THE MARKOFF-FIBONACCI NUMBERS

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Abstract. This article is a survey on results and topics related to the occurrences of the Fibonacci numbers in the Markoff sequence.

1. Introduction

The Markoff equation is given by

$$x^2 + y^2 + z^2 = 3xyz \tag{1.1}$$

in positive integers $x \leq y \leq z$. A Markoff number is any positive integer which is a component of some solution triple to the Markoff equation. The first few Markoff numbers are

$$\mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{13}, 29, \mathbf{34}, \mathbf{89}, 169, 194, \mathbf{233}, 433, \mathbf{610}, \ldots$$

appearing as the maximal coordinates of the Markoff triples

$$(1,1,1), (1,1,2), (1,2,5), (1,5,13), (2,5,29), (1,13,34), (1,34,89),$$

$$(2,29,169)(5,13,194), (1,89,233), (5,29,433), (1,233,610), \ldots.$$

The Fibonacci sequence $\{F_m\}_{m\geq 0}$ starts as $F_0 = 0$, $F_1 = 1$ and satisfies the recurrence $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$. Its first few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots.$$

We call the numbers appearing in both sequences above the Markoff-Fibonacci numbers.

The Markoff equation is one of the most fascinating equations, not only because of the beautiful structure of its set of solution triples, but also because of its astonishing connections to numerous areas (see [16] and the references there).

Cassini's identity for the Fibonacci numbers with even indices states that

$$F_{2n}^2 + 1 = F_{2n-1}F_{2n+1},$$

which gives $(F_{2n+1} - F_{2n-1})^2 = F_{2n-1}F_{2n+1} - 1$. On simplification we get

$$1 + F_{2n-1}^2 + F_{2n+1}^2 = 3F_{2n-1}F_{2n+1} \tag{1.2}$$

and hence $(1, F_{2n-1}, F_{2n+1})$ is a Markoff triple for all $n \geq 0$. Thus all odd indexed Fibonacci numbers are Markoff numbers.

Both the Markoff and the Fibonacci sequences are endowed with a plethora of identities and structure, making the study of the Markoff-Fibonacci numbers especially enjoyable. For instance, (1.2) is an identity for Fibonacci numbers that probably would not appear on most lists of properties for this popular sequence.

2. The Markoff and the Markoff-Rosenberger equations

In this section we present a brief background for the two equations in question. We also present an elementary proof of an old fact on the coefficient 3 in the Markoff equation that does not seem to be available in the literature.

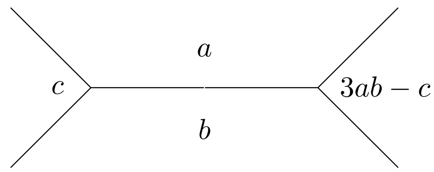

Figure 1. The Markoff tree rule.

2.1. **The Markoff tree.** Markoff showed a simple way to generate all Markoff numbers that may then be represented by a tree. The two triples $(1, 1, 1)$ and $(1, 1, 2)$ are called singular Markoff triples. All other ordered triples (a, b, c) satisfy $a < b < c$, and are called non-singular.

Given a Markoff triple (a, b, c), it is easy to see that there is exactly one other Markoff triple with a and b as two components. To see this, suppose that (a, b, x) is a Markoff triple. Then $a^2 + b^2 + x^2 = 3abx$, which is a quadratic in x, and hence we obtain two solutions, $x = c$ and $x = 3ab - c$. We have thus found a new triple $(a, b, 3ab - c)$. In the same manner, fixing a and c, or b and c, we obtain the triples $(a, c, 3ac - b)$ and $(b, c, 3bc - a)$. Thus, from a non-singular Markoff triple (a, b, c) we obtain three Markoff triples, called the neighbours of (a, b, c). For example, the neighbours of the triple $(1, 2, 5)$ are $(1, 1, 2)$, $(1, 5, 13)$ and $(2, 5, 29)$.

