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Frequency Analysis of Extreme Events Using the Univariate Beta Family Probability Distributions

Cornel Ilinca * **and Cristian Gabriel Anghel**

Faculty of Hydrotechnics, Technical University of Civil Engineering Bucharest, Lacul Tei, nr. 122–124, 020396 Bucharest, Romania; cristian.anghel@utcb.ro
* Correspondence: cornel@utcb.ro; Tel.: +40-723-071-247

Abstract: This manuscript presents three families of distributions, namely the Beta, Beta Prime and Beta Exponential families of distributions. From all the distributions of these families, 14 statistical distributions of three, four and five parameters are presented that have applicability in the analysis of extreme phenomena in hydrology. These families of distributions were analyzed regarding the improvement of the existing legislation for the determination of extreme events, specifically the elaboration of a norm regarding frequency analysis in hydrology. To estimate the parameters of the analyzed distributions, the method of ordinary moments and the method of linear moments were used; the latter conforms to the current trend for estimating the parameters of statistical distributions. The main purpose of the manuscript was to identify other distributions from these three families with applicability in flood frequency analysis compared to the distributions already used in the literature from these families, such as the Log–logistic distribution, the Dagum distribution and the Kumaraswamy distribution. The manuscript does not exclude the applicability of other distributions from other families in the frequency analysis of extreme values, especially since these families were also analyzed within the research carried out in the Faculty of Hydrotechnics and presented in other materials. All the necessary elements for their use are presented, including the probability density functions, the complementary cumulative distribution functions, the quantile functions and the exact and approximate relations for estimating parameters. A flood frequency analysis case study was carried out for the Prigor RiverRiver, to numerically present the proposed distributions. The performance of this distributions were evaluated using the relative mean error, the relative absolute error and the L-skewness–L-kurtosis diagram. The best fit distributions are the Kumaraswamy, the Generalized Beta Exponential and the Generalized Beta distributions, which presented a stability related to both the length of the data and the presence of outliers.

Keywords: frequency estimates; frequency analysis; extreme value statistics; beta; Pearson VI; Pearson XII; Lomax; estimation parameters; approximate form; method of ordinary moments; method of linear moments

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1. Introduction

An important part of the study of extreme events in hydrology involves frequency analysis. Hydrological frequency analysis is important because it determines extreme values with certain exceeding probabilities; these have a defining role in the design of dams and in water management [1–5].

Analyzing the international literature [6–12], the most frequently used distributions in flood frequency analysis are the distributions from the Gamma family (Pearson III, Generalized Gamma, Log Pearson), and the distributions from the GEV family (Weibull, Gumbel, Fréchet). From the Beta distribution family, the most used distributions for flood frequency analysis are the Log–logistic distribution [11,12], the Dagum distribution [13], the Burr distribution [14], the Kumaraswamy distribution [15].

In hydrological frequency analysis, to estimate the parameters of the distributions, the method of ordinary moments (MOM) and three-parameter distributions are used, as well as the method of linear moments (L-moments) using distributions chosen based on the L-skewness (τ_3)–L-kurtosis (τ_4) variation criterion, recommended for distributions of at least four parameters. These parameter estimation methods are two of the most frequently used parameter estimation methods in hydrology, which is the reason for presenting only these two [3,6,11,16].

In the case of MOM estimation, it is recommended to use three-parameter distributions because they can only calibrate moments up to order 3, namely the skewness that represents the $m_3/m_2^{1.5}$ ratio. In the case of series smaller than 100 values, the skewness requires correction or can be chosen, as is the hydrological practice in Romania, depending on the origin of the extreme values [2–5]. For moments of higher order, namely kurtosis (m_4/m_2^2), this cannot be corrected or chosen for small series, because the distributions would generate in some cases unrealistic values, especially in the area of small exceeding probabilities where there is no observed data [3,17–19].

In the case of parameter estimation with L-moments, it is recommended to use distributions that have τ_3 and τ_4 natural values and that are very close to those of the observed data; distributions of at least four parameters are recommended, because the method requires the calibration of moments up to the fourth order. An advantage of using the L-moments method is that the method is more stable, being less affected by small data lengths, although in some cases it requires a certain correction. The correction of the statistical indicators obtained based on the L-moments method can be done using the least squares method. Taking into account the distributions from the Beta family applied in the literature regarding the determination of maximum flows [11–15], one of the objectives of this manuscript is the analysis of the applicability of other distributions belonging to the same family of distributions.

Thus, this manuscript presents 14 statistical distributions of three, four and five parameters that are part of the families of Beta, Beta Prime and Beta Exponential distributions, namely the Generalized Beta distribution of five parameters (BG5), the Generalized Beta distribution of four parameters (BG4), the Kumaraswamy distribution (KUM4), the Pearson XII distribution (PXII), the Five-parameter Generalized Beta Prime distribution (BPG5), the Pearson VI distribution (PVI), the Lomax distribution (LMX), the Log–logistic distribution (LL), the Dagum distribution (DG), the four-parameter Burr distribution (BR4), the Paralogistic distribution (PR), the Inverse Paralogistic distribution (IPR), the Beta Exponential distribution (BE), and the Exponential Exponentiated (EE) distribution [11–15]. The estimation methods of the parameters of these distributions are MOM and L-moments, with the latter being necessary to solve some nonlinear systems of equations, which leads to some difficulties in using these distributions. Thus, for the ease of application, parameter approximation relations are presented for some of these distributions using polynomial [20], exponential or rational functions. All the mathematical elements necessary to use these distributions in the analysis of extreme events, especially in flood frequency analysis, are presented here.

