ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2022. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1291 Proposed by Diego Rattaggi, Realgymnasium Rämibühl, Zürich, Switzerland.

Let $m \in \mathbb{N}$. Express the value of

$$\sum_{n=1}^{\infty} \frac{(\alpha^{2n-1}+1)(\alpha^{2n-1}-1)}{(2n-1)\cdot \alpha^{4m(2n-1)}}$$

in terms of Lucas numbers.

B-1292 Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Bazău, Romania.

For $x, y, z > 0$, prove that

$$\frac{x^2}{(5F_{2n}^2 y + 2z)(5F_{2n}^2 z + 2y)} + \frac{y^2}{(5F_{2n}^2 z + 2x)(5F_{2n}^2 x + 2z)} + \frac{z^2}{(5F_{2n}^2 x + 2y)(5F_{2n}^2 y + 2x)} \geq \frac{3}{L_{4n}^2}.$$

B-1293 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n prove that

(A) $\sqrt{F_1^3+\sqrt{F_2^3+\cdots+\sqrt{F_n^3}}}<2$;

(B) $\sqrt{L_1^3+\sqrt{L_2^3+\cdots+\sqrt{L_n^3}}}<3$.

B-1294 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Gran Canaria, Spain and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

Let $A(x)$ and $B(x)$ be polynomials of degree n such that $A(i)=F_i$ and $B(i)=L_i$, respectively, for every i with $0\le i\le n$. Find the values of $A(n+1)$ and $B(n+1)$.

B-1295 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Given an even integer r, prove that

$$\sum_{n=0}^{\infty}\frac{L_{rn}}{2n+1}\left(\frac{4}{L_r}\right)^n\binom{2n}{n}^{-1}=\frac{L_r\pi}{2}.$$

SOLUTIONS

The Fifth Oldie from the Vault

B-415 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.
(Vol. 17.4, December 1979)

The circumference of a circle in a fixed plane is partitioned into n arcs of equal length. In how many ways can one color these arcs if each arc must be red, white, or blue? Colorings which can be rotated into one another should be considered to be the same.

Editor's Note: This is another old problem from 40 years ago. No solutions have appeared, so we feature the problem again, and invite the readers to solve it.

Solution by Albert Stadler, Herrliberg, Switzerland.

The number of rotationally distinct colorings of n arcs of a circle is calculated by means of Burnside's lemma, which states:

> Let G be a finite group that acts on a set X. For each $g\in G$, let X^g denote the set of elements in X that are fixed by g (or left invariant by g); in other words, $X^g=\{x\in X\mid gx=x\}$. Then the number of orbits is
>
> $$\frac{1}{|G|}\sum_{g\in G}|X^g|.$$

In this problem, X is the set of colored circles, which is a set of size 3^n, and G is the cyclic rotation group of order n. Let a be a generator of G, for instance, the rotation by one arc in the positive direction. Thus, $G = \{a, a^2, \ldots, a^n\}$, where a^n is the identity element of the group. For each $k \in \{1, 2, \ldots, n\}$, the rotation a^k is of order $n/\gcd(k, n)$. It partitions the set of n arcs into $\gcd(k, n)$ orbits, each of size $n/\gcd(k, n)$. A coloring is invariant under a^k if and only if it is constant on each orbit. Thus, with three colors, the number of invariant colorings for a^k is $3^{\gcd(k,n)}$. According to Burnside's lemma, the number S_n of rotationally distinct colorings is obtained by averaging the number of invariant colorings over all elements of the group. Therefore,

$$S_n = \frac{1}{n}\sum_{k=1}^{n} 3^{\gcd(k,n)} = \frac{1}{n}\sum_{d|n} 3^{\frac{n}{d}} \sum_{\substack{1\le k\le n \\ \gcd(k,n)=\frac{n}{d}}} 1 = \frac{1}{n}\sum_{d|n} \phi(d)\cdot 3^{\frac{n}{d}},$$

where ϕ is Euler's totient function.

Editor's Note: It is obvious that the result can be generalized to colorings with m colors.

Also solved by Michel Bataille, Luke Paulso (student), Raphael Schumacher (graduate student), J. N. Senadheera, Paul K. Stockmeyer, David Terr, and the proposer.

