THE LICHTENBERG SEQUENCE

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Abstract. The discovery of two passages from 1769 by the German Georg Christoph Lichtenberg and the Japanese Yoriyuki Arima, respectively, sheds some new light on the early history of integer sequences and mathematical induction. Both authors deal with the solution of the ancient *Chinese rings* puzzle, where metal rings are moved up and down on a very sophisticated mechanical arrangement. They obtain the number of (necessary) moves to solve it in the presence of n rings. While Lichtenberg considers all moves, Arima concentrates on the down moves only of the first ring. We will present a unified view on integer sequences and discuss some of their most fundamental representatives before collecting properties of the *Lichtenberg sequence* ℓ_n, defined mathematically by the recurrence $\ell_n + \ell_{n-1} = 2^n - 1$, and related sequences such as the *Jacobsthal sequence*, which is the sequence of differences of ℓ. And, of course, at some point Fibonacci numbers will enter the scene.

> Die Mathematik ist eine gar herrliche Wissenschaft,
> aber die Mathematiker taugen oft den Henker nicht.
>
> Georg Christoph Lichtenberg [12, p. 287]

1. Integer Sequences

An *integer sequence* is a mapping

$$a:\ \mathbb{N}_0 \to \mathbb{Z},\ n \mapsto a_n\,(:= a(n))\,.$$

As a convention we define $a_{-n} := 0$ for $n \in \mathbb{N}$. However, there will be *no* $a(-n)$, which means that $a \in \mathbb{Z}^{\mathbb{N}_0}$.

Elementary examples for integer sequences are $\widehat{0}$, defined by $\widehat{0}(n) = 0^n$ (A000007)[1], and 1 with $1(n) = 1$ (A000012). We also define αa for $\alpha \in \mathbb{Z}$ by $(\alpha a)(n) = \alpha a(n)$ and $a + b$ by $(a + b)(n) = a(n) + b(n)$; in particular, 0 denotes the *trivial sequence* $0_n = 0$ (A000004).

The most important statement about integer sequences is the following.

Lemma 1.1. *Let $\alpha \in \mathbb{Z}$, $a, b \in \mathbb{Z}^{\mathbb{N}_0}$. Then*

$$\forall\, n \in \mathbb{N}_0:\ b_n = a_n - \alpha\, a_{n-1} \Leftrightarrow \forall\, n \in \mathbb{N}_0:\ a_n = \sum_{k=0}^{n} \alpha^{n-k}\, b_k\,.$$

Proof. The proof of "$\Rightarrow$" is via a telescoping sum, while "$\Leftarrow$" is trivial. □

If $\alpha = 1$ in Lemma 1.1, then a is the sequence Σb of *partial sums* (or the *integral*) of b and b is the sequence $\overline{a}$ of *differences* (or the *derivative*) of a. As is desirable, we have the *Fundamental Theorem of Integer Sequences*:

$$\Sigma\overline{a} = a = \overline{\Sigma a}\,.$$

As an example let us consider the *identity sequence* (A001477)

$$\mathrm{id}:\ \mathbb{N}_0 \to \mathbb{N}_0,\ n \mapsto n\,, \tag{id}$$

for which $\overline{\mathrm{id}} = 1 - \widehat{0} = 0, 1, 1, \ldots$ (cf. A057427), i.e. it is the *characteristic function* of $\mathbb{N}$ in $\mathbb{N}_0$. For the sequence $\Delta := \Sigma\,\mathrm{id} = 0, 1, 3, 6, 10, 15, 21, \ldots$ (A000217) we obtain, from $\overline{\Delta} = \mathrm{id}$,

[1] Throughout, this refers to [16] as of 2016–06–22.

$$\Delta_n = \Delta_{n-1} + n\,; \tag{Δ}$$

these are the *triangular numbers*, known for ages and somehow related to Carl Friedrich Gauss (1777–1855) (cf. [7, p. 34–36]).