In this manuscript, new elements such as the expressions of the cumulative complementary functions; the inverse functions for BG5, BG4, PXII, BPG5, PVI, LMX, BE, EE; the approximation relations for parameter estimation for PXII, PR, IPR, EE; and the relations for parameter estimation with MOM for LMX, PR, IPR, facilitate the ease of using these distributions in flood frequency analysis. Some of the quantiles of the analyzed distributions do not have explicit forms; they are represented in this manuscript with the help of the predefined function from Mathcad, which is equivalent to other functions from other dedicated programs (e.g., the Beta.Inv function from Excel, etc.).

Thus, all these new elements for these distributions presented in Table 1 will help hydrology researchers to use these distributions easily.

Table 1. Novelty elements.

New Elements	Distribution
Inverse function	BG5, BG4, PXII, BPG5, PVI, LMX, PR, IPR, BE, EE
Complementary cumulative distribution function	PXII, PVI, PMX, PR, IPR, EE
The characteristic function which generates moments	BG5, BPG5, BR4,
Approximate estimate of the parameters	PXII, LMX, LL, PR, IPR, EE
The exact estimate of the parameters	PXII, PVI, LMX, PR, IPR

This is the first time that the BG5, BPG5, LMX, PR, IPR, BE and EE distributions are used in flood frequency analysis.

The raw and central moments of the analyzed distributions were determined using the methodology presented in the Supplementary Materials, based on the probability density functions.

In the case of estimation with L-moments, the determination of L-moments uses the substitution methodology (variable change) using the expression of the inverse function. This parameter estimation method is based exclusively on the inverse function of the distribution; the presentation of the expressions of the inverse functions of some distributions that have not been presented so far are novel elements, and are essential in the application of distributions using the L-moments method.

In order to verify the performances of the proposed distributions, a flood frequency analysis was carried out, using the Prigor River as a case study. All results are presented in comparison with the Pearson III distribution, which is the parent distribution in flood frequency analysis in Romania [2–5,21].

Comparing the results and choosing the best distribution was based on the performance indicators: relative mean error (RME) [22], relative absolute error (RAE) [22] and L-skewness–L-kurtosis diagrams [3,8,11].

The manuscript is organized as follows: The description of the statistical distributions by presenting the density function, the complementary cumulative function and the quantile function, is given in Section 2.1. The presentation of the relations for exact calculation and the approximate relations for determining the parameters of the distributions is given in Section 2.2. A case study applying these distributions to flood frequency analysis for the Prigor River is presented in Section 3. Results, discussions and conclusions are presented in Sections 3–5, respectively.

2. Methods

This section presents 14 probability distributions from Beta families with applicability in flood frequency analysis. All the mathematical elements necessary for the use of these distributions are presented.

The parameter estimation methods are the method of ordinary moments and the method of linear moments. Both the exact equations and the approximate relations for estimating the parameters of the distributions are presented. To estimate the parameters using the L-moments method, in the case of distributions where the inverse function does not have a close form and depends on more than two parameters, the estimation of the distribution parameters is carried out numerically using the Gaussian Quadrature method [23–25].

Figure 1 shows the membership of each analyzed distribution relative to the Beta family of which it is a part [26].

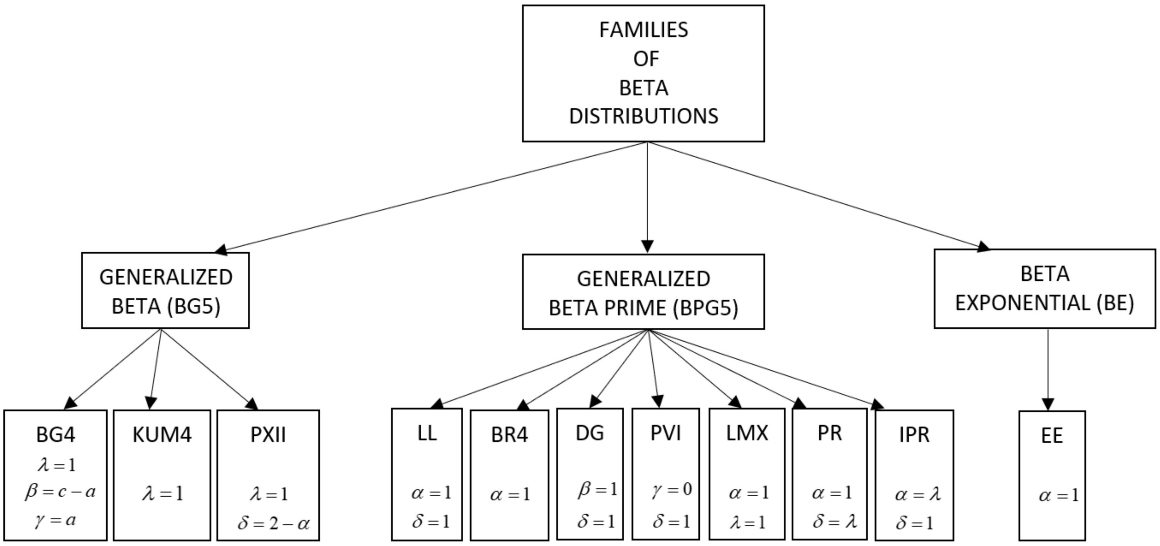


Figure 1. The relationships of Beta distribution families with other distributions.

