FENCE TILING DERIVED IDENTITIES INVOLVING THE METALLONACCI NUMBERS SQUARED OR CUBED

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Abstract. We refer to the generalized Fibonacci sequence $(M_n^{(c)})_{n\geq 0}$, where $M_{n+1}^{(c)} = cM_n^{(c)} + M_{n-1}^{(c)}$ for $n > 0$ with $M_0^{(c)} = 0$, $M_1^{(c)} = 1$, for $c = 1, 2, \dots$ as the c-metallonacci numbers. We consider the tiling of an n-board (an $n \times 1$ rectangular board) with c colours of $1/p \times 1$ tiles (with the shorter sides always aligned horizontally) and $(1/p, 1-1/p)$-fence tiles for $p \in \mathbb{Z}^+$. A (w, g)-fence tile is composed of two $w \times 1$ sub-tiles separated by a $g \times 1$ gap. The number of such tilings equals $(M_{n+1}^{(c)})^p$ and we use this result for the cases $p = 2, 3$ to devise straightforward combinatorial proofs of identities relating the metallonacci numbers squared or cubed to other combinations of metallonacci numbers. Special cases include relations between the Pell numbers cubed and the even Fibonacci numbers. Most of the identities derived here appear to be new.

1. Introduction

The metallic means (also known as the metallic ratios) are defined by

$$\phi^{(c)} = c + \cfrac{1}{c + \cfrac{1}{c + \cfrac{1}{c + \cfrac{1}{c + \ddots}}}} = \frac{c + \sqrt{c^2+4}}{2}, \qquad c = 1, 2, \dots, \tag{1.1}$$

as the $c = 1, 2, 3$ cases are known, respectively, as the golden, silver, and bronze means (or ratios). Since $\lim_{n\to\infty} M_{n+1}^{(c)}/M_n^{(c)} = \phi^{(c)}$, where $M_n^{(c)}$ satisfies the generalized Fibonacci recurrence relation,

$$M_n^{(c)} = cM_{n-1}^{(c)} + M_{n-2}^{(c)} + \delta_{n,1}, \quad M_{n<1}^{(c)} = 0, \tag{1.2}$$

in which $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise (and $\phi^{(c)}$ is the larger root of the recurrence relation auxiliary equation, $\lambda^2 - c\lambda - 1 = 0$), we will refer to the sequence $(M_n^{(c)})_{n\geq 0} = 0, 1, c, c^2+1, \dots$ as the *c-metallonacci numbers*. The c-metallonacci numbers for $c = 1, \dots, 20$ are, respectively, the Fibonacci numbers (F_n), the Pell numbers (P_n), and sequences A006190, A001076, A052918, A005668, A054413, A041025, A099371, A041041, A049666, A041061, A140455, A041085, A154597, A041113, A178765, A041145, A243399, and A041181 in the OEIS [15].

Comparing the identity $F_{jn} = L_j F_{j(n-1)} + (-1)^{j+1} F_{j(n-2)} + F_j \delta_{n,1}$ [7], where L_j is the j-th Lucas number ($L_j = L_{j-1} + L_{j-2} + 2\delta_{j,0} - \delta_{j,1}$, $L_{j<0} = 0$), with (1.2) we see that

$$M_n^{(L_j)} = F_{jn}/F_j, \quad \text{if } j \text{ is odd and positive.} \tag{1.3}$$

We will later use $M_n^{(4)} = \frac{1}{2}F_{3n}$, the $j = 3$ instance of (1.3), in relating the Pell numbers to the even Fibonacci numbers.

There are F_{n+1} ways to tile an n-board using squares and dominoes [5], where an n-board is an $n \times 1$ rectangular board divided into n equal cells numbered 1 to n. If there are c colours of square tiles available, the number of tilings is $M_{n+1}^{(c)}$. To obtain a combinatorial interpretation of $(M_{n+1}^{(c)})^p$ for integer p larger than 1, we tile an n-board using fence tiles in addition to ordinary (1-piece rectangular) tiles. A (w, g)-*fence* is a tile composed of two sub-tiles (called *posts*) of dimensions $w \times 1$ separated by a gap of width g [9].

For $p \in \mathbb{Z}^+$, the number of ways to tile an n-board using $1/p \times 1$ tiles (with the shorter sides always aligned with the long direction of the board and which we will denote by r) and $(1/p, 1-1/p)$-fences (which we will denote by f) is F_{n+1}^p [11]. This result is easy to generalize to the case where there are c colours of r to choose from as we now show in the following lemma and theorem (which are, respectively, special cases of the bijection given in the proof of Theorem 2.2 in [1] and Corollary 2.2 therein). As in [11], we regard each cell as being divided into p equal *slots* which can be filled with either an r or a post.

Lemma 1.1. *There is a bijection between the tilings of an n-board using $1/p \times 1$ tiles (r) of which there are c colours available and $(1/p, 1-1/p)$-fences (f) and the tilings of an ordered p-tuple of n-boards using squares available in c colours and dominoes, where $p, c \in \mathbb{Z}^+$.*

Proof. If an r (left post of an f) occupies the j-th slot of cell k, place a square of the same colour (domino) starting on cell k of the j-th n-board. As the posts of a fence occupy two consecutive j-th slots they correspond to a single domino in the j-th board of the p-tuple. The process is clearly reversible and so the mapping is a bijection. □

Theorem 1.2. *Let B_n be the number of ways to tile an n-board using $1/p \times 1$ tiles which come in c colours and $(1/p, 1-1/p)$-fences, where $p, c \in \mathbb{Z}^+$. Then $B_n = (M_{n+1}^{(c)})^p$.*

Proof. There are $M_{n+1}^{(c)}$ ways to tile an n-board using squares of which there are c different colours and dominoes [5]. From Lemma 1.1, B_n is the same as the number of ways to tile an ordered p-tuple of n-boards using squares of c possible colours and dominoes. □

Although we will not use it in our proofs of identities, for completeness we present the following generalization of Theorem 6.2 in [11] which gives an alternative combinatorial interpretation of $(M_{n+1}^{(c)})^p$. The proof mirrors that of the original theorem; the only difference is that in the bijection, an r of a particular colour corresponds to a $(1/2p, (1-1/p)/2)$-fence of the same colour.

