \tag{6.6.2}$$

Indeed the combinatorics of (6.2) were discussed (at least when $t = 1$) in [3; p. 144]. However, it would be of interest to prove (6.2) using the operator methods presented here. The method of Church and Bicknell [8] (cf. [14; pp. 232-234]) would seem to suggest a natural method to emulate, but, so far, this path has failed for me.

In furtherance of such a project, I would note that one may prove in the same way Corollary 2 is proved that

$$\mathcal{S}_n(t,q) = \sum_{0 \le 2j \le n-1} q^{j^2} t^j \begin{bmatrix} n-1-j \\ j \end{bmatrix}. \tag{6.6.3}$$

¿From (6.3) one may prove (6.2) by comparing coefficients of t^j on both sides and invoking the q-analog of the Chu-Vandermonde summation [4; eq. (3.3.10), p. 37].

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