The determination of the maximum flows was carried out in stages according to Figure 2. The verification of the character of outliers, normality and homogeneity were carried out in the data curation phase.

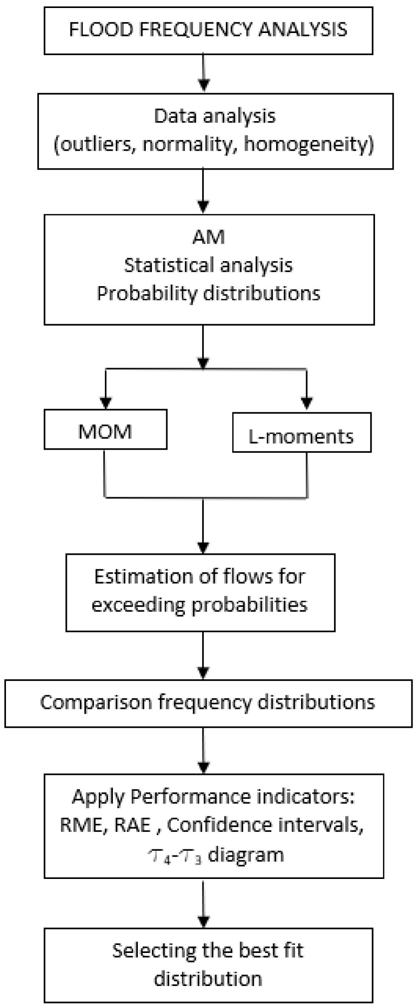


Figure 2. Methodological approach.

2.1. Probability Distributions

Table 2 presents the probability density function, $f(x)$, the complementary cumulative distribution function, $F(x)$, and quantile function, $x(p)$, for the analyzed distributions. All $F(x)$ and $x(p)$ of the analyzed distributions were determined using the methodology presented in the Supplementary Materials, using only $f(x)$ [11–15,26].

Table 2. The analyzed probability distributions.