Theorem 1.3. *For $p, c \in \mathbb{Z}^+$, the number of ways to tile an n-board using $(1/2p, (1-1/p)/2)$-fences of which there are c possible colours and $(1/2p, 1-1/2p)$-fences is $(M_{n+1}^{(c)})^p$.*

To obtain identities via combinatorial proof we use the result that all tilings of n-boards can be expressed as tilings using metatiles. A *metatile* is a grouping of tiles that exactly covers an integral number of cells and cannot be split to make smaller metatiles [8]. The concept of a metatile is only useful if the tiles have gaps (as is the case with fences) or are of non-integer length; otherwise the tiles themselves are metatiles.

When tiling with r and f, the simplest types of metatile are a single cell filled with r (which we denote by r^p) and the gapless arrangement of p interlocking fences (f^p) which we refer to as a *bifence* (*trifence*) in the $p = 2$ ($p = 3$) case [10, 11]. Since each r can be any one of c colours, the number of types of r^p metatile is c^p. The lengths of r^p and f^p are 1 and 2, respectively.

We refer to a metatile that contains both r and f as being *mixed* [10]. There is an infinite number of possible mixed metatiles. In the case $p = 2$, there are four infinite families of mixed metatiles and each mixed metatile contains two r, one in the first cell and the other in the final cell of the metatile [10, 12]. The number of mixed metatiles of a given length when there are c possible colours for the r is therefore straightforward to determine as we will show at the start of Section 2 before using the result to obtain various identities involving $(M_n^{(c)})^2$.

In Section 3 we consider the family of Pascal-like triangles whose (n,k)-th entries are the number of tilings of an n-board that use k fences in the $p=2$ case. The entries are used in one of the identities we prove in Section 2 and we identify Riordan arrays the triangles are simply related to.

When tiling with r and f in the $p=3$ case, there are no easily discernible families of mixed metatiles. Enumeration of the number of mixed metatiles of a given length requires a different approach; in [11] we managed to do this (when $c=1$) by considering the possible endings of a mixed metatile and obtaining recursion relations for the numbers of metatiles with a given ending and length. On combining the recursion relations one finds that the total number of mixed metatiles with a given length is a multiple of a Pell number. We apply the same procedure adapted for the more general case of r occurring in different colours in Section 4, and use the resulting expression to derive identities in Section 5.

2. Identities involving the metallonacci numbers squared

In the $p=2$ case, the four families of mixed metatiles expressed symbolically are $r(ff)^j fr$ (e.g., cells 4–5 and 14–17 in Fig. 1), $fr(ff)^j r$ (e.g., cells 6–7 and 18–21), $r(ff)^{j+1}r$ (e.g., cells 8–10), and $fr(ff)^j fr$ (e.g., cells 11–13) where $j \geq 0$ and their respective lengths are $2j+2$, $2j+2$, $2j+3$, and $2j+3$ [10, 12]. Given that each r can be one of c colours this means that there are $2c^2$ mixed metatiles of each length $l = 2, 3, \ldots$.

In some of the proofs we use the concept of a *filled fence* which is a fence with its gap filled by an r and thus has the symbolic representation fr. It is not a metatile by itself since its length is 3/2. There is a filled fence at the start and/or end of all but one of the four families of mixed metatiles.

The identities in this section are generalizations of various identities in [10, 12], reduce to them when $c=1$, and are proved in a similar way.

Lemma 2.1. *For all $n \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$B_n = \delta_{n,0} + c^2 B_{n-1} + (2c^2+1)B_{n-2} + 2c^2 \sum_{l=3}^{n} B_{n-l}, \quad B_{n<0} = 0. \tag{2.1}$$

Proof. Consider tiling an n-board with r and f in the $p=2$ case. As in [5, 9], we condition on the final metatile; if this final metatile is of length l then there are B_{n-l} ways to tile the rest of the board. In the present case there are c^2 metatiles of length 1 (r^2), $2c^2+1$ metatiles of length 2 (i.e, rfr, frr, and f^2), and $2c^2$ metatiles of each length $l \geq 3$. The $\delta_{n,0}$ results from

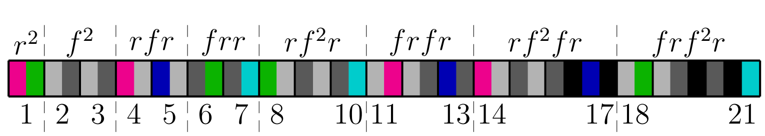

FIGURE 1. A 21-board tiled with all metatiles of length less than 5 (but excluding those differing only by the colour of the r tiles) when $p=2$. Dashed lines indicate the boundaries between metatiles. The symbolic representation is shown above the metatile and the cell numbers below.

the fact that if $l = n$ there is one way in which the metatile fills the board and so we require $B_0 = 1$. $\square$

For the rest of the paper we write M_n rather than $M_n^{(c)}$ for ease of reading.

Identity 2.2. *For all $n \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$M_n^2 = \delta_{n,1} + c^2 M_{n-1}^2 + (2c^2+1)M_{n-2}^2 + 2c^2 \sum_{l=3}^{n-1} M_{n-l}^2.$$

Proof. After replacing n by $n-1$ in (2.1), the identity follows from Theorem 1.2. $\square$

When $c = 1$, Identity 2.2 reduces to Identity 4.1 in [10].