Lemma 1.1 has another useful consequence:

Corollary 1.2. *Let $\alpha, \beta \in \mathbb{Z}$; then*

$$(\beta - \alpha) \sum_{k=0}^{n} \alpha^{n-k}\, \beta^k = \beta^{n+1} - \alpha^{n+1}.$$

Proof. Put $a_k = \beta^{k+1} - \alpha^{k+1}$ and $b_k = (\beta - \alpha)\beta^k$ in Lemma 1.1. □

2. The Dyadic Number System

In his (draft of a) letter[2] dated 1697–01–02 (the "New Year's Letter") to Duke Rudolph August, Prince of Brunswick-Wolfenbüttel, Gottfried Wilhelm Leibni(t)z (1646–1716) presented an explicit list of the first 15 entries of the *dyadic sequence* (A000079) given by

$$D_n = 2^n \tag{D}$$

in both decimal

$$D = 1, 2, 4, 8, 16, 32, 64, \ldots$$

and binary representation

$$D = 1, 10, 100, 1000, 10000, 100000, 1000000 \ldots ;$$

they will be identified with each other by writing $D_n = (10^n)_2$.

Of equal interest is the *Mersenne sequence* (A000225)

$$M := D - 1 = 0, 1, 3, 7, 15, 31, 63, \ldots,$$

named for Marin Mersenne (1588–1648), a member of the order of the Minims. He had tried, mostly in vain, to devise methods to test the primality of *Mersenne numbers*[3] $M_n = D_n - 1 = (1^n)_2$ in order to find (even) *perfect* numbers. The latter are known to be those members of the, also otherwise interesting, sequence[4] $\Delta \circ M$ (A006516) for which M_n is prime (*Euclid-Euler theorem*), namely a *Mersenne prime*; it was the French number theorist Édouard Lucas (1842–1891) who first came up with a satisfactory test; cf. [20].

For technical reasons it is often useful to employ the sequence (A131577)

$$\overline{M} = 0, 1, 2, 4, 8, 16, 32, \ldots,$$

where obviously $\overline{M}_n = D_{n-1} = (\omega^{(n)})_2$, with $\omega^{(0)}$ being the empty (binary) string and $\omega^{(n)} = 10^{n-1}$ for $n \in \mathbb{N}$. Moreover, the sequence (A011782)

$$\overline{D} = 1, 1, 2, 4, 8, 16, 32, \ldots$$

fulfills $\overline{D}_n = M_{n-1} + 1$. Finally, from Corollary 1.2 it follows that $\Sigma D_n = M_{n+1} = D_{n+1} - 1$ and $\Sigma M_n = M_{n+1} - (n+1) = 2M_n - n$, the latter because $M_{n+1} = 2M_n + 1$. The sequence

$$E := \Sigma M = 0, 1, 4, 11, 26, 57, 120, \ldots$$

is called (an) *Eulerian sequence* (cf. A000295) for Leonhard Euler (1707–1783), who in 1755 considered a certain list of polynomials whose coefficients have found a modern combinatorial interpretation as follows. Let S_n be the set of permutations on $[n] := \{1, \ldots, n\}$ and for $\sigma \in S_n$ let $\operatorname{exc}(\sigma) = |\{i \in [n] \mid \sigma_i > i\}|$ ($\in [n]_0 := \{0, \ldots, n-1\}$) denote the number of *excedances* of σ; then $\left\langle {n \atop k} \right\rangle := |\{\sigma \in$

[2] This draft is preserved in the State Library of Lower Saxony in Hanover (LBr II/15); a facsimile can be found between pages 24 and 25 in: R. Loosen, F. Vonessen (eds.), G. W. Leibniz, Zwei Briefe über das binäre Zahlensystem und die chinesische Philosophie, Belser, Stuttgart, 1968.

[3] so named in [14, p. 230].

[4] as usual, $\circ$ stands for the composition of mappings.

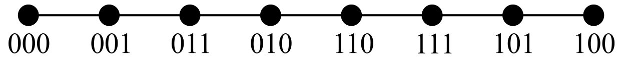

FIGURE 1. The state graph for the three-ring game.

$S_n \mid \mathrm{exc}(\sigma) = k\}|$. This opens a whole new world of *Eulerian numbers*; cf. [17]. Our sequence is then given by $E_n = \left\langle {n+1 \atop 1} \right\rangle$ (see [17, Chapter 1]). Compare this with $\Delta_n = \binom{n+1}{2}$.