Distr.	$f(x)$	$F(x)$	$x(p)$
BG5	$\frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)\cdot\Gamma(\delta)}\cdot\left\lvert\frac{\lambda}{\beta}\right\rvert\cdot\left(\frac{x-\gamma}{\beta}\right)^{\alpha\cdot\lambda-1}\cdot\left(1-\left(\frac{x-\gamma}{\beta}\right)^{\lambda}\right)^{\delta-1}$	$ibeta\left(\left(\frac{x-\gamma}{\beta}\right)^{\lambda},\alpha,\delta\right)$	$\gamma+\beta\cdot qbeta(1-p,\alpha,\delta)^{\frac{1}{\lambda}}$
BG4	$\frac{\Gamma(\alpha+\beta)}{(c-a)\cdot\Gamma(\alpha)\cdot\Gamma(\beta)}\cdot\left(1-\frac{x-a}{c-a}\right)^{\beta-1}\cdot\left(\frac{x-a}{c-a}\right)^{\alpha-1}$	$1-ibeta\left(\frac{x-a}{c-a},\alpha,\beta\right)$	$a+(c-a)\cdot qbeta(1-p,\alpha,\beta)$
Kum4	$\frac{\alpha\cdot\beta\cdot\left(\frac{x-\gamma}{\lambda-\gamma}\right)^{\alpha-1}\cdot\left(1-\left(\frac{x-\gamma}{\lambda-\gamma}\right)^{\alpha}\right)^{\beta-1}}{\lambda-\gamma}$	$1-\left(1-\left(\frac{x-\gamma}{\lambda-\gamma}\right)^{\alpha}\right)^{\beta}$	$(\lambda-\gamma)\cdot\left(1-p^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}+\gamma$
PXII	$\frac{\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\cdot\left(1-\frac{x-\gamma}{\beta}\right)^{1-\alpha}}{\Gamma(\alpha)\cdot\Gamma(2-\alpha)\cdot\lvert\beta\rvert}$	$1-ibeta\left(\frac{x-\gamma}{\beta},\alpha,2-\alpha\right)$	$\gamma+\beta\cdot\ln(qbeta(1-p,\alpha,2-\alpha))$
BPG5	$\frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)\cdot\Gamma(\delta)}\cdot\left\lvert\frac{\lambda}{\beta}\right\rvert\cdot\left(\frac{x-\gamma}{\beta}\right)^{\alpha\cdot\lambda-1}\cdot\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\lambda}\right)^{-\delta-\alpha}$	$ibeta\left(\frac{1}{1+\left(\frac{x-\gamma}{\beta}\right)^{-\lambda}},\alpha,\delta\right)$	$\gamma+\beta\cdot\left(\frac{qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)}\right)^{\frac{1}{\lambda}}$
PVI	$\frac{\Gamma(\alpha+\lambda)}{\beta\cdot\Gamma(\alpha)\cdot\Gamma(\lambda)}\cdot\frac{\left(1+\frac{x}{\beta}\right)^{-(\alpha+\lambda)}}{\left(\frac{x}{\beta}\right)^{1-\alpha}}$	$1-ibeta\left(\frac{\frac{x}{\beta}}{1+\frac{x}{\beta}},\alpha,\lambda\right)$	$\frac{\beta\cdot qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)}$
LMX	$\frac{\left(\frac{x-\gamma}{\beta}\right)^{-1}\cdot\left(1+\frac{x-\gamma}{\beta}\right)^{-1-\lambda}}{\lambda\cdot\beta}$	$1-ibeta\left(\frac{1}{1+\left(\frac{x-\gamma}{\beta}\right)^{-1}},1,\lambda\right)$	$\frac{(\gamma+\beta)\cdot qbeta(1-p,1,\lambda)-\gamma}{qbeta(1-p,1,\lambda)-1}$
LL	$\frac{\alpha\cdot\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\cdot\left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha}+1\right)^{-2}}{\beta}$	$\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-1}$	$\gamma+\beta\cdot\left(\frac{1}{p}-1\right)^{\frac{1}{\alpha}}$
DG	$\frac{\alpha\cdot\gamma\cdot\left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta\cdot\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{\gamma+1}}$	$1-\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{-\gamma}$	$\beta\cdot\left((1-p)^{-\frac{1}{\gamma}}-1\right)^{-\frac{1}{\alpha}}$
BR4	$\frac{\frac{\beta\cdot\alpha}{x-\gamma}\cdot\left(\frac{\lambda}{x-\gamma}\right)^{\beta}}{\left(1+\left(\frac{\lambda}{x-\gamma}\right)^{\beta}\right)^{\alpha+1}}$	$1-\frac{1}{\left(1+\left(\frac{\lambda}{x-\gamma}\right)^{\beta}\right)^{\alpha}}$	$\gamma+\lambda\cdot\left(\frac{1}{\left(\frac{1}{1-p}\right)^{\frac{1}{\alpha}}-1}\right)^{\frac{1}{\beta}}$
PR	$\frac{\alpha^{2}\cdot\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}}{\lvert\beta\rvert\cdot\left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha}+1\right)^{\alpha+1}}$	$\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-\alpha}$	$\gamma+\beta\cdot\left(p^{-\frac{1}{\alpha}}-1\right)^{\frac{1}{\alpha}}$
IPR	$\frac{\alpha^{2}\cdot\left(\frac{x-\gamma}{\beta}\right)^{\alpha^{2}-1}}{\lvert\beta\rvert\cdot\left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha}+1\right)^{\alpha+1}}$	$1-\frac{\left(\frac{x-\gamma}{\beta}\right)^{\alpha^{2}}}{\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{\alpha}}$	$\gamma+\beta\cdot\left(\frac{(1-p)^{\frac{1}{\alpha}}}{1-(1-p)^{\frac{1}{\alpha}}}\right)^{\frac{1}{\alpha}}$
BE	$\frac{\Gamma(\alpha+\lambda)}{\beta\cdot\Gamma(\alpha)\cdot\Gamma(\lambda)}\cdot\exp\left(-\alpha\cdot\frac{x-\gamma}{\beta}\right)\cdot\left(1-\exp\left(-\frac{x-\gamma}{\beta}\right)\right)^{\lambda-1}$	$1-ibeta\left(\exp\left(-\frac{x-\gamma}{\beta}\right),\alpha,\lambda\right)$	$\gamma-\beta\cdot\ln(qbeta(p,\alpha,\lambda))$
EE	$\frac{\lambda}{\beta}\cdot\exp\left(-\frac{x-\gamma}{\beta}\right)\cdot\left(1-\exp\left(-\frac{x-\gamma}{\beta}\right)\right)^{\lambda-1}$	$1-ibeta\left(\exp\left(-\frac{x-\gamma}{\beta}\right),1,\lambda\right)$	$\gamma-\beta\cdot\ln(qbeta(p,1,\lambda))$

where $\Gamma(x)$ returns the value of the Euler gamma function of x; $\Gamma(a,x)$ returns the value of the incomplete gamma function of x with parameter a; $ibeta(a,x,y)$ returns the value of the incomplete beta function of x and y with parameter a; and $qbeta(p,s_1,s_2)$ returns the inverse cumulative probability distribution for probability p, for beta distribution. All predefined functions are presented in Appendix D.

2.2. Parameter Estimation

The parameter estimation of the analyzed statistical distributions is presented for MOM and L-moments, two of the most used methods in hydrology for parameter estimation [3,9,11,16].

