ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-654 Proposed by Slavko Simic, Belgrade, Yugoslavia

Let $x = \{x_i\}_{i=1}^n$ be a sequence of real numbers and $p = \{p_i\}_{i=1}^n$ be a sequence of positive numbers with $\sum_{i=1}^n p_i = 1$. Define $S_k = \sum_{i=1}^k p_i x_i^k - \left(\sum_{i=1}^k p_i x_i\right)^k$, for $k = 1, 2, 3, \ldots$ Prove that $S_3^2 \leq \frac{3}{2} S_2 S_4$. Is it true that the inequality $S_{2m+1}^{2m} \leq \frac{(2m+1)m^{2m}}{(m+1)^{2m-1}} S_2 S_{2m+2}^{2m-1}$ holds for all $m \geq 1$?

H-655 Proposed by Slavko Simic, Belgrade, Yugoslavia

Let $\{c_i\}_{i=1}^n$ be a finite sequence of distinct positive integers and $q > 1$ be a natural number. Prove that $\left\lfloor \frac{\sum_{i=1}^n c_i q^{c_i}}{\sum_{i=1}^n q^{c_i}} \right\rfloor = c$, where $c = \max\{c_i : i = 1, \ldots, b\}$. Is it true that $\left\lfloor \frac{(q-1)\sum_{i=1}^n c_i q^{c_i}}{\sum_{i=1}^n q^{c_i}} \right\rfloor = c(q-1) - 1$?

H-656 Proposed by Andrew Cusumano, Great Neck, NY

Let $A_n = \sum_{k=1}^n k^k$. Show that $\lim_{n\to\infty} \left(\frac{A_{n+2}}{A_{n+1}} - \frac{A_{n+1}}{A_n}\right) = e$. Show that the same holds for the sequence of general term $A_n = (n+1)^{n+1} - n^n$.

H-657 Proposed by Paul S. Bruckman, Sointula, Canada

Show that the equation $(a + b\alpha)^4 + (a + b\beta)^4 = c^4$ has no nonzero integer solutions a, b, c, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

SOLUTIONS

A determinant with Fibonacci, Lucas and Pell numbers

H-636 Proposed by Charles K. Cook, Sumter, SC
(Vol. 44, No. 1, February 2006)

Evaluate the determinant of the matrix

$$\begin{vmatrix} F_n^2+L_n^2-P_n^2-R_n^2 & 2(L_nP_n-F_nR_n) & 2(F_nP_n+L_nR_n) \\ 2(F_nR_n+L_nP_n) & F_n^2-L_n^2+P_n^2-R_n^2 & 2(P_nR_n-F_nL_n) \\ 2(L_nR_n-F_nP_n) & 2(F_nL_n+P_nR_n) & F_n^2-L_n^2-P_n^2+R_n^2 \end{vmatrix}$$

where F_n, L_n, P_n and R_n are the Fibonacci, Lucas, Pell and Pell-Lucas numbers, respectively.

Solution by the proposer

Letting M stand for the original matrix, we get, using an idea presented by B. Jansson [1] in the solution of a more general problem proposed by C.W. Trigg [2], after some simplification that

$$\det(M)^2 = \det(M)\det(M^T) = \det(MM^T)$$

$$= \det\begin{vmatrix} (F_n^2+L_n^2+P_n^2+R_n^2)^2 & 0 & 0 \\ 0 & (F_n^2+L_n^2+P_n^2+R_n^2)^2 & 0 \\ 0 & 0 & (F_n^2+L_n^2+P_n^2+R_n^2)^2 \end{vmatrix}$$

$$= (F_n^2+L_n^2+P_n^2+R_n^2)^6.$$

Thus, $\det(M) = \pm(F_n^2+L_n^2+P_n^2+R_n^2)^3$. To decide on the sign, note that since $\det(M)$ is a sum of signed products of binary recurrent sequences, it follows that $\det(M)$ is a linearly recurrent sequence of n (of some large order) itself, and so is each of $\pm(F_n^2+L_n^2+P_n^2+R_n^2)^3$. It follows from known facts about zeros of linearly recurrent sequences that in order to decide which sign should we pick, it suffices to compute the given determinant in a particular value. Computing it at $n=0$, we get that the determinant is positive, so the formula $\det(M) = (F_n^2+L_n^2+P_n^2+R_n^2)^3$ holds for all $n \geq 0$.

[1] B. Jansson. "Solution to Problem 750." Mathematics Magazine **43.4** (1970):230.

[2] C.W. Trigg. "Problem 730." Mathematics Magazine **43.1** (1970):48.

Also solved by Kenneth B. Davenport and Paul S. Bruckman.

