THE NUMBER OF DIFFERENT PARTS IN THE PARTITIONS OF n

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ABSTRACT. Consider the partitions of n. Each partition contains some number of different parts. We study the statistical distribution of the number of different parts across all the partitions of n.

1. Introduction

Consider the partitions of n. Each partition contains some number of different parts. We study the statistical distribution of the number of different parts across all the partitions of n.

We will see that the distribution is roughly normal with mean and variance given by

$$\mu \approx \frac{\sqrt{6n}}{\pi} - \left(\frac{1}{2} - \frac{3}{\pi^2}\right) \text{ and } \sigma^2 \approx \frac{3}{\pi^2}\left(\frac{\pi}{\sqrt{6}} - \frac{\sqrt{6}}{\pi}\right)\sqrt{n}.$$

2. Exact calculations

If X denotes the number of different parts in a partition of $n \geq 1$, $p_{m,n}$ the number of partitions with $X = m$, f_m the relative frequency with which $X = m$, then

$$f_m = \frac{p_{m,n}}{p(n)}$$

and

$$E(X) = \mu = \sum_{m\geq 1} m f_m = \sum_{m\geq 1} m \frac{p_{m,n}}{p(n)} = \frac{1}{p(n)} \sum_{m\geq 1} m p_{m,n},$$

$$E(X^2) = \sum_{m\geq 1} m^2 f_m = \sum_{m\geq 1} m^2 \frac{p_{m,n}}{p(n)} = \frac{1}{p(n)} \sum_{m\geq 1} m^2 p_{m,n}$$

and, of course,

$$\sigma^2 = E(X^2) - E(X)^2.$$

(All this is very straightforward.)

We now show that

$$E(X) = \frac{p(n-1) + \cdots + p(0)}{p(n)}$$

and

$$E(X^2) = \frac{p(n-1) + p(n-2) + 3p(n-3) + 3p(n-4) + 5p(n-5) + \cdots}{p(n)}.$$

We start with the observation that

$$1+\sum_{\substack{m\geq 1\\ n\geq 1}} p_{m,n}a^m q^n = \prod_{n\geq 1}\left(1+a\left(q^n+q^{2n}+\cdots\right)\right)$$
$$=\prod_{n\geq 1}\left(1+\frac{aq^n}{1-q^n}\right)$$
$$=\prod_{n\geq 1}\left(\frac{1+(a-1)q^n}{1-q^n}\right).$$

It follows by differentiation with respect to a that

$$\sum_{\substack{m\geq 1\\ n\geq 1}} mp_{m,n}a^{m-1}q^n = \prod_{n\geq 1}\left(\frac{1+(a-1)q^n}{1-q^n}\right)\sum_{n\geq 1}\frac{q^n}{1+(a-1)q^n}$$

and

$$\sum_{\substack{m\geq 2\\ n\geq 1}} m(m-1)p_{m,n}a^{m-2}q^n = \prod_{n\geq 1}\left(\frac{1+(a-1)q^n}{1-q^n}\right)$$
$$\times\left\{\left(\sum_{n\geq 1}\frac{q^n}{1+(a-1)q^n}\right)^2-\sum_{n\geq 1}\frac{q^{2n}}{(1+(a-1)q^n)^2}\right\}$$

and hence,

$$\sum_{\substack{m\geq 1\\ n\geq 1}} m^2p_{m,n}a^{m-1}q^n = \prod_{n\geq 1}\left(\frac{1+(a-1)q^n}{1-q^n}\right)$$
$$\times\left\{a\left(\sum_{n\geq 1}\frac{q^n}{1+(a-1)q^n}\right)^2-a\sum_{n\geq 1}\frac{q^{2n}}{1+(a-1)q^n}+\sum_{n\geq 1}\frac{q^n}{1+(a-1)q^n}\right\}.$$