2.2.1. Beta Generalized by Five Parameters (BG5)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + \beta \cdot \frac{\Gamma\left(\alpha + \frac{1}{\lambda}\right)}{\Gamma\left(\alpha + \frac{1}{\lambda} + \delta\right)} \cdot \frac{\Gamma(\alpha + \delta)}{\Gamma(\alpha)} \tag{1}$$

$$\sigma^2 = \beta^2 \cdot \frac{\Gamma(\alpha + \delta)}{\Gamma(\alpha)} \cdot \left(\frac{\Gamma\left(\alpha + \frac{2}{\lambda}\right)}{\Gamma\left(\alpha + \frac{2}{\lambda} + \delta\right)} - \frac{\Gamma\left(\alpha + \frac{1}{\lambda}\right)^2}{\Gamma\left(\alpha + \frac{1}{\lambda} + \delta\right)^2} \cdot \frac{\Gamma(\alpha + \delta)}{\Gamma(\alpha)} \right) \tag{2}$$

where μ, σ^2 represents the expected value and variance.

Because they are too long, the relations for estimating skewness and kurtosis are presented in Appendix C.

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

$$L_1 = \gamma + \beta \cdot \frac{\Gamma\left(\alpha + \frac{1}{\lambda}\right)}{\Gamma\left(\alpha + \frac{1}{\lambda} + \delta\right)} \cdot \frac{\Gamma(\alpha + \delta)}{\Gamma(\alpha)} \tag{3}$$

$$L_2 = \int_0^1 \left(\beta \cdot qbeta(1 - p, \alpha, \delta)^{\frac{1}{\lambda}}\right) \cdot (1 - 2 \cdot p) \cdot dp \tag{4}$$

$$L_3 = \int_0^1 \left(\beta \cdot qbeta(1 - p, \alpha, \delta)^{\frac{1}{\lambda}}\right) \cdot \left(6 \cdot p^2 - 6 \cdot p + 1\right) \cdot dp \tag{5}$$

$$L_4 = \int_0^1 \left(\beta \cdot qbeta(1 - p, \alpha, \delta)^{\frac{1}{\lambda}}\right) \cdot \left(1 - 20 \cdot p^3 + 30 \cdot p^2 - 12 \cdot p\right) \cdot dp \tag{6}$$

where L_1, L_2, L_3, L_4 represents the first four linear moments.

2.2.2. Beta Generalized by Four Parameters (BG4)

The equations needed to estimate the parameters with MOM have the following expressions [26]:

$$\mu = \frac{\alpha \cdot c + \beta \cdot a}{\alpha + \beta} \tag{7}$$

$$\sigma^2 = \left(\frac{c - a}{\alpha + \beta}\right)^2 \cdot \frac{\alpha \cdot \beta}{\alpha + \beta + 1} = \frac{(\mu - a) \cdot (c - \mu)}{7} \tag{8}$$

$$C_s = \frac{2 \cdot (\beta - \alpha)}{\alpha + \beta + 2} \cdot \sqrt{\frac{\alpha + \beta + 1}{\alpha \cdot \beta}} \tag{9}$$

$$C_k = \frac{3 \cdot (\beta + \alpha + 1) \cdot \left[2 \cdot (\alpha + \beta)^2 + \alpha \cdot \beta \cdot (\alpha + \beta - 6)\right]}{\alpha \cdot \beta \cdot (\beta + \alpha + 2) \cdot (\beta + \alpha + 3)} \tag{10}$$

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

$$L_1 = \frac{\alpha \cdot c + \beta \cdot a}{\alpha + \beta} \tag{11}$$

$$\tau_3 = \frac{\int_0^1 qbeta(1-p,\alpha,\beta)\cdot\left(6\cdot p^2-6\cdot p+1\right)\cdot dp}{\int_0^1 qbeta(1-p,\alpha,\beta)\cdot(1-2\cdot p)\cdot dp} \tag{12}$$

$$\tau_4 = \frac{\int_0^1 qbeta(1-p,\alpha,\beta)\cdot\left(1-20\cdot p^3+30\cdot p^2-12\cdot p\right)\cdot dp}{\int_0^1 qbeta(1-p,\alpha,\beta)\cdot(1-2\cdot p)\cdot dp} \tag{13}$$

where τ_3, τ_4 represents the L-skewness and L-kurtosis, respectively.

2.2.3. Kumaraswamy (KUM4)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + (\lambda-\gamma)\cdot\beta\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} \tag{14}$$

$$\sigma^2 = (\lambda-\gamma)^2\cdot\beta\cdot\Gamma(\beta)\cdot\left(\frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} - \beta\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2}\right) \tag{15}$$

Because they are too long, the relations for estimating skewness and kurtosis are presented in Appendix C.