A Fibonacci Triangle

H-637 Proposed by Ovidiu Furdui, Kalamazoo, MI
(Vol. 44, no. 1, February 2006)

Prove that

$$\sqrt{\frac{1}{2}+\frac{F_{2n}}{2\sqrt{F_{2n}^2+1}}},\ \sqrt{\frac{1}{2}+\frac{1}{2\sqrt{F_{2n+1}^2+1}}}\ \text{and}\ \sqrt{\frac{1}{2}+\frac{1}{2\sqrt{F_{2n+2}^2+1}}}$$

are the sides of a triangle whose circumradius is $1/2$ for all $n \geq 0$.

Solution by Paul S. Bruckman, Sointula, Canada

The formula

$$R=\frac{abc}{4K} \tag{1}$$

for the circumradius of a triangle with sides a, b, c can be found in many elementary geometry texts, where K is the area of the triangle. By Heron's formula,

$$K=\sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s=(a+b+c)/2. \tag{2}$$

Setting $R=1/2$ in (1) and using (2), we get that the required relation is equivalent to

$$a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2)+4a^2b^2c^2=0. \tag{3}$$

We take

$$a^2=\frac{1}{2}\left(1+\frac{F_{2n}}{(F_{2n}^2+1)^{1/2}}\right),\ b^2=\frac{1}{2}\left(1+\frac{1}{(F_{2n+1}^2+1)^{1/2}}\right),\ c^2=\frac{1}{2}\left(1+\frac{1}{(F_{2n+2}^2+1)^{1/2}}\right).$$

Thus,

$$2a^2-1=F_{2n}(F_{2n}^2+1)^{-1/2},\ 2b^2-1=(F_{2n+1}^2+1)^{-1/2},\ 2c^2-1=(F_{2n+2}^2+1)^{-1/2}.$$

Multiplying the above relations we get

$$\left((2a^2-1)(2b^2-1)(2c^2-1)\right)^{-2}=(F_{2n}^2+1)(F_{2n+1}^2+1)(F_{2n+2}^2+1)/F_{2n}^2$$

$$=(F_{2n+1}F_{2n-1})(F_{2n+3}F_{2n-1})(F_{2n+2}F_{2n+1})F_{2n}^{-2}=(F_{2n+3}F_{2n+1}F_{2n-1}/F_{2n})^2.$$

Since $2a^2-1,\ 2b^2-1,\ 2c^2-1$ are positive, upon extracting square-roots we get

$$\left((2a^2-1)(2b^2-1)(2c^2-1)\right)^{-1} = F_{2n+3}F_{2n+1}F_{2n-1}/F_{2n}. \tag{4}$$

However,

$$(2a^2-1)(2b^2-1)(2c^2-1) = 8a^2b^2c^2 - 4(a^2b^2+a^2c^2+b^2c^2) + 2(a^2+b^2+c^2) - 1.$$

Comparing the last relation with (3), we see that we must prove the following:

$$(2a^2-1)(2b^2-1)(2c^2-1) = 2(a^2+b^2+c^2) - 2(a^4+b^4+c^4) - 1, \tag{5}$$

which in light of (4) is equivalent to

$$2(a^2+b^2+c^2) - 2(a^4+b^4+c^4) - 1 = \frac{F_{2n}}{F_{2n+3}F_{2n+1}F_{2n-1}}.$$

This is equivalent to

$$(2a^2-1)^2 + (2b^2-1)^2 + (2c^2-1)^2 = 1 - \frac{2F_{2n}}{F_{2n+3}F_{2n+1}F_{2n-1}}. \tag{6}$$

Now

$$(2a^2-1)^2 = \frac{F_{2n}^2}{F_{2n+1}F_{2n-1}},\ (2b^2-1)^2 = \frac{1}{F_{2n+3}F_{2n-1}},\ (2c^2-1)^2 = \frac{1}{F_{2n+3}F_{2n-1}}.$$

Thus,

$$(2a^2-1)^2 + (2b^2-1)^2 + (2c^2-1)^2 = \frac{F_{2n}^2F_{2n+3} + F_{2n+1} + F_{2n-1}}{F_{2n+3}F_{2n+1}F_{2n-1}}. \tag{7}$$

Comparing (6) and (7), we see that it suffices to prove the following identity:

$$F_{2n+3}F_{2n+1}F_{2n-1} - 2F_{2n} = F_{2n}^2F_{2n+3} + F_{2n+1} + F_{2n-1}. \tag{8}$$

Since $F_{2n+1}F_{2n-1} - 1 = F_{2n}^2$, identity (8) simplifies to $F_{2n+3} = 2F_{2n} + F_{2n+1} + F_{2n-1}$. This last one is indeed an identity since $2F_{2n} + F_{2n+1} + F_{2n-1} = (F_{2n} + F_{2n+1}) + (F_{2n} + F_{2n-1}) = F_{2n+2} + F_{2n+1} = F_{2n+3}$, as wanted. This proves that indeed the given a, b and c form a triangle of circumradius radius $1/2$.