3. The Chinese Rings

In the modern mathematical theory of the *Chinese rings* (cf. [7, Chapter 1]), this ancient puzzle, whose (European) tradition can be traced back at least to the beginning of the 16th century (cf. [5]), is modeled by its *state graph* which turns out to be a path graph on 2^n vertices. Here n is the number of rings which are arranged mechanically on or off a bar (or loop). A state of the puzzle, i.e. a vertex of the state graph R^n, is represented by an $s \in \{0,1\}^n$ and written as $s = s_n \dots s_1$, where s_r is 0 or 1 depending on whether ring $r \in [n]$ is off or on the bar, respectively. An edge is then between vertices $s_n \dots s_{r+1} s_r \omega^{(r-1)}$ and $s_n \dots s_{r+1}(1 - s_r)\omega^{(r-1)}$ and represents a move of ring r. See Figure 1 for the example where $n = 3$.

The classical task is to get from 1^n to 0^n (or vice versa), and the fundamental mathematical problem is to find the minimal number of ring moves ℓ_n to achieve this task.

Since 1^n lies on the path from 0^n to 10^{n-1} of length M_n, it is clear from the rules of the game that the following holds (cf. [7, (1.2)], where ℓ is denoted by β):

$$\ell_n + \ell_{n-1} = M_n \,. \tag{L.0}$$

Thus, (recall that by convention $\ell_{-1} = 0$ and consequently $\ell_0 = M_0 = 0$)

$$\ell = 0, 1, 2, 5, 10, 21, 42, 85, 170, 341, \dots$$

is the sequence (A000975). It has been studied as early as 1769 by Georg Christoph Lichtenberg (1742–1799) in [11], so we propose to call it the *Lichtenberg sequence*[5].

4. Lichtenberg's Analysis

The title of Lichtenberg's article, which addressed a more general audience and was therefore written in the local language German, translates as "On the game with elaborately intertwined rings, commonly known as Nürnberger Tand". The latter expression cannot easily be translated, not even into modern German. It seems to have been originally attached to what we now call the Chinese rings[6] and has later been carried over to other toys from the German city which demonstrates its ongoing importance in the toy industry by its annual toy trade fair. Another German name at the time was "Zankeisen", somehow related to the English "tiring irons". The article is written clearly and with an unusual, for the time, mathematical rigor.

After pronouncing his goal of a mathematical examination into the law of times needed to get a certain (positive) quantity of rings down, the author presents, without proof as he admits, four statements which can be summarized in modern terms as

1. Because of the composition of the "machine", a ring can be brought down from the bar only if all other rings before it, except its immediate neighbor, are off the bar. The same applies to the move of a ring onto the bar.
2. The task $1^n \to 0^n$ needs the same amount of time as the task $0^n \to 1^n$, just the procedure is in reverse.[7]

[5] We use small ℓ because capital L is reserved for the *Lucas sequence* (A000032).

[6] This name has not been reported earlier than 1872 in [2], where ℓ_{60} was asked for; hint by A. Heeffer.

[7] Of course, Lichtenberg does not employ our formal notation, but he *does* use the letter n for the number of rings!

3. From statement 1 one deduces that after the removal from the bar of ring n in the task $1^n \to 0^n$ state 010^{n-2} is reached. Thereafter, in order to get ring $n-1$ off the bar, one has to bring back the first $n-2$ rings onto the bar, i.e. one has to pass the state 01^{n-1}.[8]
4. The time needed to solve the task $1^n \to 0^n$ is assumed to be proportional to the number of moves made, i.e. to ℓ_n.

After this preparation, Lichtenberg sets off to calculate ℓ_n without recourse to the extreme state 10^{n-1} (if $n \geq 2$) by deducing, from statement 3 and with explicit reference to statement 2, what we would write as

$$\ell_n = \ell_{n-2} + 1 + \ell_{n-2} + \ell_{n-1}\,;$$

note that this applies only for $n \in \mathbb{N}$. In a lengthy footnote, Lichtenberg explicates what is meant if $n = 1$, when $n-2$ has a "negated" value: he defends statement 1 in this case mathematically by introducing a ring 0 on the bar and a ring -1 off the bar and bearing in mind that $\ell_0 = 0 = \ell_{-1}$. This is in perfect accordance with our convention!

In order to match our definition of an integer sequence having offset 0, we summarize Lichtenberg's recurrence as

$$\ell_0 = 0,\ \ell_{n+1} = \ell_n + 2\ell_{n-1} + 1\,. \tag{L.1}$$

Let us note in passing that by combination of (L.1) and (L.0) we obtain

$$\ell_0 = 0,\ \ell_{n+1} = D_n + \ell_{n-1}\,. \tag{L.2}$$

Lichtenberg establishes a table with the numerical values from ℓ_1 to ℓ_9 and identifies from this list yet another recursive law for his sequence, namely[9]

$$\ell_n = 2\ell_{n-1} + n_0, \tag{L.3}$$

which can easily be established by induction: the case $n = 0$ is clear, and for the induction step we have

$$2\ell_n + (n+1)_0 = \ell_n + \ell_n + 1 - n_0 = \ell_n + 2\ell_{n-1} + 1 = \ell_{n+1},$$

the latter equality following from (L.1).