The equations needed to estimate the parameters with L-mom have the following expressions:

$$L_1 = \gamma + B_1 \tag{16}$$

$$L_2 = B_1 - 2\cdot B_2 \tag{17}$$

$$L_3 = 6\cdot B_3 - 6\cdot B_2 + B_1 \tag{18}$$

$$L_4 = B_1 - 12\cdot B_2 + 30\cdot B_3 - 20\cdot B_4 \tag{19}$$

where $B_r = (\lambda-\gamma)\cdot\beta\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(r\cdot\beta)}{\Gamma\left(1+r\cdot\beta+\frac{1}{\alpha}\right)}; r = 1,2,3,4.$

2.2.4. Pearson XII (PXII)

The equations needed to estimate the parameters with MOM have the following expressions [26]:

$$\mu = \gamma + \frac{\beta\cdot\alpha}{2} \tag{20}$$

$$\sigma^2 = \beta^2\cdot\frac{\alpha}{6}\left(1-\frac{\alpha}{2}\right) \tag{21}$$

$$C_s = \frac{2\cdot\alpha - 3\cdot\alpha^2 + \alpha^3}{24\cdot\left(\frac{\alpha}{6}\cdot\left(1-\frac{\alpha}{2}\right)\right)^{1.5}} \tag{22}$$

The shape parameter can be obtained approximately depending on the skewness coefficient, using the following rational function ($0.1 < C_s \leq 9$):

$$\alpha = \frac{1.0003631 + 0.2101096 \cdot C_s + 0.0136862 \cdot C_s^2 - 0.0007348 \cdot C_s^3}{1 + 0.7921435 \cdot C_s + 0.4505945 \cdot C_s^2 + 0.2041907 \cdot C_s^3} \quad (23)$$

$$\beta = \frac{\sigma}{\sqrt{\frac{\alpha}{6} \cdot \left(1 - \frac{\alpha}{2}\right)}} \quad (24)$$

$$\gamma = \mu - \frac{\beta \cdot \alpha}{2} \quad (25)$$

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

An approximate form can be adopted based on the parameter estimation depending on L-skewness ($0 < \tau_3 \leq 1$), as follows:

$$\alpha = 1.000119258 - 1.999964052 \cdot \tau_3 - 0.110940785 \cdot \tau_3^2 + 4.915113247 \cdot \tau_3^3 - 7.287825998 \cdot \tau_3^4 + 4.545489583 \cdot \tau_3^5 - 1.059896657 \cdot \tau_3^6 \quad (26)$$

$$\beta = \frac{L_2}{z} \quad (27)$$

$$\gamma = L_1 - \frac{\beta \cdot \alpha}{2} \quad (28)$$

where $z = \int_0^1 qbeta(1-p, \alpha, 2-\alpha) \cdot (1 - 2 \cdot p) \cdot dp$, which can be approximated with the following equation:

$$z = 0.000068524 + 0.495907748 \cdot \alpha - 0.768887496 \cdot \alpha^2 + 1.02025564 \cdot \alpha^3 - 1.043420944 \cdot \alpha^4 + 0.624074705 \cdot \alpha^5 - 0.161412167 \cdot \alpha^6 \quad (29)$$

2.2.5. Beta Prime Generalized by Five Parameters (BPG5)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + \beta \cdot \frac{\Gamma\left(\alpha + \frac{1}{\lambda}\right) \cdot \Gamma\left(\delta - \frac{1}{\lambda}\right)}{\Gamma(\alpha) \cdot \Gamma(\delta)} \quad (30)$$

$$\sigma^2 = \frac{\beta^2}{\Gamma(\alpha) \cdot \Gamma(\delta)} \cdot \left(\Gamma\left(\alpha + \frac{2}{\lambda}\right) \cdot \Gamma\left(\delta - \frac{2}{\lambda}\right) - \frac{\Gamma\left(\alpha + \frac{1}{\lambda}\right)^2 \cdot \Gamma\left(\delta - \frac{1}{\lambda}\right)^2}{\Gamma(\alpha) \cdot \Gamma(\delta)}\right) \quad (31)$$

Because they are too long, the relations for estimating skewness and kurtosis are presented in Appendix C.

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

$$L_1 = \gamma + \beta \cdot \frac{\Gamma\left(\alpha + \frac{1}{\lambda}\right) \cdot \Gamma\left(\delta - \frac{1}{\lambda}\right)}{\Gamma(\alpha) \cdot \Gamma(\delta)} \quad (32)$$

$$\tau_3 = \frac{\int_0^1 \left(\frac{qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)}\right)^{\frac{1}{\lambda}} \cdot \left(6 \cdot p^2 - 6 \cdot p + 1\right) \cdot dp}{\int_0^1 \left(\frac{qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)}\right)^{\frac{1}{\lambda}} \cdot (1 - 2 \cdot p) \cdot dp} \quad (33)$$

$$\tau_4 = \frac{\int_0^1 \left(\frac{qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)}\right)^{\frac{1}{\lambda}} \cdot \left(1 - 20 \cdot p^3 + 30 \cdot p^2 - 12 \cdot p\right) \cdot dp}{\int_0^1 \left(\frac{qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)}\right)^{\frac{1}{\lambda}} \cdot (1 - 2 \cdot p) \cdot dp} \tag{34}$$