Also solved by the proposer.

An Inequality with Fibonacci Logarithms

H-638 **Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 44, no. 1, February 2006)**

Let n be a positive integer. Prove that

$$4+2\sum_{k=1}^{n}\frac{F_{k+1}}{\log\left(1+F_{k+1}/F_k\right)} < F_{n+1}+3F_{n+2}.$$

Note: This problem has appeared also as B-1009. The proposer apologizes for the oversight and wishes to retract his proposal H-638. However, the editor has already received solutions from Gokcen Alptekin, Paul S. Bruckman, G. C. Greubel, H.-J. Seiffert, Naim Tuglu, and the proposer.

Identities with Fibonacci and Lucas Polynomials

H-639 **Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 44, no. 2, May 2006)**

The sequences of the Fibonacci and Lucas polynomials are defined by

$$F_0(x)=0,\ F_1(x)=1,\ \text{and } F_{n+1}(x)=xF_n(x)+F_{n-1}(x) \text{ for } n\geq 1,$$

$$L_0(x)=2,\ L_1(x)=x,\ \text{and } L_{n+1}(x)=xL_n(x)+L_{n-1}(x) \text{ for } n\geq 1,$$

respectively. Prove that, for all non-zero complex numbers x and all positive integers n,

(a)

$$\sum_{k=0}^{2n-1}\binom{4n-1-k}{k}2^{4n-1-2k}x^kF_k(x)=x^{2n-1}L_{2n-1}(x)F_{2n}(4/x),$$

(b)

$$\sum_{k=0}^{2n-1}\binom{4n-1-k}{k}2^{4n-1-2k}x^kL_k(x)=x^{2n-1}(x^2+4)F_{2n-1}(x)F_{2n}(4/x),$$

(c)

$$\sum_{k=0}^{2n}\binom{4n+1-k}{k}2^{4n+2-2k}x^kF_k(x)=x^{2n+1}F_{2n}(x)L_{2n+1}(4/x),$$

(d)

$$\sum_{k=0}^{2n}\binom{4n+1-k}{k}2^{4n+2-2k}x^kL_k(x)=x^{2n+1}L_{2n}(x)L_{2n+1}(4/x).$$

Solution by the proposer

It is known that

$$F_n(x) = (\alpha(x)^n - \beta(x)^n)/\sqrt{x^2+4} \quad \text{and} \quad L_n(x) = \alpha(x)^n + \beta(x)^n, \tag{1}$$

where $\alpha(x) = (x + \sqrt{x^2+4})/2$ and $\beta(x) = (x - \sqrt{x^2+4})/2$, and that

$$F_{2n}(x) = \sum_{k=0}^{n-1} \binom{2n-1-k}{k} x^{2n-1-2k}. \tag{2}$$

It suffices to prove the result when $x > 0$ is rational such that $\sqrt{x^2+4}$ is irrational. For such x, we put $y = 2\sqrt{-\beta(x)/x}$. Since $x - \beta(x) = \alpha(x)$, (1) gives

$$F_{2n}(y) = \frac{\sqrt{x}}{2x^n\sqrt{\alpha(x)}} \left(\left(\sqrt{-\beta(x)} + \sqrt{\alpha(x)}\right)^{2n} - \left(\sqrt{-\beta(x)} - \sqrt{\alpha(x)}\right)^{2n} \right),$$

so that, by

$$\left(\sqrt{-\beta(x)} + \sqrt{\alpha(x)}\right)^2 = \sqrt{x^2+4} + 2 = x\alpha(4/x)$$

and

$$\left(\sqrt{-\beta(x)} - \sqrt{\alpha(x)}\right)^2 = \sqrt{x^2+4} - 2 = -x\beta(4/x),$$

$$F_{2n}(y) = \frac{\sqrt{x}}{2\sqrt{\alpha(x)}} \left(\alpha(4/x)^n - (-\beta(4/x))^n\right).$$

On the other hand, from (2), we find

$$F_{2n}(y) = \frac{\sqrt{x\alpha(x)}}{(x\alpha(x))^n} \sum_{k=0}^{n-1} \binom{2n-1-k}{k} 2^{2n-1-2k} x^k \alpha(x)^k,$$

because $-\beta(x) = 1/\alpha(x)$. Combining both identities gives

$$\sum_{k=0}^{n-1} \binom{2n-1-k}{k} 2^{2n-1-2k} x^k \alpha(x)^k = \frac{1}{2} x^n \alpha(x)^{n-1} (\alpha(4/x)^n - (-\beta(4/x))^n).$$

The desired identities (a) and (b) follow by replacing n by $2n$, using (1) and the relation $\alpha(x)^k = (L_k(x) + \sqrt{x^2+4}F_k(x))/2$. Similarly, with n replaced by $2n+1$, one gets (c) and (d).

Also solved by Paul S. Bruckman and Harris Kwong.