Finally, Lichtenberg notices from this law (put $\alpha = 2$, $a = \ell$ and $b_n = n_0$ in Lemma 1.1) that if written according to Leibniz's dyadic system his sequence fulfills

$$\ell_n = \sum_{k=0}^{n-1} (n-k)_0 \cdot 2^k, \tag{L.4}$$

i.e. the *Lichtenberg number* ℓ_n is the number with binary (or dyadic) representation of length n and alternating bits; in particular, $(\ell_n)_0 = n_0$. A list of ℓ_1 to ℓ_9 is given in this form and Lichtenberg concludes: "From this results a similarity of this machine to a calculating machine for Leibnitz's dyadic system".

Possibly to impress the reader, the article ends with some numerical examples for the time needed to solve the task. Here the physicist Lichtenberg starts with an apparently empirical value of 11 to 12 minutes needed for (the classical) 9 rings, such that a move takes approximately 2 seconds. He claims that, if one devotes 6 hours per day to the solution, a game with "only" 20 rings would take more than 64 days to accomplish. His calculation for 30 rings (2760 years)[10] is erroneous, however, whereas his statement that for 50 rings one would need "many million years" is, of course, correct.

[8] It would be interesting to know Lichtenberg's argument for this (true) statement, because it contains the minimality of his solution; private communication by Paul K. Stockmeyer, 2016–09–09.

[9] Every $n \in \mathbb{N}_0$ can be identified with the *binary sequence* of its coefficients in base 2 representation. We therefore write n_0 for $n \bmod 2$. Note that $n_0 + (n+1)_0 = 1$.

[10] This error is a bit surprising because the times are essentially doubling with every ring added, such that going from 20 to 30 would roughly result in a factor of 1000, leading to less than 200 years.

5. More Properties of the Lichtenberg Sequence

Since $D_{n-1} \leq \ell_n < D_n$, it follows from (L.4) that ℓ_n is the smallest l greater than ℓ_{n-1} with the property that $\mathrm{d}_1(l, \ell_{n-1}) = n$, where l and ℓ_{n-1} are interpreted as their corresponding (infinite) bit strings and d_1 is the distance function on $\left\{\alpha \in \{0,1\}^{\mathbb{N}_0} \mid \sum_{k=0}^{\infty} \alpha_k < \infty\right\}$ given by $\mathrm{d}_1(\alpha, \beta) = |\{k \in \mathbb{N}_0 \mid \alpha_k \neq \beta_k\}|$. So the Lichtenberg numbers have the same relation to the d_1 (or *Hamming*[11]) metric as the triangular numbers to the canonical one; cf. (Δ).

Explicit formulas for the Lichtenberg numbers are collected in the following proposition.

Proposition 5.1.

$$\ell_n = \left\lceil \frac{2}{3} M_n \right\rceil = \frac{1}{3}\left(M_{n+1} - 1 + n_0\right) = \left\lfloor \frac{2}{3} D_n \right\rfloor . \tag{L.5}$$

Proof. Putting $\alpha = -1$, $a = \ell$ and $b = M$ in Lemma 1.1, we get

$$\ell_n = \sum_{k=0}^{n} (-1)^{n-k} M_k = \sum_{k=0}^{n} (-1)^{n-k} 2^k - \sum_{k=0}^{n} (-1)^{n-k},$$

such that from Corollary 1.2 we arrive at

$$\ell_n = \frac{1}{3}\left(2^{n+1} - (-1)^{n+1}\right) - \frac{1}{2}\left(1 - (-1)^{n+1}\right) = \frac{1}{3}\left(M_{n+1} - 1 + n_0\right).$$

This can also be expressed as

$$\frac{1}{3}\left(2M_n + n_0\right) = \ell_n = \frac{1}{3}\left(2D_n - 2 + n_0\right),$$

from which the other two expressions in (L.5) follow. $\square$

Note in passing that $\ell_{2m} = \frac{2}{3}(4^m - 1)$ is $(10)^m$ in binary representation ("power" m meaning m-fold repetition). Therefore, $.(10)^m = 4^{-m}\ell_{2m} = \frac{2}{3}(1 - 4^{-m})$, so that $\frac{2}{3} = (.101010\ldots)_2$. This fact is noteworthy because $\frac{2}{3}$ was a special fraction in Ancient Egypt (cf. [7, p. 85]). The connection between $\frac{2}{3}$ and ℓ was observed by Paul K. Stockmeyer, who explored the mathematical properties of the sequence further; see [19].