2.2.6. Pearson VI (PVI)

The equations needed to estimate the parameters with MOM have the following expressions [26]:

$$\mu = \frac{\beta \cdot \alpha}{\lambda - 1} \tag{35}$$

$$\sigma^2 = \frac{\beta^2 \cdot \alpha \cdot (\alpha + \lambda - 1)}{(\lambda - 1)^2 \cdot (\lambda - 2)} \tag{36}$$

$$C_s = \left(\frac{\lambda - 2}{\alpha \cdot (\alpha + \lambda - 1)}\right)^{0.5} \cdot \frac{2 \cdot (2 \cdot \alpha + \lambda - 1)}{\lambda - 3} \tag{37}$$

$$C_k = \frac{3 \cdot (\lambda - 2)}{(\lambda - 3) \cdot (\lambda - 4)} \cdot \left(\frac{2 \cdot (\lambda - 1)^2}{\alpha \cdot (\alpha + \lambda - 1)} + \lambda + 5\right) \tag{38}$$

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

$$\tau_2 = \frac{(\lambda - 1)}{\alpha} \cdot \int_0^1 \frac{qbeta(1-p,\alpha,\lambda)}{1 - qbeta(1-p,\alpha,\lambda)} \cdot (1 - 2 \cdot p) \cdot dp \tag{39}$$

$$\tau_3 = \frac{\int_0^1 \frac{qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)} \cdot \left(6 \cdot p^2 - 6 \cdot p + 1\right) \cdot dp}{\int_0^1 \frac{qbeta(1-p,\alpha,\lambda)}{1-qbeta(1-p,\alpha,\lambda)} \cdot (1 - 2 \cdot p) \cdot dp} \tag{40}$$

$$\beta = \frac{L_1 \cdot (\lambda - 1)}{\alpha} \tag{41}$$

2.2.7. Lomax (LMX)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + \frac{\beta}{\lambda - 1} \tag{42}$$

$$\sigma^2 = \frac{\beta^2 \cdot \lambda}{(\lambda - 2) \cdot (\lambda - 1)} \tag{43}$$

$$C_s = \frac{2 \cdot \lambda \cdot (\lambda + 1)}{(\lambda - 3) \cdot (\lambda - 2) \cdot (\lambda - 1)^3 \cdot \left(\frac{\lambda}{(\lambda-2)\cdot(\lambda-1)^2}\right)^{1.5}} \tag{44}$$

The parameter λ can be obtained approximately depending on the skewness coefficient, using the following rational functions ($2 < C_s \leq 9$):

$$\lambda = \frac{3.9284194 - 3.0914822 \cdot C_s + 6.9762716 \cdot C_s^2}{1 - 5.2706252 \cdot C_s + 2.385356 \cdot C_s^2} \tag{45}$$

$$\beta = \sigma \cdot (\lambda - 1) \cdot \sqrt{1 - \frac{\lambda}{2}} \tag{46}$$

$$\gamma = \mu - \frac{\beta}{\lambda - 1} \tag{47}$$

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

$$L_1 = \gamma + \frac{\beta}{\lambda - 1} \tag{48}$$

$$L_2 = \int_0^1 \frac{(\gamma + \beta) \cdot qbeta(1 - p, 1, \lambda) - \gamma}{qbeta(1 - p, 1, \lambda) - 1} \cdot (1 - 2 \cdot p) \cdot dp \tag{49}$$

$$L_3 = \int_0^1 \frac{(\gamma + \beta) \cdot qbeta(1 - p, 1, \lambda) - \gamma}{qbeta(1 - p, 1, \lambda) - 1} \cdot \left(6 \cdot p^2 - 6 \cdot p + 1\right) \cdot dp \tag{50}$$

An approximate form can be adopted based on the parameter estimation depending on L-skewness, as follows ($\tau_3 > 0.35$):

$$\lambda = \frac{7.7817436 + 257.3810019 \cdot \tau_3 + 330.3408356 \cdot \tau_3^2}{1 - 296.9737484 \cdot \tau_3 + 882.1944808 \cdot \tau_3^2} \tag{51}$$

$$\beta = \frac{L_2}{z} \tag{52}$$

$$\gamma = L_1 - \frac{\beta}{\lambda - 1} \tag{53}$$

where the exact form of the z parameter is:

$$z = -\int_0^1 \frac{qbeta(1 - p, 1, \lambda)}{qbeta(1 - p, 1, \lambda) - 1} \cdot (1 - 2 \cdot p) \cdot dp \tag{54}$$

or a simplified form can be adopted using a rational function:

$$z = \frac{-0.3686598 + 0.4493708 \cdot \lambda - 0.0010038 \cdot \lambda^2}{1 - 1.8121107 \cdot \lambda + 0.8536406 \cdot \lambda^2} \tag{55}$$

2.2.8. Log–Logistic (LL)

The equations needed to estimate the parameters with MOM have the following expressions [11,26,27]:

$$\mu = \gamma + \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \tag{56}$$

$$\sigma^2 = \beta^2 \cdot \left(\Gamma\left(1 + \frac{2}{\alpha}\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right)^2 \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)^2\right) \tag{57}$$

Because it is too long, the relationship for estimating skewness is presented in Appendix C.

