ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-662 *Proposed by Philip L. Mana, Albuquerque, NM*

For fixed n, find all m such that $L_nF_m - F_{m+n} = (-1)^n$.

B-623 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{2n-1} L_{n+k}L_k.$$

Prove that $S(n)$ is an integral multiple of L_n for all positive integers n.

B-624 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T_n = \sum_{i=1}^{n} L_{2(n+i)-1}.$$

For every positive integer n, prove that either $F_n | T_n$ or $L_n | T_n$.

B-625 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geqq 2.$$

Let $G_n = F_nP_n$ and $H_n = L_nP_n$. Show that (G_n) and (H_n) satisfy

$$K_{n+4} - 2K_{n+3} - 7K_{n+2} - 2K_{n+1} + K_n = 0.$$

B-626 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let G_n and H_n be as in B-625. Express the generating functions

$$G(z) = \sum_{n=0}^{\infty} G_n z^n \quad \text{and} \quad H(z) = \sum_{n=0}^{\infty} H_n z^n$$

as rational functions of z.

B-627 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let

$$C_{n,k} = (F_n^3 + F_{n+1}^3 + \cdots + F_{n+k-1}^3)/k.$$

Find the smallest k in $\{2, 3, 4, \ldots\}$ such that $C_{n,k}$ is an integer for every n in $\{0, 1, 2, \ldots\}$.

SOLUTIONS

2 Problems on Pythagorean Triples

B-598 *Proposed by Herta T. Freitag, Roanoke, VA*

For which positive integers n is $(2L_n, L_{2n} - 3, L_{2n} - 1)$ a Pythagorean triple? For which of these n's is the triple primitive?

B-599 *Proposed by Herta T. Freitag, Roanoke, VA*

Do B-598 with the triple now $(2L_n, L_{2n} + 1, L_{2n} + 3)$.

Solutions by Thomas M. Green, Contra Costa College, San Pablo, CA

It is known that $L_{2n} = L_n^2 + 2(-1)^{n+1}$.

For n odd, we have $L_{2n} = L_n^2 + 2$ and the triple

$$(2L_n, L_{2n} - 3, L_{2n} - 1) = (2L_n, L_n^2 - 1, L_n^2 + 1)$$

which is a Pythagorean triple. Furthermore, a Pythagorean triple of the type $(2m, m^2 - 1, m^2 + 1)$ is primitive if m is even. Thus, if $L_n = m$, an even number, then $(2L_n, L_n^2 - 1, L_n^2 + 1)$ is primitive. But, if n is odd, L_n is even only when n is an odd multiple of three.

Similarly, for n even (B-599), the triple

$$(2L_n, L_{2n} + 1, L_{2n} + 3) = (2L_n, L_n^2 - 1, L_n^2 + 1)$$

is Pythagorean and will be primitive if L_n is even. In this case, however, if n is even, L_n is even only when n is an even multiple of three.

Also solved by Paul S. Bruckman, Frank Conliffe, Richard Dry, Piero Filipponi & Adina Di Porto, C. Georghiou, L. Kuipers, Bob Prielip, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Paul Tzermias, and the proposer.

Fibonacci Multiples of 121160

B-600 *Proposed by Philip L. Mana, Albuquerque, NM*

Let n be any positive integer and $m = n^{13} - n$. Prove that F_m is an integral multiple of 30290.

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

We prove a more general result, namely: F_m is an integral multiple of 121,160, where $m = n^{13} - n$; n being a positive integer.

We can express

$$n^{13} - n = (n^7 - n)(n^6 + 1) = (n^5 - n)(n^8 + n^4 + 1)$$
$$= (n^3 - n)(n^{10} + n^8 + n^6 + n^4 + n^2 + 1).$$

By Fermat's theorem: $n^p - n \equiv 0 \pmod{p}$, where p is prime and n is a positive integer.

Thus, we conclude that:

$$n^{13} - n \equiv 0 \pmod{13};\ n^{13} - n \equiv 0 \pmod{7};\ n^{13} - n \equiv 0 \pmod{5}.$$

Since $n^3 - n$ is a factor of $n^{13} - n$ and $n^3 - n$ is a product of three consecutive integers, $n - 1$, n, $n + 1$, we have:

$$n^3 - n \equiv 0 \pmod{6} \Rightarrow n^{13} - n \equiv 0 \pmod{6}$$
$$\Rightarrow F_5 \cdot F_6 \cdot F_7 \cdot F_{13} \text{ divides } F_m$$

(by the fact that r divides s implies F_r divides F_s)

$$\Rightarrow 5 \cdot 8 \cdot 13 \cdot 233 \text{ is a factor of } F_m.$$

Thus, we are done.