Three Atypical Solutions

B-1271 Proposed by Ivan V. Fedak, Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n, prove that

$$\frac{\alpha^n - \beta^n}{\alpha^{n+3} - \beta^{n+3}} + \frac{\alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)(\alpha^{n+1} + \beta^{n+1})} > \frac{1}{2}.$$

Solution 1 by Won Kyun Jeong, Kyungpook National University, Daegu, Korea.

Because of Binet's formulas, we find

$$\frac{\alpha^n - \beta^n}{\alpha^{n+3} - \beta^{n+3}} + \frac{\alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)(\alpha^{n+1} + \beta^{n+1})} = \frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{L_{n+1}}.$$

It is easy to verify the inequality when $n = 1$. For $n > 1$, it follows from $L_m = F_{m+2} - F_{m-2}$ that

$$\frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{L_{n+1}} = \frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{F_{n+3} - F_{n-1}} > \frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{F_{n+3}} = \frac{F_{n+2}}{F_{n+3}} > \frac{1}{2}.$$

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

Because we cannot have $F_n = F_{n+1} = F_{n+2}$ for $n \ge 1$, it follows from Nesbitt's inequality that

$$\begin{aligned}&\frac{\alpha^n - \beta^n}{\alpha^{n+3} - \beta^{n+3}} + \frac{\alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)(\alpha^{n+1} + \beta^{n+1})} + 1 \\ &\quad = \frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{L_{n+1}} + \frac{F_{n+2}}{F_{n+2}} = \frac{F_n}{F_{n+1} + F_{n+2}} + \frac{F_{n+1}}{F_{n+2} + F_n} + \frac{F_{n+2}}{F_n + F_{n+1}} > \frac{3}{2}.\end{aligned}$$

Solution 3 by Raphael Schumacher (graduate student), ETH Zurich, Switzerland.

The inequality is seen to be equivalent to $\frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{L_{n+1}} > \frac{1}{2}$. We will prove that, for $n \geq 1$, the stronger inequality

$$\frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{L_{n+1}} \geq \frac{2}{3}$$

holds. It suffices to prove that

$$F_n L_{n+1} + F_{n+1} F_{n+3} - \frac{2}{3} F_{n+3} L_{n+1} \geq 0.$$

By employing the two identities $L_{n+1} = 2F_n + F_{n+1}$ and $F_{n+3} = F_n + 2F_{n+1}$, we find

$$\begin{aligned} & F_n L_{n+1} + F_{n+1} F_{n+3} - \frac{2}{3} F_{n+3} L_{n+1} \\ = \ & F_n(2F_n + F_{n+1}) + F_{n+1}(F_n + 2F_{n+1}) - \frac{2}{3}(F_n + 2F_{n+1})(2F_n + F_{n+1}) \\ = \ & \frac{2}{3}(F_n^2 + F_{n+1}^2) - \frac{4}{3} F_n F_{n+1}. \end{aligned}$$

To complete the proof, apply the AM-GM inequality to obtain

$$\frac{2}{3}(F_n^2 + F_{n+1}^2) - \frac{4}{3} F_n F_{n+1} \geq \frac{2}{3} \cdot 2F_n F_{n+1} - \frac{4}{3} F_n F_{n+1} = 0.$$

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Brian Bradie, Kenny B. Davenport (two solutions), Steve Edwards, Fort Hays State University Problem Solving Group, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Russell Jay Hendel, Wei-Kai Lai, Luke Paluso (student), Ángel Plaza, Albert Stadler, and the proposer.

A Binomial Sum of Cosine and Sine Functions

B-1272 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any integer $n \geq 0$, prove that

(i) $\displaystyle\sum_{k=0}^{n} \beta^k \binom{n}{k} \cos\frac{k\pi}{5} = (-\beta)^n \cos\frac{n\pi}{5}$,

(ii) $\displaystyle\sum_{k=0}^{n} \beta^k \binom{n}{k} \sin\frac{k\pi}{5} = -(-\beta)^n \sin\frac{n\pi}{5}$.