As Lichtenberg mentions, rings go up and down many times during execution of the solution. Let us call λ_n the number of up-moves in the $1^n \to 0^n$ task or, equivalently, the down-moves in $0^n \to 1^n$. (This is related to the *paper folding sequence*; cf. [7, p. 62f].) Then

$$\ell = 2\lambda + \mathrm{id}, \tag{Λ.1}$$

because what goes up must come down and there are n rings down in 0^n which had been up in 1^n. So we have $\ell_{n+1} - \ell_n = 2\overline{\lambda}_{n+1} + 1$, whence from (L.1) we get $\overline{\lambda}_{n+1} = \ell_{n-1}$; moreover, $\overline{\lambda}_0 = \lambda_0 = 0$. But then

$$\lambda_{n+1} = \Sigma\overline{\lambda}_{n+1} = \sum_{k=0}^{n} \overline{\lambda}_{k+1} = \sum_{k=0}^{n} \ell_{k-1} = \Sigma\ell_{n-1}. \tag{Λ.2}$$

The sequence μ describing the number of down-moves in $1^n \to 0^n$ or of up-moves in $0^n \to 1^n$, is (cf. A086445)

$$\mu = \lambda + \mathrm{id} = 0, 1, 2, 4, 7, 13, 24, 46, 89, \ldots;$$

its differences are given by (cf. A005578)

$$\overline{\mu} = 0, 1, 1, 2, 3, 6, 11, 22, 43, \ldots.$$

Here λ has been evaluated from the sequence of partial sums of ℓ which is (cf. A178420)

$$\Sigma\ell = 0, 1, 3, 8, 18, 39, 81, 166, 336, \ldots,$$

[11] cf. the remark of V. Shevelev (2012) in [16, A000975].

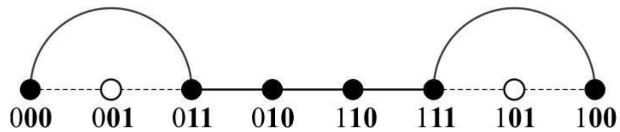

Figure 2. The state graph for the accelerated three-ring game.

because from (Λ.2), (Λ.1) and (L.2) we obtain

$$\Sigma\ell_n = \frac{1}{2}\left(\ell_{n+2} - (n+2)\right) = \frac{1}{2}\left(\ell_n + D_{n+1} - (n+2)\right) = \frac{1}{2}(\ell_n + E_n)\,,$$

i.e. $\Sigma\ell$ is the arithmetic mean of Lichtenberg's and Euler's sequences. Moreover, it fulfills the recurrence

$$\Sigma\ell_0 = 0,\ \Sigma\ell_{n+1} = \Sigma\ell_{n-1} + M_{n+1}\,. \tag{L.2bis}$$

An alternative way to approach the sequence $A := \overline{\mu}$, and consequently the sequence $\mu = \Sigma A$, is obtained from the recurrence

$$A_0 = 0,\ A_1 = 1,\ A_{n+2} = 2A_{n+1} - 1 + n_0, \tag{L.3bis}$$

which follows via (Λ.2) from (L.3). The same recurrence is fulfilled for the number of down moves of ring 1 in the $1^n \to 0^n$ task. This is so because for $n+2$ rings, ring 1 has to move down after *each* move, up or down, of ring 2 and there are A_{n+1} of these going down and $A_{n+1} - 1$ going up; for odd n the first move of ring 1 is an extra move down allowing ring 3 to move down directly without ring 2 being involved. This immediately leads to a practical solution (cf. [7, Proposition 1.6]): ring 1 is transferred in every move whose parity is the same as that of n. (Obviously, in the task(s) $0^n \to 1^n$ (or 10^{n-1}), ring 1 moves in every odd move.)