Also solved by Paul S. Bruckman, David M. Burton, Frank H. Conliffe, Piero Filipponi, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.

Integral Arithmetic Means

B-601 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $A_{n,k} = (F_n + F_{n+1} + \cdots + F_{n+k-1})/k$. Find the smallest k in {2, 3, 4, ...} such that $A_{n,k}$ is an integer for every n in {0, 1, 2, ...}.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that 24 is the value of k that is being sought.

Our solution will use the following known information:

(1) $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$, $n \geq 1$, and

(2) $F_{n+t} - F_{n-t} = L_n F_t$, t even.

[(1) is (I_1) on p. 52 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr., Houghton Mifflin, Boston, 1969, and (2) is (I_{24}) on p. 59, ibid.]

Since

$$F_n + F_{n+1} + \cdots + F_{n+k-1}$$
$$= (F_1 + F_2 + \cdots + F_{n+k-1}) - (F_1 + F_2 + \cdots + F_{n-1})$$
$$= (F_{n+k+1} - 1) - (F_{n+1} - 1) \quad \text{[by (1)]}$$
$$= F_{n+k+1} - F_{n+1},$$

$$A_{n,k} = (F_{n+k+1} - F_{n+1})/k.$$

Let n be an arbitrary nonnegative integer. If $k = 24$,

$$F_{n+k+1} - F_{n+1} = F_{(n+13)+12} - F_{(n+13)-12} = L_{n+13}F_{12} \quad \text{[by (2)]}$$
$$= L_{n+13} \cdot 144 \equiv 0 \pmod{24}.$$

Thus, $A_{n,24}$ is an integer for each nonnegative integer n.

$A_{0,2} = (F_3 - F_1)/2 = (2 - 1)/2 = 1/2$. Proceeding in this same manner, it can be shown that $A_{0,k}$ is NOT an integer for $k = 2, 3, 5, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22$, and 23 and that $A_{1,k}$ is NOT an integer for $k = 4, 6, 9, 11$, and 19. Therefore, 24 is the smallest k in $\{2, 3, 4, \ldots\}$ such that $A_{n,k}$ is an integer for every nonnegative integer n.

Also solved by David M. Burton, C. Georghiou, L. Kuipers, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.

Fibonacci Infinite Series

B-602 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Let H_n represent either F_n or L_n.

(a) Find a simplified expression for $\dfrac{1}{H_n} - \dfrac{1}{H_{n+1}} - \dfrac{1}{H_{n+2}}$.

(b) Use the result of (a) to prove that

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + 2\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}.$$

Solution by C. Georghiou, University of Patras, Greece

(a) After some simple algebra it is easy to see that

$$\frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_{n+2}} = \frac{H_{n+1}^2 - H_nH_{n+2}}{H_nH_{n+1}H_{n+2}}$$

(b) For $H_n = F_n$, we have $F_{n+1}^2 - F_nF_{n+2} = (-1)^n$, and since $F_n = 0(\alpha^n)$ it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_nF_{n+1}F_{n+2}} = \sum_{n=1}^{\infty}\left(\frac{1}{F_{2n}F_{2n+1}F_{2n+2}} - \frac{1}{F_{2n-1}F_{2n}F_{2n+1}}\right)$$
$$= -2\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}.$$

On the other hand, we have

$$\sum_{i=1}^{\infty}\left(\frac{1}{F_n} - \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}}\right) = -\sum_{i=1}^{\infty} \frac{1}{F_n} + \frac{2}{F_1} + \frac{1}{F_2}.$$

By equating the two sums we get the given expression.

Also solved by Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Tzermias, and the proposer.

Lucas Analogue

B-603 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Do the Lucas analogue of B-602(b).

Solution by C. Georguiou, University of Patras, Greece

For $H_n = L_n$, we have $L_{n+1}^2 - L_nL_{n+2} = 5(-1)^{n+1}$, and since $L_n = 0(\alpha^n)$ it follows that

$$\sum_{n=1}^{\infty} \frac{5(-1)^{n+1}}{L_nL_{n+1}L_{n+2}} = 10\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}L_{2n+1}L_{2n+2}}$$

On the other hand, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{L_n} - \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}}\right) = -\sum_{n=1}^{\infty}\frac{1}{L_n} + \frac{2}{L_1} + \frac{1}{L_2}.$$

By equating the two sums, we get

$$\sum_{n=1}^{\infty}\frac{1}{L_n} = \frac{7}{3} - 10\sum_{n=1}^{\infty}\frac{1}{L_{2n-1}L_{2n+1}L_{2n+2}}.$$

Also solved by Piero Filipponi, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Tzermias, and the proposer.

◆◇◆◇◆