If we now set $a=1$, we obtain

$$\sum_{\substack{m\geq 1\\ n\geq 1}} mp_{m,n}q^n = \prod_{n\geq 1}\frac{1}{1-q^n}\sum_{n\geq 1}q^n=\frac{q}{1-q}\sum_{n\geq 0}p(n)q^n$$
$$=\sum_{n\geq 1}\left(p(n-1)+\cdots+p(0)\right)q^n$$

and

$$\begin{aligned}
\sum_{\substack{m\geq 1\\ n\geq 1}} m^2 p_{m,n} q^n &= \prod_{n\geq 1} \frac{1}{1-q^n} \left\{ \left(\sum_{n\geq 1} q^n \right)^2 - \sum_{n\geq 1} q^{2n} + \sum_{n\geq 1} q^n \right\} \\
&= \left\{ \frac{q^2}{(1-q)^2} + \frac{q}{1-q} - \frac{q^2}{1-q^2} \right\} \sum_{n\geq 0} p(n) q^n \\
&= \left\{ \frac{q^2}{(1-q)^2} + \frac{q}{1-q^2} \right\} \sum_{n\geq 0} p(n) q^n \\
&= \sum_{n\geq 2} \left(p(n-2) + 2p(n-3) + 3p(n-4) + \cdots \right) q^n \\
&\quad + \sum_{n\geq 1} \left(p(n-1) + p(n-3) + p(n-5) + \cdots \right) q^n \\
&= \sum_{n\geq 1} \left(p(n-1) + p(n-2) + 3p(n-3) + 3p(n-4) + 5p(n-5) + \cdots \right) q^n.
\end{aligned}$$

The two stated results are immediate.

Note that the mean number of different parts in the partitions of n is precisely the same as the mean number of 1's in the partitions of n, a rather remarkable fact! To drive this point home, consider the following table.

partition of 4	number of different parts	number of 1's
4	1	0
$3+1$	2	1
$2+2$	1	0
$2+1+1$	2	2
$1+1+1+1$	1	4
total	7	7
mean	$\frac{7}{5}$	$\frac{7}{5}$

3. Approximate Calculations

We show that

$$\mu \approx \frac{\sqrt{6n}}{\pi} - \left(\frac{1}{2} - \frac{3}{\pi^2} \right), \quad \sigma^2 \approx \frac{3}{\pi^2} \left(\frac{\pi}{\sqrt{6}} - \frac{\sqrt{6}}{\pi} \right) \sqrt{n}.$$

We begin with the approximation

$$p(n) \approx \frac{\exp\{K\sqrt{n}\}}{4n\sqrt{3}} \left(1 - \left(\frac{1}{K} + \frac{K}{48} \right) \frac{1}{\sqrt{n}} \right)$$

where

$$K = \pi\sqrt{\frac{2}{3}},$$

(which can be derived from the Hardy–Ramanujan–Rademacher–Selberg formula for $p(n)$).