Identity 2.3. *For all $n \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$, $M_n^2 = \delta_{n,1} - \delta_{n,2} + (c^2+1)(M_{n-1}^2 + M_{n-2}^2) - M_{n-3}^2$.*

Proof. Subtracting (2.1) with n replaced by $n-1$ from (2.1), replacing n by $n-1$ in the result, and then using Theorem 1.2 gives the identity. $\square$

Identity 2.4. *For $n \geq 0$ and $c \in \mathbb{Z}^+$,*

$$M_{n+1}^2 - c^{2n} = 2\sum_{k=0}^{n-2}\sum_{i=1}^{k+1}(1 + \delta_{i,k+1}/2c^2)c^{2(n-k-1)}M_i^2. \tag{2.2}$$

Proof. How many ways are there to tile an n-board using at least 1 fence? *Answer 1*: $B_n - c^{2n}$ since this corresponds to all tilings except the all-r tiling. *Answer 2*: condition on the location of the last fence. Suppose this fence lies on cells $k+1$ and $k+2$ ($k = 0, \ldots, n-2$). Then cells $k+3$ up to n must be filled with r and there are $c^{2(n-k-2)}$ ways this can be done. If just a bifence lies on cells $k+1$ and $k+2$ then there are $c^{2(n-k-2)}B_k$ ways to tile the remaining cells. The other possibility is that cell $k+2$ is at the end of a mixed metatile and, as there are $2c^2$ mixed metatiles of length 2 or more, there are then $2c^2(B_{k+2-2} + B_{k+2-3} + \cdots + B_0)c^{2(n-k-2)}$ ways to tile the board. Hence, equating the two answers,

$$B_n - c^{2n} = \sum_{k=0}^{n-2} 2c^{2(n-k-1)}(B_0 + B_1 + \cdots + B_{k-1} + (1 + 1/2c^2)B_k).$$

The identity then follows from Theorem 1.2. $\square$

Identity 2.5. *For $n \geq 0$ and $c \in \mathbb{Z}^+$,*

$$M_{2n+2}^2 = c^2\left(1 + \sum_{k=1}^{n}\left(M_{2k+1}^2 + 2\sum_{i=1}^{2k} M_i^2\right)\right). \tag{2.3}$$

Proof. How many ways are there to tile an $(2n+1)$-board? *Answer 1*: B_{2n+1}. *Answer 2*: an odd-length board must have at least one r and the final r must be on an odd cell since the cells to the right must be filled with bifences which are of length 2. Condition on the location of the last r. Suppose that the last r is in cell $2k+1$ ($k = 0, \ldots, n$). Either it is part of an r^2 metatile in which case there are $c^2 B_{2k}$ ways to tile the board, or, if $k = 1, \ldots, n$, it is part of a mixed metatile and so there are $2c^2(B_{2k+1-2} + B_{2k+1-3} + \cdots + B_0)$ ways to tile the board. Hence

$$B_{2n+1} = \sum_{k=0}^{n} c^2 B_{2k} + 2c^2\sum_{k=1}^{n}(B_0 + B_1 + \cdots + B_{2k-1}).$$

The identity then follows from Theorem 1.2 and the fact that $M_1 = 1$. $\square$

Identity 2.6. *For all $n, c \in \mathbb{Z}^+$, $M_{n+1}^2 = (c^2+2)M_n^2 - M_{n-1}^2 + 2(-1)^n$.*

Proof. The $n = 1$ case can be verified by substituting in the values of M_0, M_1, and M_2. For $n > 1$, we identify a near bijection between (i) the tilings of an n-board and an $(n-2)$-board and (ii) the tilings of c^2+2 $(n-1)$-boards. There is an obvious bijection between the tilings of the n-board that end with r^2 (of which there are $c^2 B_{n-1}$) and the first c^2 $(n-1)$-boards. We let $\mathcal{B}_n$ denote the bijection between the tilings of an n-board that end in a fence and are not the all-bifence tiling and the tilings of an $(n-1)$-board which is not the all-bifence tiling. If the n-board ends in a bifence, find the final r; if it is inside a filled fence, remove the fence, otherwise replace the r and the bifence to the right of it by a filled fence. This gives all r-containing tilings of an $(n-1)$-board that end with a fence. The tilings ending in r are obtained from the n-board tilings ending in a filled fence by removing the fence. This leaves the tilings of the n-board that end with an r (but not r^2). Not counting this final r, find the final r in the tiling (as there must be at least one other r) and then obtain the corresponding $(n-1)$-board by using the same procedure as for n-boards ending in a fence. This generates all tilings of the final $(n-1)$-board ending in a free r. The bijection between the remaining r-containing tilings of this board (i.e., those ending in a fence) and the all r-containing tilings of an $(n-2)$-board is simply $\mathcal{B}_{n-1}$. When n is even, the n and $(n-2)$-boards both have an all-bifence tiling and so $B_n + B_{n-2} = (c^2+2)B_{n-1} + 2$. When n is odd, the second and third $(n-1)$-boards have all-bifence tilings which do not correspond to any of the r-containing tilings of the n or $(n-2)$-boards and so we must subtract 2. Thus overall, $B_n + B_{n-2} = (c^2+2)B_{n-1} + 2(-1)^n$, and the required identity is obtained from using Theorem 1.2. ☐

Identity 2.7. *For $n \geq 0$ and $c \in \mathbb{Z}^+$,*

$$M_{n+1}^2 = \delta_{n \bmod 2,0} + c^2 \sum_{k=1}^{n} k M_{n+1-k}^2.$$

Proof. How many tilings of an n-board contain at least two r? *Answer 1*: B_n when n is odd and $B_n - 1$ when n is even since the all-bifence tiling only occurs for even n. *Answer 2*: following the method introduced in [4], we condition on the location of the second r. As the symbolic representation of all non-bifence metatiles end in r, if the k-th cell in the n-board contains the second r, the symbolic representation of the tiling of the first k cells must end in r. This leaves one r that may be placed anywhere among the $k-1$ fences and so there are $c^2 k$ ways to tile these first k cells. There are B_{n-k} ways to tile the rest of the board. Summing over all possible k gives $\sum_{k=1}^{n} c^2 k B_{n-k}$. After equating this to Answer 1 the identity follows from Theorem 1.2. ☐

To generalize Identity 2.7 we first need the following definition and lemma. Let $H_n^{(q)}$ be the number of tilings of an n-board where the number of r (in this case half-squares) is $2q$. Thus for $n \geq 1$, $H_n^{(0)}$ is 1 if n is even (the all-bifence tiling) and 0 if n is odd. We must have $H_n^{(q)} = 0$ if $n < q$ and for convenience we set $H_0^{(0)} = 1$.