Actually, A_{n-r+1} is the number of down moves of ring $r \in [n]$. This follows from the fact that the moves of rings 2 to n form an optimal solution for $n-1$ rings and $A_{n-r+1} = A_{(n-1)-(r-1)+1}$.

The smart idea to reduce the question of the length of an optimal solution to the analysis of down moves appears in the book "Shūki Sanpō" (Vol. 2, p. 20r–21r; for a modern transcription, see [21, p. 27f])[12] by the Japanese mathematician Yoriyuki Arima (1714–1783) from 1766/9 (cf. the *Historical note* in [10, p. 679]). The author uses an inductive argument based on the recurrence relation

$$\forall\, n \in \mathbb{N}:\ A_{n+1} = 2A_n - n_0 \tag{A.1}$$

and presents tables for both, A_r and A_{9-r+1} for $r \in [9]$. (The value for A_8 is reproduced incorrectly in [21], but properly in the original.) We therefore call A the *Arima sequence*; see [3].

By induction one can show that $A_n = \frac{1}{3}\left(D_{n-1} + 1 + n_0\right)$ (and therefore $A_{n+1} = A_{n-1} + \overline{D}_{n-1}$) for $n \in \mathbb{N}$. As a consequence,

$$\left\lceil \frac{1}{3}\overline{M} \right\rceil = A = \left\lfloor \frac{1}{3}(\overline{M} + 2) \right\rfloor .$$

It follows that for $n \in \mathbb{N}_0$ the domination number of R^n (cf. [7, p. 270]) is $\gamma(R^n) = A_{n+1} = \ell_{n-1} + 1$, the latter identity being a direct consequence of (Λ.2).

6. The Purkiss Sequence

Playing the Chinese rings one realizes that rings 1 and 2 can be moved simultaneously, either both up or both down. If this is counted as *one* move (or rather "movement" to distinguish it from the normal rules), then we get what has been called the *accelerated run*; see [7, p. 56]. This counting was already employed by Cardano in the middle of the 16th century; cf. [5]. Looking for a *shortest* path, we may delete all vertices of the state graph modeling this situation ending in 01, because they subdivide those edges which correspond to the simultaneous movements of rings 1 and 2; cf. Figure 2. The result is again a path graph joining 0^n with 10^{n-1}, but now of length $\overline{D}_{n-1} + M_{n-1}$ (cf. [7, p. 269]; the cases

[12] I thank Osanobu Yamada (Kusatsu) and Steffen Döll (Hamburg) for their support in translating this passage.

$n = 0$ and $n = 1$ are also covered by our formula, for $n \geq 2$ the value is $3 \cdot 2^{n-2} - 1$), so that for the number p_n of move(ment)s to get from 1^n to 0^n or vice versa, we now have

$$p_n + p_{n-1} = \overline{D}_{n-1} + M_{n-1}\,. \tag{P.0}$$

From this we get $\pi_n + \pi_{n-1} = 1$ for the sequence $\pi := \overline{D} - p$, whence $\pi_n = 1 - n_0$ and $p_n = \overline{D}_n - 1 + n_0 = M_{n-1} + n_0$. Therefore, $p = 0, 1, 1, 4, 7, 16, 31, 64, 127, 256, \ldots$, which is essentially (A051049). Note that by (L.5) we have

$$p_{n+1} = 3\ell_{n-1} + 1. \tag{PL.1}$$

Already in 1865, Henry John Purkiss (1842–1865[13]), who calls the Chinese rings, following Wallis, the *complicati annuli* or common ring-puzzle, observed in [18] (cf. the *Historical note* in [10, p. 679]) that (in our notation)

$$(p_0 = 0,)\ p_1 = 1 = p_2, \quad \forall\, n \geq 2:\ p_{n+1} = p_n + 2p_{n-1} + 1, \tag{P.1}$$

which is the same as (L.1), except that now only $p_2 = 1$ move(ment) is necessary for just two rings. As he was considering the case of general n, we propose to name the sequence *Purkiss sequence*. In [13, p. 42], Lucas attributes this sequence to Théodore Parmentier (1821–1910), so that the letter "p" is justified in any case.

Note that for $n \geq 2$ the movements of discs 1 and 2 together correspond precisely to those moves in the normal run where disc 2 changes position, i.e. in the accelerated run disc 2 never walks alone. We therefore have

$$\forall\, n \geq 2:\ p_n = \ell_n - 2A_{n-1} + 1. \tag{PLA}$$

The sequence of partial sums is $\Sigma p_n = M_n - \frac{1}{2}(n - n_0)$ (cf. A173009). The differences are given by $\overline{p}_n = M_{n-1} - \overline{D}_{n-1} + 2n_0$ (cf. A062510).