Solution by Michel Bataille, Rouen, France.

For $0 \leq k \leq n$, we have $\alpha^n \beta^k = \alpha^{n-k}(\alpha\beta)^k = (-1)^k \alpha^{n-k}$. Multiplication by α^n to both sides of (i) and (ii) reveals that they are equivalent to

$$\sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} (-1)^k \cos\frac{k\pi}{5} = \cos\frac{n\pi}{5},$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} (-1)^k \sin\frac{k\pi}{5} = -\sin\frac{n\pi}{5},$$

respectively. We can prove both statements by showing that

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}(-1)^k e^{ik\pi/5} = e^{-in\pi/5}.$$

Now, using first the binomial theorem and then $\alpha = 2\cos\frac{\pi}{5}$, we obtain

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}(-1)^k e^{ik\pi/5} &= (\alpha - e^{i\pi/5})^n \\
&= \left(\cos\frac{\pi}{5} - i\,\sin\frac{\pi}{5}\right)^n \\
&= (e^{-i\pi/5})^n \\
&= e^{-in\pi/5}.
\end{aligned}$$

The desired result follows.

Also solved by Khristo N. Boyadzhiev, Brian Bradie, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Luke Paluso (student), Ángel Plaza, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, David Terr, Dan Weiner, and the proposer.

A Summation Formula for an Arithmetic Progression

B-1273 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Let $\{u_n\}_{n\geq 0}$ be a generalized Fibonacci sequence defined by $u_n = u_{n-1} + u_{n-2}$ with u_0 and u_1 not both being zero. Let further $\{a_n\}_{n\geq 1}$ be an arithmetic progression, that is, $a_n = a_1 + (n-1)d$, where $a_1, d > 0$. Show that

$$\sum_{k=1}^{n}\frac{u_{k+2}}{a_{k+1}\sqrt{a_k} + a_k\sqrt{a_{k+1}}} = \frac{1}{d}\left(\frac{u_3}{\sqrt{a_1}} - \frac{u_{n+2}}{\sqrt{a_{n+1}}} + \sum_{k=1}^{n-1}\frac{u_{k+1}}{\sqrt{a_{k+1}}}\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland.

We have

$$\begin{aligned}
\sum_{k=1}^{n}\frac{u_{k+2}}{a_{k+1}\sqrt{a_k} + a_k\sqrt{a_{k+1}}} &= \sum_{k=1}^{n}\frac{u_{k+2}}{a_{k+1}\sqrt{a_k} + a_k\sqrt{a_{k+1}}}\cdot\frac{a_{k+1}\sqrt{a_k} - a_k\sqrt{a_{k+1}}}{a_{k+1}\sqrt{a_k} - a_k\sqrt{a_{k+1}}} \\
&= \sum_{k=1}^{n}\frac{u_{k+2}(a_{k+1}\sqrt{a_k} - a_k\sqrt{a_{k+1}}\,)}{a_k a_{k+1}(a_{k+1} - a_k)} \\
&= \frac{1}{d}\sum_{k=1}^{n}\frac{u_{k+2}}{\sqrt{a_k}} - \frac{1}{d}\sum_{k=1}^{n}\frac{u_{k+2}}{\sqrt{a_{k+1}}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d}\sum_{k=0}^{n-1}\frac{u_{k+3}}{\sqrt{a_{k+1}}} - \frac{1}{d}\sum_{k=1}^{n}\frac{u_{k+2}}{\sqrt{a_{k+1}}} \\
&= \frac{u_3}{d\sqrt{a_1}} - \frac{u_{n+2}}{d\sqrt{a_{n+1}}} + \frac{1}{d}\sum_{k=1}^{n-1}\frac{u_{k+3}-u_{k+2}}{\sqrt{a_{k+1}}} \\
&= \frac{u_3}{d\sqrt{a_1}} - \frac{u_{n+2}}{d\sqrt{a_{n+1}}} + \frac{1}{d}\sum_{k=1}^{n-1}\frac{u_{k+1}}{\sqrt{a_{k+1}}}.
\end{aligned}$$

Also solved by Michel Bataille, Brian Bradie, Matthew Daugomah (student), Steve Edwards, I. V. Fedak, Dmitry Fleischman, Hideyuki Ohtsuka, Luke Paluso (student), Ángel Plaza, Ben Race (student), Raphael Schumacher (graduate student), Daniel Văcaru, and the proposer.