All the non-singular Markoff triples may be represented on a tree, where each vertex denotes a Markoff triple. If two Markoff triples are neighbours, then there is an edge between the two vertices that represent these triples. Given that each non-singular Markoff triple has three neighbours, three edges intersect at any vertex of the tree. Each vertex therefore is the point of intersection of three regions, where the three regions represent the three numbers of the Markoff triple represented by this vertex. Figure 1 shows the two neighbours (a, b, c) and $(a, b, 3ab - c)$. Starting from the Markoff triple $(1, 2, 5)$ we may then construct the Markoff tree (Figure 2) using the rule given in Figure 1.

Observe that the lower branch of the Markoff tree consists of the odd indexed Fibonacci numbers while the upper branch has all the odd indexed Pell numbers.

A well known tantalising claim is the Markoff conjecture.

Conjecture 2.1. *If c is a Markoff number and (a, b, c) and (a', b', c) are two Markoff triples such that $a \le b \le c$ and $a' \le b' \le c$, then $a = a'$ and $b = b'$.*

In other words, given a Markoff number c, there is exactly one ordered triple with maximal element equal to c. On the Markoff tree the conjecture simply says that no Markoff number appears more than once.

2.2. **Why the coefficient 3?** During the problem session of the 19th International Fibonacci Conference, a question arose of whether the number 3 in the Markoff equation could be replaced by any other number. This question has often been asked by admirers of the Markoff equation, and one response points to the article by Hirzebruch and Zagier [7, page 162]. The proof given in [7] while not elementary, does not use any deep theorems. Moreover, the authors therein [7, pages vii and 162] remark that while they digress to give a proof to illustrate their techniques, this result may easily be proved in an elementary fashion using the method of descent. Hurwitz in [8] looks at the general equation $x_1^2 + x_2^2 + \dots x_n^2 = x x_1 x_2 \dots x_n$, and shows that $x x_3 \dots x_n \le n$, which proves that there are no solutions for $n < x$. For the Markoff equation ($n = 3$) this gives us that the coefficient 3 may be replaced only by 1 (we show below that 2 is not possible). Baragar [2, Lemma 0.1] gives a different proof of Hurwitz's result stated above. Both these proofs use the idea of neighbouring solutions. Our proof below uses the method of descent on neighbouring solutions, which we think is worthwhile to record, since

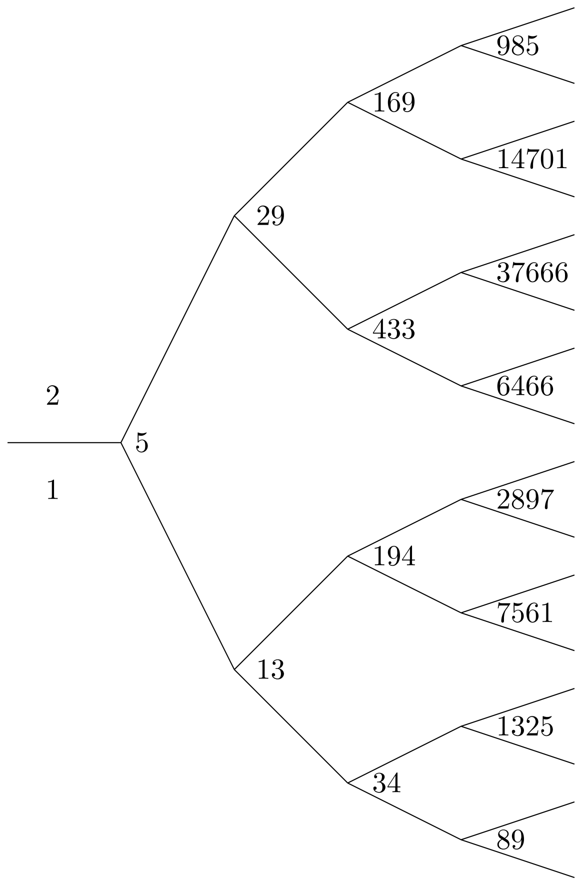

FIGURE 2. The Markoff tree.

it is precisely what Markoff used to generate the tree with all solution triples (as explained by Cassels in [3, pages 27-28]).