We have by the trapezoidal rule,

$$\begin{aligned}
&\frac{1}{2}p(n)+p(n-1)+\cdots+p(0)\\
&\approx \int_1^n \frac{\exp\{K\sqrt{x}\}}{4x\sqrt{3}}-\left(\frac{1}{K}+\frac{K}{48}\right)\frac{\exp\{K\sqrt{x}\}}{4x^{\frac{3}{2}}\sqrt{3}}\,dx\\
&\approx \frac{1}{4\sqrt{3}}\int_1^n \frac{1}{\sqrt{x}}\cdot\frac{\exp\{K\sqrt{x}\}}{\sqrt{x}}\,dx-\left(\frac{1}{K}+\frac{K}{48}\right)\cdot\frac{1}{4\sqrt{3}}\int_1^n\frac{\exp\{K\sqrt{x}\}}{x^{\frac{3}{2}}}\,dx\\
&\approx \frac{1}{4\sqrt{3}}\left\{\frac{2}{K}\frac{\exp\{K\sqrt{n}\}}{\sqrt{n}}+\frac{1}{K}\int_1^n\frac{\exp\{K\sqrt{x}\}}{x^{\frac{3}{2}}}\,dx\right\}\\
&\quad-\left(\frac{1}{k}+\frac{K}{48}\right)\frac{1}{4\sqrt{3}}\int_1^n\frac{\exp\{K\sqrt{x}\}}{x^{\frac{3}{2}}}\,dx\\
&\approx \frac{1}{2K\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{\sqrt{n}}-\frac{K}{192\sqrt{3}}\int_1^n\frac{1}{x}\cdot\frac{\exp\{K\sqrt{x}\}}{\sqrt{x}}\,dx\\
&\approx \frac{1}{2K\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{\sqrt{n}}-\frac{K}{192\sqrt{3}}\left\{\frac{2}{K}\frac{\exp\{K\sqrt{n}\}}{n}+\frac{2}{K}\int_1^n\frac{\exp\{K\sqrt{x}\}}{x^2}\,dx\right\}\\
&\approx \frac{1}{2K\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{\sqrt{n}}-\frac{1}{96\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{n}+\frac{1}{2}\cdot\frac{\exp\{K\sqrt{n}\}}{4n\sqrt{3}}\\
&\approx \frac{1}{2K\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{\sqrt{n}}+\frac{11}{96\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{n}
\end{aligned}$$

and

$$p(n-1)+\cdots+p(0)\approx\frac{1}{2K\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{\sqrt{n}}-\frac{13}{96\sqrt{3}}\frac{\exp\{K\sqrt{n}\}}{n}.$$

This can be written

$$\begin{aligned}
p(n-1)+\cdots+p(0)&\approx\frac{\exp\{K\sqrt{n}\}}{2K\sqrt{3n}}\left(1-\frac{13K}{48}\frac{1}{\sqrt{n}}\right)\\
&\approx\frac{\exp\{K\sqrt{n}\}}{2\pi\sqrt{2n}}\left(1-\frac{13K}{48}\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

while

$$p(n)\approx\frac{\exp\{K\sqrt{n}\}}{4n\sqrt{3}}\left(1-\left(\frac{1}{K}+\frac{K}{48}\right)\frac{1}{\sqrt{n}}\right).$$

It follows that

$$\begin{aligned}
E(X)&\approx\frac{\sqrt{6n}}{\pi}\cdot\frac{1-\frac{13K}{48}\frac{1}{\sqrt{n}}}{1-\left(\frac{1}{K}+\frac{K}{48}\right)\frac{1}{\sqrt{n}}}\\
&\approx\frac{\sqrt{6n}}{\pi}\left(1-\left(\frac{K}{4}-\frac{1}{K}\right)\frac{1}{\sqrt{n}}\right)\\
&\approx\frac{2\sqrt{n}}{K}\left(1-\left(\frac{K}{4}-\frac{1}{K}\right)\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

$$\approx \frac{2\sqrt{n}}{K} - \left(\frac{1}{2} - \frac{2}{K^2}\right)$$
$$\approx \frac{\sqrt{6n}}{\pi} - \left(\frac{1}{2} - \frac{3}{\pi^2}\right),$$

as claimed.

Now let

$$g(n) = p(n-1) + \cdots + p(0).$$

Then

$$g(n) \approx \frac{1}{2K\sqrt{3}} \frac{\exp\{K\sqrt{n}\}}{\sqrt{n}} - \frac{13}{96\sqrt{3}} \frac{\exp\{K\sqrt{n}\}}{n}.$$