7. The Jacobsthal Sequence

Even more interesting, and in fact more popular (cf., e.g., [8, 1]), is the sequence which we obtain as the differences in the Lichtenberg sequence (A001045):

$$J := \overline{\ell} = 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \ldots;$$

as can be seen from (PL.1) it can also be written as

$$J_n = \frac{1}{3}\overline{p}_{n+2}\,, \tag{JP.1}$$

or, making use of (L.1) and (PL.1),

$$\ell_n - p_n = J_{n-1}\,. \tag{JP.2}$$

This means that the number of moves we "save" when employing the accelerated counting for n rings is just J_{n-1}. By (L.0) this sequence fulfills

$$J_n + J_{n-1} = \overline{M}_n = D_{n-1}\,. \tag{J.0}$$

It is usually called the *Jacobsthal sequence* for Ernst Jacobsthal (1882–1965), but the relation to the latter's article [9] is a bit vague. In that paper, Jacobsthal considers a special type of *Fibonacci polynomials* f_n given for every $x \in \mathbb{R}$ by the recurrence

$$f_{-1}(x) = 0,\ f_0(x) = 1,\ f_{n+1}(x) = f_n(x) + xf_{n-1}(x).$$

Letting $x = 0, 1, 2$ we get

$$f_{n-1}(0) = \overline{\mathrm{id}}_n,\ f_{n-1}(1) = F_n,\ f_{n-1}(2) = J_n, \tag{IFJ}$$

respectively; here F is, of course, the *Fibonacci sequence* and the last identity follows from

$$J_0 = 0,\ J_1 = 1,\ J_{n+2} = J_{n+1} + 2J_n, \tag{J.1}$$

[13] "who was drowned while bathing in the Cam" (from an obituary in: The Oxford, Cambridge, and Dublin Messenger of Mathematics 3 (1866)).

which in turn is a consequence of (J.0) and (L.2); (L.2) also leads to

$$J_0 = 0,\ J_1 = 1,\ J_{n+2} = J_n + D_n. \tag{J.2}$$

Relations between the Jacobsthal and Lichtenberg sequences can easily be obtained from (L.1)

$$J_0 = 0,\ J_{n+1} = 2\ell_{n-1} + 1 \tag{JL.1}$$

and (L.3)

$$J_n = \ell_{n-1} + n_0, \tag{JL.2}$$

which in turn combine to

$$J_0 = 0,\ J_{n+1} = 2J_n + (-1)^n. \tag{J.3}$$

This means that for $n \geq 2$ the binary representation of J_n has length $n-1$ and bits alternating from left to right, but with the rightmost bit being always 1; in other words, it is an alternating bit string of length $n-2$ followed by a 1:

$$J_n = 1 + \sum_{k=1}^{n-2} (n-1-k)_0 \cdot 2^k. \tag{J.4}$$

Another consequence of (J.3) is

$$J_n = A_{n+1} - 1 + n_0, \text{ or either } A_n = J_{n-1} + n_0, \tag{JA.1}$$

and

$$J_{n+1} = 2A_{n+1} - 1. \tag{JA.2}$$

To see this, put $\widetilde{J}_n = A_{n+1} - 1 + n_0$; then, using (L.3bis),

$$\widetilde{J}_0 = A_1 - 1 = 0,\ \widetilde{J}_{n+1} = A_{n+2} - n_0 = 2A_{n+1} - 1 = 2\widetilde{J}_n + (-1)^n,$$

such that (JA.1) and implicitly (JA.2) follow from (J.3).