Let us consider the equation

$$x^2 + y^2 + z^2 = nxyz \tag{2.1}$$

where n is a positive integer. Suppose that (x, y, z) with $x \leq y \leq z$ is a solution triple. Observe from (2.1) that if $x = y = z$, then $n = 1$ or 3. Otherwise, we have $z > 1$ and $nxyz < 3z^2$ and thus

$$nxy < 3z. \tag{2.2}$$

It is easy to verify (in an identical manner to finding the neighbours of a Markoff triple, as seen in Section 2.1) that $(x, y, nxy - z)$ is also a solution triple. We will now show that $z' = nxy - z < z$. Re-writing (2.1) as

$$x^2 + y^2 = z(nxy - z) \tag{2.3}$$

we see that $z' > 0$. Next, dividing (2.1) by xyz, we have

$$\begin{aligned} n &= \frac{x}{yz} + \frac{y}{xz} + \frac{z}{yx} \\ &< \frac{3}{2} + \frac{z}{yx} \end{aligned}$$

since $\frac{x}{yz} + \frac{y}{xz} < \frac{3}{2}$ (as $z > 1$). It follows that $nxy - z < \frac{3}{2}xy < \frac{9z}{2n}$ using (2.2), and thus if $n > 4$, we have $z' < z$. We have shown above that if we have an ordered solution (x, y, z) with maximal element $z > 1$ and $n > 4$, then the triple $(x, y, nxy - z)$ has maximal element smaller than z. This leads us by descent to the solution $(1, 1, 1)$ and hence $n = 3$. Thus equation (2.1) has no solution for $n > 4$.

In the case when $n = 4$, looking at (2.1) modulo 4, we note that x, y and z are all even. Hence

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = xyz = 8\frac{x}{2}\frac{y}{2}\frac{z}{2}$$

which is not possible, as we have shown above that for $n > 4$ there are no solutions.

In a similar manner we can eliminate the case $n = 2$ and we are left with the cases $n = 1$ or $n = 3$.

If $n = 1$, then looking at (2.1) modulo 3, it is easy to see that x, y and z are divisible by 3. Dividing the equation by 9, we see that $(x/3, y/3, z/3)$ is a solution of the Markoff equation. Conversely, if (x, y, z) is solution of the Markoff equation, then $(3x, 3y, 3z)$ is a solution of (2.1) with $n = 1$. Hence, there is a one to one correspondence between the solutions of (2.1) with $n = 1$ and $n = 3$.

2.3. **The Markoff-Rosenberger equation.** Rosenberger generalized the Markoff equation to

$$ax^2 + by^2 + cz^2 = dxyz \tag{2.4}$$

where a, b and c are positive integers that divide d.

As with the Markoff equation the solution triples form a tree, where each solution triple (x, y, z) has the following three neighbours:

$$\left(x, y, \frac{d}{c}xy - z\right), \ \left(x, \frac{d}{b}xz - y, z\right), \ \left(\frac{d}{a}zy - x, y, z\right). \tag{2.5}$$

Note that as the Markoff-Rosenberger equation lacks the symmetry of the Markoff equation, we cannot permute the three components of a solution triple to get another solution.

Rosenbeger [14] proved that (2.4) has non-trivial solutions if and only if (a, b, c, d) is one of the following:

$$\{(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 4), (1, 2, 3, 6), (1, 1, 2, 2), (1, 1, 5, 5)\}. \tag{2.6}$$

He showed that every solution triple is connected via the neighbour rule (2.5) to a fundamental solution triple. Moreover, Rosenberger also listed these fundamental solutions. The first equation in (2.6) has the fundamental solution $(3, 3, 3)$. The second (the Markoff equation), the third and the fourth equation have the fundamental solution $(1, 1, 1)$. For $(a, b, c, d) = (1, 1, 2, 2)$ the fundamental solution triple is $(2, 2, 2)$, and the last equation given in (2.6) has the fundamental solution triples $(1, 2, 1)$ and $(2, 1, 1)$.