Lemma 2.8. *For $n \geq q > 0$ and $c \in \mathbb{Z}^+$,*

$$H_n^{(q)} = H_{n-2}^{(q)} + c^{2q} \binom{n+q-1}{2q-1}. \tag{2.4}$$

Proof. The symbolic representation of a tiling must end in either r or ff. If it ends in r, we are free to place the remaining $2q-1$ half-squares and $n-q$ fences in any order; this gives $c^{2q}\binom{n+q-1}{2q-1}$ possibilities overall. If it ends in ff, there are $H_{n-2}^{(q)}$ ways to tile the remaining cells. ☐

We will show in Section 3 that $H_n^{(q)}$ is, for $n \geq q \geq 0$, the (n,q)-th entry of a Riordan array.

Identity 2.9. *For $q > 0$, $n \geq q$, and $c \in \mathbb{Z}^+$,*

$$M_{n+1}^2 = \sum_{j=0}^{q-1} H_n^{(j)} + c^{2q} \sum_{k=q}^{n} \binom{k+q-1}{2q-1} M_{n+1-k}^2.$$

Proof. How many tilings of an n-board contain at least $2q$ half-squares? *Answer 1*: the total number of tilings minus the tilings that contain less than $2q$ half-squares, i.e., $B_n - \sum_{j=0}^{q-1} H_n^{(j)}$. *Answer 2*: we condition on the location of the $2q$-th half-square. If it occurs in the k-th cell, the symbolic representation of the tiling up to that cell must end in r. There are $\binom{k+q-1}{2q-1}$ ways to order the remaining $2q-1$ half-squares and $k-q$ fences and B_{n-k} ways to tile the rest of the board. Summing over all possible k and equating the result to Answer 1 gives

$$B_n - \sum_{j=0}^{q-1} H_n^{(j)} = c^{2q} \sum_{k=q}^{n} \binom{k+q-1}{2q-1} B_{n-k},$$

and the identity follows from Theorem 1.2. □

Identity 2.10. *For $n > 3$ and $c \in \mathbb{Z}^+$, $M_{n+1}^2 = c^2 M_n^2 + 2(c^2+1)M_{n-1}^2 + c^2 M_{n-2}^2 - M_{n-3}^2$.*

Proof. For $n > 3$, how many tilings of an n-board are there? *Answer 1*: B_n. *Answer 2*: condition on the end tiles. If the first and last tiles are both r there are B_{n-1} ways to tile the remaining cells in between. If the first tile is an r and the tiling ends with a filled fence or vice versa there are B_{n-2} ways to tile the remaining cells. If the tiling starts and ends with a filled fence, there are B_{n-3} ways to tile the remaining cells. In the cases so far, there is an r at both ends so we must include a factor of c^2 when counting the tilings. The remaining possibility is that the tiling starts or ends with a bifence. In each case this leaves B_{n-2} tilings for the remaining cells. However, we have counted tilings that start and end in a bifence (of which there are B_{n-4}) twice and so we must subtract these to leave, on equating both answers, $B_n = c^2 B_{n-1} + 2(c^2+1)B_{n-2} + c^2 B_{n-3} - B_{n-4}$. The identity follows from Theorem 1.2. □

In the following identity we use the fact that the number of ways to tile an n-board using only r^2 and f^2 is $M_{n+1}^{(c^2)}$ since this is equivalent to tiling an n-board with squares which come in c^2 varieties and dominoes [6, 5]. When $c = 2$, using (1.3), the identity relates P_n^2 to F_{3n}.

Identity 2.11. *For $n \geq 0$ and $c \in \mathbb{Z}^+$,*

$$M_{n+1}^2 = M_{n+1}^{(c^2)} + 2c^2 \sum_{k=0}^{n-2} \sum_{l=2}^{n-k} M_{k+1}^{(c^2)} M_{n+1-k-l}^2.$$

Proof. How many ways are there to tile an n-board using at least 1 mixed metatile? *Answer 1*: $B_n - M_{n+1}^{(c^2)}$ since $M_{n+1}^{(c^2)}$ is the number of ways to tile an n-board without using mixed metatiles. *Answer 2*: condition on the position of the first mixed metatile. If it lies on cells $k+1$ to $k+l$ where $k = 0, \ldots, n-l$ and $l = 2, \ldots, n-k$, there are $2c^2 M_{k+1}^{(c^2)} B_{n-k-l}$ ways to tile the board. Summing over all possible k and l and equating to Answer 1 gives

$$B_n - M_{n+1}^{(c^2)} = 2c^2 \sum_{\substack{k\geq 0,\, l\geq 2,\\ k+l\leq n}} M_{k+1}^{(c^2)} B_{n-k-l}.$$

After re-expressing as a double sum, the identity then follows from Theorem 1.2. □

3. Pascal-like triangles for the $p = 2$ case

As in [8, 12], we form Pascal-like triangles by tabulating $\left[{n \atop k}\right]$, the number of tilings of an n-board that use k fences (see Fig. 2 for the $c = 2$ case). The choice $\left[{0 \atop 0}\right] = 1$ is justified in the proof of Identity 3.7.