The analogue of Proposition 5.1 is

Proposition 7.1.

$$J_n = \left\lceil \frac{1}{3} M_n \right\rceil = \frac{1}{3}\left(M_n + 2n_0\right) = \left\lfloor \frac{1}{3}(D_n + 1) \right\rfloor. \tag{J.5}$$

Proof. By virtue of (J.0) we put $\alpha = -1$, $a = J$ and $b = \overline{M}$ in Lemma 1.1 and use Corollary 1.2. The rest is as in the proof of Proposition 5.1. $\square$

For the sequence of differences we have from (JL.1)

$$\overline{J}_0 = 0,\ \overline{J}_1 = 1,\ \overline{J}_{n+2} = 2J_n.$$

There is an interesting feature in Jacobsthal's paper, namely [9, formula (9)]

$$f_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k, \tag{f}$$

which for $x = 2$ yields

$$J_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} D_k. \tag{J.6}$$

The latter identity can be proved directly by recourse to the Lichtenberg sequence. We define

$$\widetilde{\ell}_{n-1} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} D_{k-1};$$

then

$$\begin{aligned}
\widetilde{\ell}_n + 2\widetilde{\ell}_{n-1} + 1 &= \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} D_{k-1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} D_k \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \left\{ \binom{n+1-k}{k} + \binom{n+1-k}{k-1} \right\} D_{k-1} \\
&= \sum_{k=1}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n+2-k}{k} D_{k-1} \; = \; \widetilde{\ell}_{n+1}.
\end{aligned}$$

Hence, from (L.1),

$$\ell_{n-1} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} D_{k-1} \tag{L.6}$$

and (J.6) follows by virtue of (JL.1).

From (J.6) and (JA.2) we obtain

$$A_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \overline{D}_k. \tag{A.2}$$

Putting $x = 1$ in (f), we can make the connection to the beautiful formula

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}, \tag{F}$$

which has been known implicitly in Indian prosody for ages (cf. [7, p. 14]). Note that this implies

$$A_{n+1} - F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} M_{k-1} = 0, 0, 0, 0, 1, 3, 9, 22, 52, \ldots. \tag{AF}$$

Let us finally mention another beautiful connection between the Fibonacci and the Jacobsthal numbers via *Stern's diatomic sequence* $s = 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, \ldots$ (cf. [15]). The latter is named for Moritz Abraham Stern (1807–1894) and defined, e.g., by the recurrence

$$s_0 = 0,\ s_1 = 1,\ \forall\, n \in \mathbb{N}:\ (s_{2n} = s_n) \wedge (s_{2n+1} = s_n + s_{n+1}). \tag{S}$$

We then have

$$F = s \circ J = s \circ \overline{\ell}\,. \tag{FSJ}$$

This remarkable relation can be deduced from the theory of another famous mathematical game, namely the *Tower of Hanoi*. For details, see [6] and [7, Section 2.4].

As we have shown, the Arima, Purkiss, and Jacobsthal sequences and their derivates can be traced back to Lichtenberg's sequence which therefore can be considered to be the most fundamental among all of them, originating in the Chinese rings and summarized in Table 1.

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[14]The identification of the author is testified, e.g., by [4, p. 57].

name	symbol	definition	OEIS®	initial entries for $n=$														
identity	id	(id)	A001477	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
		$\overline{\text{id}}$	A057427	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
triangular	Δ	$\Sigma\,\text{id}$	A000217	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105
dyadic	D	(D)	A000079	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384
Mersenne	M	$D-1$	A000225	0	1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	16383
		$\overline{M}$	A131577	0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
		$\overline{D}$	A011782	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
Eulerian	E	ΣM	(A000295)	0	1	4	11	26	57	120	247	502	1013	2036	4083	8178	16369	32752
Lichtenberg	ℓ	(L.0)	A000975	0	1	2	5	10	21	42	85	170	341	682	1365	2730	5461	10922
		$\Sigma\ell$	(A178420)	0	1	3	8	18	39	81	166	336	677	1359	2724	5454	10915	21837
Arima	A	(L.3bis)	(A005578)	0	1	1	2	3	6	11	22	43	86	171	342	683	1366	2731
		ΣA	(A086445)	0	1	2	4	7	13	24	46	89	175	346	688	1371	2737	5468
Purkiss	p	(P.0)	(A051049)	0	1	1	4	7	16	31	64	127	256	511	1024	2047	4096	8191
		Σp	(A173009)	0	1	2	6	13	29	60	124	251	507	1018	2042	4089	8185	16376
		$\overline{p}$	(A062510)	0	1	0	3	3	9	15	33	63	129	255	513	1023	2049	4095
Jacobsthal	J	$\overline{\ell}$	A001045	0	1	1	3	5	11	21	43	85	171	341	683	1365	2731	5461
Fibonacci	F	(IFJ)	A000045	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377

TABLE 1. Important integer sequences addressed in the text (An OEIS® entry in brackets means that the offset is shifted.)