3. Current results

In this section we present the results in the literature on the topic in discussion.

Soon after Luca and Srinivasan published their paper [12], there was a burst of activity in the area, where these ideas were extended and improved, to apply to other sequences and equations. We present below all the results on the topic that we were able to uncover and do hope that there are no omissions.

(1) Luca and Srinivasan [12]: Markov equation with Fibonacci components.
The authors proved that the only Markoff triples with all Fibonacci components are given by $(1, F_{2n-1}, F_{2n+1})$ where $n \geq 0$ is an integer. Using the Binet formula for each component of a Markoff-Fibonacci triple (F_i, F_j, F_n) and the Markoff equation, they showed that $i = 2, 3, 5, 7$. Then for each i using properties of the Markoff and Fibonacci numbers, they were able to show the desired result.

(2) Kafle, Srinivasan and Togbe [9]: Markoff equation with Pell components.
This paper follows closely (1) above.

(3) Rayaguru, Sahukar and Panda [13]: Markov equation with components of some binary recurrence sequences.
The authors here prove independently the same results as in (2) above using similar methods.

(4) Tengely [17]: Markoff-Rosenberger triples with Fibonacci components.
The author lists the finite number of solution triples with all Fibonacci components for the Markoff-Rosenbeger equations other than the Markoff equation. He first bounds i as in (1) above. Then he bounds $n-j$ similarly and examines equation (2.4) for each case given in (2.6), and for each value of i and $n-j$. Using modular considerations and by reducing the equation to a quartic genus 1 curve, he is able to finish his proof using Magma for obtaining integral points on this curve.

(5) Altassan and Luca [1]: Markov type equations with solutions in Lucas sequences.
These authors study equation (2.4) without any conditions on a, b, c, d other than that they are positive integers. They look for solution triples (u_i, u_j, u_n) from the Lucas sequence defined as $u_n = ru_{n-1} + su_{n-2}$, where $u_0 = 0, u_1 = 1$ with $r \geq 1$ and $s = \pm 1$. Their main theorem bounds $i, n-j$ and r. As a result they are able to show that for the Markoff equation the solution triples must be the Markoff-Fibonacci or Markoff-Pell triples. They also show that for the other Markoff-Rosenberger equations there are only a finite number of solution triples. The methods used are elementary relying mainly upon the Binet formula.

(6) Kafle, Srinivasan and Togbe [10]: Markoff-Rosenberger triples with Pell components.
The authors give a proof similar to the one in (1), using only the properties of the Pell numbers, and generalizing a property of the Markoff equation to work for the Markoff-Rosenberger equations.

(7) Hashim, Szalay and Tengely [6]: Markoff-Rosenberger triples and generalized Lucas sequences.
The authors apply the methods in (4) above, to generalized Lucas sequences defined as $u_n = pu_{n-1} - qu_{n-2}$, where $(u_0, u_1) = (0, 1)$ or $(2, p)$, and show that if (u_i, u_j, u_n) is a solution triple to any Markoff-Rosenberger equation (2.4), other than the Markoff equation, then i is bounded above. They then apply this bound to two sequences, the Balancing numbers and the Jacobsthal numbers, and give the finite list of solution triples with all components from the said sequences.

(8) Gómez, Gómez and Luca [5]: Markov triples with k-generalized Fibonacci components.
The authors use properties of the k-generalized Fibonacci sequence to show that there are no other solution triples from this sequence, besides the Markoff-Fibonacci ones.