Identity 3.1. *For $n \geq 0$ and $c \in \mathbb{Z}^+$,*

$$M_{n+1}^2 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}. \tag{3.1}$$

Proof. The right-hand side of (3.1) is the sum of row n which gives all possible ways to tile an n-board. The result then follows from Theorem 1.2. □

Identity 3.2. *For $n \geq k \geq 0$ and $c \in \mathbb{Z}^+$,*

$$\begin{bmatrix} n \\ k \end{bmatrix} = c^{2(n-k)} \sum_{j=k-m}^{m} \binom{n-j}{j} \binom{n-(k-j)}{k-j},$$

where $m = \min(\lfloor n/2 \rfloor, k)$.

Proof. From Lemma 1.1, $\left[{n \atop k}\right]$ is also the number of square-domino tilings (with c colours of squares to choose from) of an ordered pair of n-boards that use k dominoes in total. The number of ways to tile an n-board with j dominoes (and $n - 2j$ squares that come in c colours) is $c^{n-2j}\binom{n-j}{j}$. If one of the n-boards has j dominoes the other will have $k - j$ dominoes and $n - 2(k - j)$ squares. Hence there are $c^{n-2j}c^{n-2(k-j)}\binom{n-j}{j}\binom{n-(k-j)}{k-j}$ ways to tile the n-boards if the first board has j dominoes. Evidently j cannot exceed k or $\lfloor n/2 \rfloor$ and so $m \geq j \geq k - m$. We then sum over all possible values of j. □

The next three identities can be obtained directly from Identity 3.2. However, the combinatorial proofs we give instead are quick and intuitively appealing.

Identity 3.3. *For $n \geq 0$ and $c \in \mathbb{Z}^+$, $\left[{n \atop 0}\right] = c^{2n}$.*

Proof. This corresponds to the all-r tiling of an n-board. □

Identity 3.4. *For $n \geq 0$ and $c \in \mathbb{Z}^+$, $\left[{n \atop n}\right]$ is 1 if n is even and is 0 otherwise.*

Proof. A bifence is of length 2 (and is composed of 2 fences). Thus the fence-only tiling can only occur when n is even. □

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	1								
1	4	0							
2	16	8	1						
3	64	64	16	0					
4	256	384	176	24	1				
5	1024	2048	1408	384	36	0			
6	4096	10240	9472	3968	736	48	1		
7	16384	49152	57344	32768	9472	1280	64	0	
8	65536	229376	323584	235520	93952	20096	2080	80	1

Figure 2. The start of a Pascal-like triangle with entries $\left[{n \atop k}\right]$ when $c = 2$. It is also the start of the row-reversed $(1/(1 - x^2), 4x/(1 - x)^2)$ Riordan array.

Identity 3.5. *For $n \geq 1$ and $c \in \mathbb{Z}^+$, $\left[{n \atop 1} \right] = 2(n-1)c^{2(n-1)}$.*

Proof. Only the rfr and frr metatiles contain 1 fence. Both metatiles are length 2. There are $n-1$ ways to place a length-2 metatile on an n-board (with the remaining $n-1$ cells occupied by r^2 metatiles). □

Identity 3.6. *For $n \geq q \geq 0$ and $c \in \mathbb{Z}^+$, $\left[{n \atop n-q} \right] = H_n^{(q)}$.*

Proof. The result follows from the definition of $H_n^{(q)}$ since $\left[{n \atop n-q} \right]$ is also the number of tilings containing $2q$ half-squares. □

Identity 3.7. *For $n, k \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \delta_{n,0}\delta_{k,0} - \delta_{n,1}\delta_{k,1} + c^2 \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + c^2 \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}, \tag{3.2}$$

where $\left[{n \atop k} \right] = 0$ if $n < k$ or $k < 0$.

Proof. We condition on the last metatile. If that metatile is of length l and contains j fences, there are $\left[{n-l \atop k-j} \right]$ ways to tile the remaining cells using $k-j$ fences. Considering all possible metatiles gives

$$\begin{bmatrix} n \\ k \end{bmatrix} = \delta_{n,0}\delta_{k,0} + c^2 \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + 2c^2 \sum_{j=1}^{\infty} \begin{bmatrix} n-j-1 \\ k-j \end{bmatrix}. \tag{3.3}$$

If $n = l$ and $k = j$ there is exactly one way to tile the whole board (i.e., by using that single metatile) and so we make $\left[{0 \atop 0} \right] = 1$. Replacing n by $n-1$ and k by $k-1$ in (3.3) and subtracting the result from (3.3) gives (3.2). □

A $(P(x), Q(x))$ *Riordan array* is a lower triangular matrix whose (n,k)-th entry is the coefficient of x^n in the series for $P(x)(Q(x))^k$ where $P(x) = P_0 + P_1 x + P_2 x^2 + \cdots$ and $Q(x) = Q_1 x + Q_2 x^2 + \cdots$ [14, 3]. The k-th column gives the power series for $P(x)(Q(x))^k$ and so a Riordan array can be regarded as a geometric series of generating functions with the first ($k = 0$) column giving the coefficients of the generating function $P(x)$. The n-th element along the leading diagonal is $P_0 Q_1^n$. Some familiar examples of Riordan arrays are the $(1, x)$ Riordan array which is the (infinite dimensional) identity matrix and the $(1/(1-x), x/(1-x))$ Riordan array which is Pascal's triangle.

A *row-reversed* $(P(x), Q(x))$ *Riordan array* has the elements up to and including the main diagonal of each row of the $(P(x), Q(x))$ Riordan array placed in reverse order [2]. As we shall prove shortly, an example of such an array appears in Fig. 2. For a row-reversed array, it is the expansion of $P(x)$ that appears along the main diagonal and the geometric series with common ratio Q_1 starting at P_0 that is the leftmost column. With the obvious exception of Pascal's triangle (due to its symmetry), a row-reversed Riordan array is not, in general, a Riordan array.