We have, again by the trapezoidal rule,

$$\begin{aligned}
&p(n-1) + p(n-2) + 3p(n-3) + 3p(n-4) + 5p(n-5) + \cdots \\
&= g(n) + 2g(n-2) + 2g(n-4) + 2g(n-6) + \cdots \\
&\approx \int_1^n \frac{1}{2K\sqrt{3}} \frac{\exp\{K\sqrt{x}\}}{\sqrt{x}} - \frac{13}{96\sqrt{3}} \frac{\exp\{K\sqrt{x}\}}{x}\, dx \\
&\approx \frac{1}{2K\sqrt{3}} \frac{2}{K} \exp\left\{K\sqrt{n}\right\} - \frac{13}{96\sqrt{3}} \int_1^n \frac{1}{\sqrt{x}} \cdot \frac{\exp\{K\sqrt{x}\}}{\sqrt{x}}\, dx \\
&\approx \frac{1}{K^2\sqrt{3}} \exp\left\{K\sqrt{n}\right\} - \frac{13}{96\sqrt{3}} \left\{\frac{2}{K} \cdot \frac{\exp\{K\sqrt{n}\}}{\sqrt{n}} + \frac{1}{K} \int_1^n \frac{\exp\{K\sqrt{x}\}}{x^{\frac{3}{2}}}\, dx\right\} \\
&\approx \frac{1}{K^2\sqrt{3}} \exp\left\{K\sqrt{n}\right\} - \frac{13}{48K\sqrt{3}} \frac{\exp\{K\sqrt{n}\}}{\sqrt{n}} \\
&\approx \frac{1}{K^2\sqrt{3}} \exp\left\{K\sqrt{n}\right\} \left(1 - \frac{13K}{48} \frac{1}{\sqrt{n}}\right).
\end{aligned}$$

It follows that

$$E(X^2) \approx \frac{4n}{K^2} \cdot \frac{1 - \frac{13K}{48} \frac{1}{\sqrt{n}}}{1 - \left(\frac{1}{K} + \frac{K}{48}\right) \frac{1}{\sqrt{n}}}$$
$$\approx \frac{4n}{K^2} \left(1 - \left(\frac{K}{4} - \frac{1}{K}\right) \frac{1}{\sqrt{n}}\right)$$
$$\approx \frac{6n}{\pi^2} \left(1 - \left(\frac{K}{4} - \frac{1}{K}\right) \frac{1}{\sqrt{n}}\right).$$

We also have

$$E(X) \approx \frac{\sqrt{6n}}{\pi} \left(1 - \left(\frac{K}{4} - \frac{1}{K}\right) \frac{1}{\sqrt{n}}\right).$$

It follows that

$$E(X)^2 \approx \frac{6n}{\pi^2} \left(1 - 2\left(\frac{K}{4} - \frac{1}{K}\right) \frac{1}{\sqrt{n}}\right),$$

and so

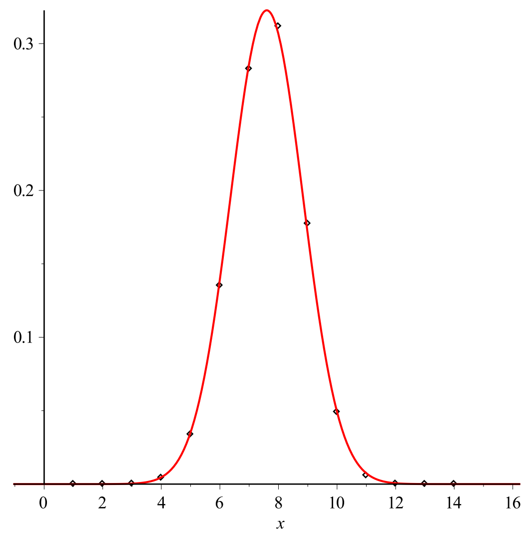
$$\sigma^2 = E(X^2) - E(X)^2 \approx \frac{6n}{\pi^2} \cdot \left(\frac{K}{4} - \frac{1}{K}\right) \frac{1}{\sqrt{n}}$$

$$\approx \frac{3}{\pi^2}\left(\frac{K}{2}-\frac{2}{K}\right)\sqrt{n}$$
$$\approx \frac{3}{\pi^2}\left(\frac{\pi}{\sqrt{6}}-\frac{\sqrt{6}}{\pi}\right)\sqrt{n},$$

again as claimed.

4. Illustration

We illustrate the foregoing with the probability distribution function for $n = 100$ together with the approximating normal $y = \dfrac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$.



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