(9) Luca [11]: Markov triples with two Fibonacci components.
In this delightful paper, the methods used outshine the main result proved. The author shows that there are only a finite number of Markoff triples with two Fibonacci components, other than the Markoff-Fibonacci triples (on the lower branch

of the Markoff tree), and the ones arising as offshoots from these triples, namely, $(5, 13, 194), (13, 34, 1325), (34, 89, 9077), \ldots$. If two components of a Markoff triple are F_m and F_n with $m < n$, then F_n appears k steps down a branch of the Markoff tree at F_m. Using linear forms in logarithms, the author first shows that if m and $n - m$ are sufficiently large, then k is bounded. Next, using the subspace theorem he shows that $k = 1$ or 2, and finally using the Corvaja-Zannier machinery he proves some amazing identities connecting F_m and F_n to conclude the proof.

4. Some open questions

One of the first questions that comes to mind is whether any even indexed Fibonacci number appears as a component of a Markoff triple. While all the odd indexed Fibonacci numbers sit on the lower branch of the Markoff tree, the even indexed ones do not seem to appear anywhere on the tree. As far as the author is aware this question has not been answered in print.

Observe that it follows immediately from the Markoff conjecture that there are no Markoff triples with an odd indexed Fibonacci component, other than $(1, F_{2n-1}, F_{2n+1})$. Recall that the Markoff conjecture states that there are no repeats in the Markoff tree. Hence the odd indexed Fibonacci numbers that appear on the lower branch will not be found anywhere else on the tree.

The Markoff-Fibonacci and the Markoff-Pell triples appear on the lower and upper branch of the Markoff tree. One may look at other branches on the Markoff tree. For example, if we look at the branch at $a = 5$, we get Markoff triples with minimal element 5, such as $(5, 29, 433)$, $(5, 433, 6466)$ and so on. The sequence of Markoff numbers at this branch is

$$\cdots 2897, 194, 13, 29, 433, 6466, \ldots.$$

This sequence a_n has the recurrence relation

$$a_0 = 1, a_1 = 2, a_{n+2} = 15a_{n-1} - a_{n-2}.$$

The question here is whether there are Markoff triples with all components coming from the sequence a_n above, other than $(5, a_n, a_{n+1})$.

This question is clearly not as appealing as our original one on Markoff-Fibonacci triples, as the sequence a_n does not share the same fame as that of the Fibonacci sequence. However, the author feels that the answer could perhaps lead us to new ways of thinking of the Markoff conjecture. Indeed the Markoff conjecture is precisely about looking for repeated occurences on the Markoff tree.

The results on the Markoff-Rosenberger triples with Fibonacci (or Pell) components in the case when $(a, b, c) \neq (1, 1, 1)$, lack the charm of Markoff-Fibonacci triples, in that there are only a finite number of these, while the Markoff-Fibonacci numbers light up a whole (infinite) branch of the tree.

It may be more interesting (as in the case of the Markoff tree) to look at other branches on a Markoff-Rosenberger tree, where we already have an infinite set of triples. For example, for the Markoff-Rosenberger equation corresponding to $(a, b, c, d) = (1, 1, 2, 4)$, the sequence

$$1, 3, 11, 41, 153, \ldots$$

appears on the branch with the third component equal to 1. They arise from the solution triples

$$(1, 1, 1), (1, 3, 1), (11, 3, 1), (11, 41, 1), \ldots$$

which are obtained by using the formula (2.5) alternatively for y and x.

The sequence mentioned above is defined as

$$b_0 = 1, b_1 = 1, b_n = 4b_{n-1} - b_{n-2}$$

and has numerous interesting interpretations that one may read in sequence A001835 in [15]. One such property stated there is that it is the number of ways of packing a $3 \times 2(n-1)$ rectangle with dominoes. Our interpretation here, of the sequence arising from components of solution triples of the Markoff-Rosenberger equation does not appear in A001835 in [15] and it would certainly be interesting to analyse how these different interpretations are connected.

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