In recent work, we have shown that various tiling problems have close links with Riordan arrays or row-reversed Riordan arrays [12, 13, 2]. The following theorem is a restatement of part of Theorem 33 in [2].

Theorem 3.8. *Suppose a triangle is constructed by letting the (n,k)-th entry be the number of ways to tile an n-board that use k fences. The triangle is a row-reversed Riordan array if and only if there is (i) a metatile of length 1 that lacks fences and (ii) for all metatiles $l - s$ is 0 or 1, where l is the length of the metatile and s is the number of fences it contains.*

When tiling with r and f in the $p=2$ case, there is a metatile of length 1 lacking f which is r^2. Each metatile contains either 0 or 2 r and so the total length of the posts of the fences in the metatile is l or $l-1$, respectively. This is also the number of fences (as the total length of the posts of one fence is 1) and hence $l-s$ is always 0 or 1. Thus each member of the family of triangles is a row-reversed Riordan array.

Theorem 3.9. *If $R(n,k)$ is the (n,k)-th entry of the $(1/(1-x^2), c^2x/(1-x)^2)$ Riordan array then for any $c \in \mathbb{Z}^+$,*

$$\begin{bmatrix} n \\ k \end{bmatrix} = R(n, n-k). \tag{3.4}$$

Proof. We have already established that the triangle is a row-reversed $(P(x), Q(x))$ Riordan array and so we just need to find $P(x)$ and $Q(x)$. Following the procedure given in Remark 36 of [2] (or Example 1 in [13]), $P(x)$ is the generating function of the leading diagonal of the triangles which is $1/(1-x^2)$. To find $Q(x)$, letting $\left[{n \atop k}\right]'$ be the (n,k)-th entry of the row-reversed triangle, we row-reverse (3.2) (by replacing $\left[{n-m \atop k-l}\right]$ by $\left[{n-m \atop n-m-k+l}\right]'$ and then replacing k by $n-k$) which gives

$$\begin{bmatrix} n \\ k \end{bmatrix}' = \delta_{n,0}\delta_{k,0} - \delta_{n,1}\delta_{k,0} + c^2 \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}' + \begin{bmatrix} n-1 \\ k \end{bmatrix}' + c^2 \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}' + \begin{bmatrix} n-2 \\ k \end{bmatrix}' - \begin{bmatrix} n-3 \\ k \end{bmatrix}',$$

replace $\left[{n-a \atop k-b}\right]'$ by $x^a P Q^{k-b}$, and divide by PQ^{k-1}. This leaves $Q = c^2x + xQ + c^2x^2 + x^2Q - x^3Q$ from which $Q = c^2x/(1-x)^2$. □

Notice that since $R(n,k) = H_n^{(k)}$ (which follows from Identity 3.6 and Theorem 3.9), we now have a combinatorial interpretation of the (n,k)-th element of the $(1/(1-x^2), c^2x/(1-x)^2)$ Riordan array: the number of tilings of an n-board using $2k$ vertically placed half-squares that come in c colours and $(\frac{1}{2}, \frac{1}{2})$-fences.

4. Metatiles when $p = 3$

When $p=3$, let μ_l be the number of mixed metatiles of length l, and $\mu_l^{[\sigma]}$ be the number of mixed metatiles of length l that have slot content σ in the final cell where σ is a length-3 binary string with 0 (1) representing an r (a post). From the mixed metatiles in Fig. 3 taken in order we see that

$$\mu_2^{[001]} = \mu_2^{[010]} = \mu_2^{[100]} = c^4, \quad \mu_2^{[011]} = \mu_2^{[101]} = \mu_2^{[110]} = c^2. \tag{4.1}$$

Lemma 4.1. *For $l \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$\mu_l^{[\sigma]} = \begin{cases} 2c\mu_{l-1}^{[\sigma]} + \mu_{l-2}^{[\sigma]} + c^4\delta_{l,2} + c^3\delta_{l,3}, & \sigma \in \{001, 010, 100\}, \\ 2c\mu_{l-1}^{[\sigma]} + \mu_{l-2}^{[\sigma]} + c^2\delta_{l,2} - c^3\delta_{l,3}, & \sigma \in \{011, 101, 110\}, \end{cases} \tag{4.2}$$

where $\mu_{l<2}^{[\sigma]} = 0$.

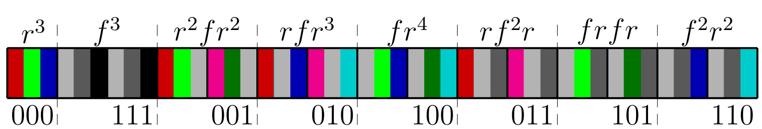

Figure 3. A 15-board tiled with metatiles of length less than 3 when $p = 3$. Symbolic representations are shown above the metatiles and final cell slot contents σ are given below. Dashed lines show boundaries between metatiles.

Proof. Given a metatile of length $l-1$ with some r in the final cell, we can create a metatile of length l by replacing one (or more) of the r by, in each instance, the left post of a fence. The corresponding right post(s) will then lie in the l-th cell and the metatile is completed by filling any empty slot in that cell with an r. It is then easily seen that for $l > 2$,

$$\mu_l^{[001]} = c(\mu_{l-1}^{[100]} + \mu_{l-1}^{[010]} + \mu_{l-1}^{[110]}), \quad \mu_l^{[010]} = c(\mu_{l-1}^{[100]} + \mu_{l-1}^{[001]} + \mu_{l-1}^{[101]}),$$
$$\mu_l^{[100]} = c(\mu_{l-1}^{[010]} + \mu_{l-1}^{[001]} + \mu_{l-1}^{[011]}), \quad \mu_l^{[011]} = \mu_{l-1}^{[100]}/c, \quad \mu_l^{[101]} = \mu_{l-1}^{[010]}/c, \quad \mu_l^{[110]} = \mu_{l-1}^{[001]}/c.$$