Apply the Triangle Inequality

B-1274 Proposed by Ivan V. Fedak, Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n, prove that

$$\sum_{k=1}^{n}\sqrt{F_{2k-1}} \geq \sqrt{F_{2n+3}-2F_{n+3}+2}.$$

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC.

Consider vectors $\vec{v_i} = (F_i, F_{i-1})$ for $i \geq 1$. Then $\|\vec{v_i}\| = \sqrt{F_i^2 + F_{i-1}^2} = \sqrt{F_{2i-1}}$, according to Identity 30 [1, p. 97]. Applying the same identity, we also notice that

$$\begin{aligned}
\sqrt{F_{2n+3}-2F_{n+3}+2} &= \sqrt{F_{n+2}^2 + F_{n+1}^2 - 2(F_{n+2}+F_{n+1}) + 2} \\
&= \sqrt{(F_{n+2}-1)^2 + (F_{n+1}-1)^2} \\
&= \sqrt{(F_1+F_2+\cdots+F_n)^2 + (F_0+F_1+\cdots+F_{n-1})^2},
\end{aligned}$$

because of Theorem 5.2 [1, p. 69]. Therefore, the proposed inequality is equivalent to

$$\sum_{k=1}^{n}\|\vec{v}_k\| \geq \left\|\sum_{k=1}^{n}\vec{v}_k\right\|,$$

which is true according to the generalized triangle inequality. The equality occurs when $n = 1$.

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

Also solved by Thomas Achammer, Michel Bataille, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, Hiduyuki Ohtsuka, Luke Paluso (student), Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, and the proposer.

An Unusual Sum of Products

B-1275 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Given a real number $c > 0$, for any integer $n \geq 0$, find a closed form expression for the sum

$$\sum_{k=0}^{n} \prod_{j=k}^{n} \frac{1}{c(L_{2^{j+1}} + 1) + L_{2^j} - 1}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Let $f(n)$ denote the sum in the problem statement. It is not difficult to see that

$$f(n+1) = \frac{1}{c(L_{2^{n+2}} + 1) + L_{2^{n+1}} - 1} \big(1 + f(n)\big).$$

We will now show, by induction, that

$$f(n) = \frac{1}{c(L_{2^{n+1}} + 1)}.$$

The identity clearly holds when $n = 0$. Assume it holds for some nonnegative integer n. It follows that

$$\begin{aligned} f(n+1) &= \frac{1}{c(L_{2^{n+2}} + 1) + L_{2^{n+1}} - 1} \left(1 + \frac{1}{c(L_{2^{n+1}} + 1)}\right) \\ &= \frac{c(L_{2^{n+1}} + 1) + 1}{c^2(L_{2^{n+2}} + 1)(L_{2^{n+1}} + 1) + c(L_{2^{n+1}}^2 - 1)}. \end{aligned}$$

Using

$$L_m^2 = \alpha^{2m} + 2\alpha^m \beta^m + \beta^{2m} = L_{2m} + 2(-1)^m,$$

we find

$$L_{2^{n+1}}^2 - 1 = L_{2^{n+2}} + 2(-1)^{2^{n+1}} - 1 = L_{2^{n+2}} + 1.$$

Therefore,

$$f(n+1) = \frac{c(L_{2^{n+1}} + 1) + 1}{c^2(L_{2^{n+2}} + 1)(L_{2^{n+1}} + 1) + c(L_{2^{n+2}} + 1)} = \frac{1}{c(L_{2^{n+2}} + 1)}.$$

This completes the induction and establishes the closed form expression for $f(n)$.

Also solved by Michel Bataille, Steve Edwards, I. V. Fedak, G. C. Greubel, Luke Paluso (student), Raphael Schumacher (graduate student), Albert Stadler, and the proposer.

Belated Acknowledgment: Illia Antypenko, a high school student, also solved problems B-1267 and B-1269.