Notice that we gain (lose) a factor of c when the number of r in the metatile increases (decreases) by one. From the above equations, their symmetry, and (4.1), we have $\mu_l^{[001]} = \mu_l^{[010]} = \mu_l^{[100]}$, $\mu_l^{[011]} = \mu_l^{[101]} = \mu_l^{[110]}$, and thus for $l > 3$, $\mu_l^{[001]} = 2c\mu_{l-1}^{[001]} + \mu_{l-2}^{[001]}$. This gives us $\mu_l^{[011]} = 2c\mu_{l-1}^{[011]} + \mu_{l-2}^{[011]}$ for $l > 3$ as well. Using symmetry, (4.1), $\mu_3^{[\sigma]} = 2c^5 + c^3$ for $\sigma \in \{001, 010, 100\}$, $\mu_3^{[\sigma]} = c^3$ for $\sigma \in \{011, 101, 110\}$, and the fact that there are no mixed metatiles of length less than 2 leads to the result (4.2). □

Lemma 4.2. *For $l \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$\mu_l = 2c\mu_{l-1} + \mu_{l-2} + 3c^2(c^2+1)\delta_{l,2}, \quad \mu_{l<2} = 0. \tag{4.3}$$

Proof. Sum (4.2) over the 6 possible σ. □

On comparing (1.2) with (4.3) we obtain

$$\mu_l = 3c^2(c^2+1)M_{l-1}^{(2c)}, \quad l \in \mathbb{Z}. \tag{4.4}$$

From (4.2), for $l \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,

$$\mu_l^{[\sigma]} = \begin{cases} c^4 M_{l-1}^{(2c)} + c^3 M_{l-2}^{(2c)}, & \sigma \in \{001, 010, 100\}, \\ c^2 M_{l-1}^{(2c)} - c^3 M_{l-2}^{(2c)}, & \sigma \in \{011, 101, 110\}. \end{cases} \tag{4.5}$$

5. Identities involving the metallonacci numbers cubed

The identities we present in this section are generalizations of those in [11] and reduce to the latter identities when $c = 1$.

Lemma 5.1. *For all $n \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$B_n = \delta_{n,0} + c^3 B_{n-1} + (3c^4 + 3c^2 + 1)B_{n-2} + \sum_{l=3}^{n} \mu_l B_{n-l}, \tag{5.1}$$

where $B_n = 0$ for $n < 0$.

Proof. As in the proof of Lemma 2.1, the result follows from conditioning on the last metatile. There are c^3 metatiles of length 1 (r^3), $1+\mu_2$ of length 2 (f^3 and the mixed metatiles of length 2), and μ_l metatiles of length l for each $l \geq 3$. □

Identity 5.2. *For all $n \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$M_n^3 = \delta_{n,1} + c^3 M_{n-1}^3 + (3c^4 + 3c^2 + 1)M_{n-2}^3 + 3c^2(c^2+1)\sum_{l=3}^{n-1} M_{l-1}^{(2c)} M_{n-l}^3. \tag{5.2}$$

Proof. The result follows from Lemma 5.1, (4.4), and Theorem 1.2. □

Identity 5.3. *For all $n \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$,*

$$M_n^3 = \delta_{n,1} - 2c\delta_{n,2} - \delta_{n,3} + (c^3 + 2c)(M_{n-1}^3 - M_{n-3}^3) + (c^4 + 3c^2 + 2)M_{n-2}^3 - M_{n-4}^3. \tag{5.3}$$

Proof. Representing (5.1) by $E(n)$, in the equation $E(n) - 2cE(n-1) - E(n-2)$ we re-index two of the sums and rearrange to give

$$B_n = \delta_{n,0} - 2c\delta_{n,1} - \delta_{n,2} + (c^3+2c)B_{n-1} + (c^4+3c^2+2)B_{n-2} - (c^3+2c)B_{n-3} - B_{n-4} + \sum_{l=5}^{n}(\mu_l - 2c\mu_{l-1} - \mu_{l-2})B_{n-l}.$$

The sum vanishes by virtue of (4.3) and, after changing n to $n-1$, we obtain the required result on using (4.4) and Theorem 1.2. □

Identity 5.4. *For $n \ge 0$, $c \in \mathbb{Z}^+$, and $j = 0, 1$,*

$$M^3_{2n+j+1} = \delta_{j,0} + c^3\delta_{j,1} + \sum_{k=1}^{n}\left(c^3 M^3_{2k+j} + 3c^2(c^2+1)\sum_{i=1}^{2k} M^{(2c)}_{2k+j-i}M^3_i\right). \tag{5.4}$$

Proof. How many ways are there to tile a $(2n+j)$-board using at least one r? *Answer 1*: $B_{2n+j} - \delta_{j,0}$ since only the all-trifence tiling has no r and this only occurs for even-length boards. *Answer 2*: the final r must lie on an even (odd) cell if j is 0 (1) since the cells after this, if any, must be filled with trifences (which are each two cells long). Condition on the location of the final r. Suppose it is in cell $2k+j$ ($k = \delta_{j,0}, \ldots, n$). Either it is part of r^3 and so there are $c^3 B_{2k+j-1}$ ways to tile cells 1 to $2k+j$, or it is part of a mixed metatile and so there are $\mu_2 B_{2k+j-2} + \mu_3 B_{2k+j-3} + \cdots + \mu_{2k+j}B_0$ ways to tile them. In the latter case, evidently, k cannot be zero. Hence, equating the answers,

$$B_{2n+j} - \delta_{j,0} = c^3\sum_{k=\delta_{j,0}}^{n} B_{2k+j-1} + \sum_{k=1}^{n}(\mu_{2k+j}B_0 + \mu_{2k+j-1}B_1 + \cdots + \mu_2 B_{2k-2+j}).$$

Then, after simplifying, (5.4) follows from (4.4) and Theorem 1.2. □

Identity 5.5. *For $n \ge 0$ and $c \in \mathbb{Z}^+$,*

$$M^3_{n+1} - c^{3n} = \sum_{k=0}^{n-2} c^{3(n-k-2)}\left(M^3_{k+1} + 3c^2(c^2+1)\sum_{i=0}^{k} M^{(2c)}_{k+1-i}M^3_{i+1}\right). \tag{5.5}$$

Proof. How many ways are there to tile an n-board using at least 1 fence? *Answer 1*: $B_n - c^{3n}$ since this corresponds to all tilings except the all-r tilings. *Answer 2*: condition on the location of the last fence. Suppose this fence lies on cells $k+1$ and $k+2$ ($k = 0, \ldots, n-2$). Either there is a trifence covering these cells and so there are $c^{3(n-k-2)}B_k$ ways to tile the board, or the cells are at the end of a mixed metatile and so there are $c^{3(n-k-2)}(\mu_2 B_{k+2-2} + \mu_3 B_{k+2-3} + \cdots + \mu_{k+2}B_0)$ ways to tile it. Hence, equating the two answers,

$$B_n - c^{3n} = \sum_{k=0}^{n-2} c^{3(n-k-2)}(B_k + \mu_{k+2}B_0 + \mu_{k+1}B_1 + \cdots + \mu_3 B_{k-1} + \mu_2 B_k).$$

The identity then follows from (4.4) and Theorem 1.2. □

The proof of the following identity is entirely analogous to the proof of Identity 2.11 (or Identity 3.6 in [11]).

Identity 5.6. *For $n \ge 0$ and $c \in \mathbb{Z}^+$,*

$$M^3_{n+1} = M^{(c^3)}_{n+1} + 3c^2(c^2+1)\sum_{k=0}^{n-2}\sum_{l=2}^{n-k} M^{(2c)}_{l-1}M^{(c^3)}_{k+1}M^3_{n+1-k-l}. \tag{5.6}$$

Before proving the remaining identities we need the following lemma.

Lemma 5.7. *For $q = 0, 1, 2, 3$ and $c \in \mathbb{Z}^+$, there are $(cM_n)^q M_{n-1}^{3-q}$ ways to tile an n-board with r (which comes in c colours) and f if the number of r in the final cell is q.*

Proof. We use the bijection described in the proof of Lemma 1.1. For each final cell slot containing an r (a post), there corresponds an n-board tiled with dominoes and c possible colours of square that ends in a square (domino) for which there remain M_n (M_{n-1}) possible tilings. □

Identity 5.8. *For $n > 0$ and $c \in \mathbb{Z}^+$,*

$$(cM_n)^2 M_{n-1} = \sum_{k=1}^{n-1} \left(c^4 M_k^{(2c)} + c^3 M_{k-1}^{(2c)}\right) M_{n-k}^3.$$

Proof. How many ways are there to tile an n-board that ends with the right post of a fence which is immediately preceded by two r? *Answer 1*: as the final cell contains 2 r, by Lemma 5.7, there are $(cM_n)^2 M_{n-1}$ ways. *Answer 2*: the number of possible final metatiles of length l is $\mu_l^{[001]}$. Hence if the final metatile has length l, there are $\mu_l^{[001]} B_{n-l}$ ways to tile the board. Summing over all possible $l = 2, \ldots, n$ and equating to Answer 1 gives

$$(cM_n)^2 M_{n-1} = \sum_{l=2}^{n} \mu_l^{[001]} B_{n-l}.$$

Replacing l by $k + 1$ and then using (4.5) and Theorem 1.2 gives the identity. □

Identity 5.9. *For $n > 0$ and $c \in \mathbb{Z}^+$,*

$$cM_n M_{n-1}^2 = \sum_{k=1}^{n-1} \left(c^2 M_k^{(2c)} - c^3 M_{k-1}^{(2c)}\right) M_{n-k}^3.$$

Proof. How many ways are there to tile an n-board that ends with the right post of a fence which is immediately preceded by an r which is itself preceded by another right post? *Answer 1*: as the final cell contains a single r, by Lemma 5.7, there are $cM_n M_{n-1}^2$ ways. *Answer 2*: the number of possible final metatiles of length l is $\mu_l^{[101]}$ and so the number of ways to tile the board is $\sum_{l=2}^{n} \mu_l^{[101]} B_{n-l}$. Replacing l by $k+1$, equating the answers, and then using (4.5) and Theorem 1.2 gives the identity. □

Our final identity is obtained by summing the previous two. It can also be obtained directly by asking how many ways there are to tile an n-board that ends with the right post of a fence which is immediately preceded by an r.

Identity 5.10. *For $n > 0$ and $c \in \mathbb{Z}^+$,*

$$M_{n+1} M_n M_{n-1} = c(c^2 + 1) \sum_{k=1}^{n-1} M_k^{(2c)} M_{n-k}^3.$$

Putting $c = 1$ into the identities in this section give us the identities obtained in [11]. Putting $c = 2$ gives us relations between the Pell numbers cubed and F_{3n}, the even Fibonacci numbers, as a result of (1.3) and (4.4). For example, from Identity 5.10 we have, for $n > 0$,

$$P_{n+1} P_n P_{n-1} = 5 \sum_{k=1}^{n-1} F_{3k} P_{n-k}^3. \tag{5.7}$$

which could be regarded as the dual of Identity 3.10 in [11].

6. Discussion

One can arrive at identities involving $(M_n^{(c)})^p$ for larger values of p in an analogous way to what was done here in the $p = 3$ case. However, the $(p-1)$-th order recursion relations one obtains for the number of mixed metatiles, unlike in the $p = 3$ case, do not correspond to any known sequences as far as we can see.

We showed that when $p = 2$, the triangle counting the number of ways to tile an n-board using k fences is a row-reversed Riordan array. By Theorem 3.8 this will not be the case for $p > 2$ as the f^p metatile has $l - s = 2 - p < 0$.

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