The shape parameter can be obtained approximately depending on the skewness coefficient, using the following rational function:

$$\alpha = \exp\begin{pmatrix} 2.2464005 - 0.8505518 \cdot \ln(C_s) + 0.1224968 \cdot \ln(C_s)^2 + \\ 0.0509751 \cdot \ln(C_s)^3 + 0.0033792 \cdot \ln(C_s)^4 - 0.0066062 \cdot \ln(C_s)^5 - \\ 0.0021326 \cdot \ln(C_s)^6 + 0.0002985 \cdot \ln(C_s)^7 + 0.0002292 \cdot \ln(C_s)^8 + 0.0000263 \cdot \ln(C_s)^9 \end{pmatrix} \quad (58)$$

$$\beta = \frac{\sigma}{\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right) \cdot \Gamma\left(1-\frac{2}{\alpha}\right) - \Gamma\left(1+\frac{1}{\alpha}\right)^2 \cdot \Gamma\left(1-\frac{1}{\alpha}\right)^2}} \quad (59)$$

$$\gamma = \mu - \frac{\beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \quad (60)$$

The equations needed to estimate the parameters with L-moments have the following expressions:

$$\alpha = \frac{1}{\tau_3} = \frac{L_2}{L_3} \quad (61)$$

$$\beta = \frac{\alpha^2 \cdot L_2}{\Gamma\left(\frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)} \quad (62)$$

$$\gamma = L_1 - \frac{\beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \quad (63)$$

2.2.9. Dagum (DG)

The equations needed to estimate the parameters with MOM have the following expressions [13,26,28]:

$$\mu = \beta \cdot \frac{\Gamma\left(\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\gamma)} \quad (64)$$

$$\sigma^2 = \frac{\beta^2}{\Gamma(\gamma)} \cdot \left(\Gamma\left(\gamma + \frac{2}{\alpha}\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) - \frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)^2 \cdot \Gamma\left(1-\frac{1}{\alpha}\right)^2}{\Gamma(\gamma)} \right) \quad (65)$$

Because they are too long, the relations for estimating skewness and kurtosis are presented in Appendix C.

The equations needed to estimate the parameters with L-moments have the following expressions:

$$L_1 = \beta \cdot \frac{\Gamma\left(\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\gamma + 1)} \quad (66)$$

$$L_2 = \gamma \cdot \beta \cdot \left(2 \cdot \frac{\Gamma\left(2 \cdot \gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \gamma + 1)} - \frac{\Gamma\left(\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\gamma + 1)}\right) \quad (67)$$

$$L_3 = \gamma \cdot \beta \cdot \left(6 \cdot \frac{\Gamma\left(3 \cdot \gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(3 \cdot \gamma + 1)} - 6 \cdot \frac{\Gamma\left(2 \cdot \gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \gamma + 1)} + \frac{\Gamma\left(\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\gamma + 1)}\right) \quad (68)$$

2.2.10. Burr of Four Parameters (BR4)

The equations needed to estimate the parameters with MOM have the following expressions [14,26]:

$$\mu = \gamma + \lambda \cdot \frac{\Gamma\left(1 - \frac{1}{\beta}\right) \cdot \Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)} \quad (69)$$

$$\sigma^2 = \frac{\lambda^2}{\Gamma(\alpha)} \cdot \left(\Gamma\left(1 - \frac{2}{\beta}\right) \cdot \Gamma\left(\alpha + \frac{2}{\beta}\right) - \frac{\Gamma\left(1-\frac{1}{\beta}\right)^2 \cdot \Gamma\left(\alpha+\frac{1}{\beta}\right)^2}{\Gamma(\alpha)} \right) \tag{70}$$

Because they are too long, the relations for estimating skewness and kurtosis are presented in Appendix C.

The equations needed to estimate the parameters with L-moments have the following expressions:

$$L_1 = \gamma + \frac{\lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)} \tag{71}$$

$$L_2 = \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)} - \frac{\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)} \right) \tag{72}$$

$$L_3 = \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{2 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(3 \cdot \alpha)} - \frac{3 \cdot \Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)} + \frac{\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)} \right) \tag{73}$$

$$L_4 = \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{5 \cdot \Gamma\left(4 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(4 \cdot \alpha)} - \frac{10 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(3 \cdot \alpha)} + \frac{6 \cdot \Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)} - \frac{\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)} \right) \tag{74}$$

2.2.11. Paralogistic (PR)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + \beta \cdot \frac{\Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{75}$$

$$\sigma^2 = \frac{\beta^2}{\Gamma(\alpha)} \cdot \left(\Gamma\left(1 + \frac{2}{\alpha}\right) \cdot \Gamma\left(\alpha - \frac{2}{\alpha}\right) - \frac{\Gamma\left(1 + \frac{1}{\alpha}\right)^2 \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)^2}{\Gamma(\alpha)} \right) \tag{76}$$

Because it is too long, the relationship for estimating skewness is presented in Appendix C.

The parameter α can be obtained approximately depending on the skewness coefficient, using the following rational functions ($0 < C_s \leq 4$):

$$\alpha = \frac{5.185064722 + 14.441621221 \cdot C_s + 7.41534611 \cdot C_s^2 + 3.251478763 \cdot C_s^3}{1 + 3.712917717 \cdot C_s + 3.852064136 \cdot C_s^2 + 1.822992333 \cdot C_s^3} \tag{77}$$

$$\beta = \frac{\sigma}{\sqrt{\frac{1}{\Gamma(\alpha)} \cdot \left(\Gamma\left(1 + \frac{2}{\alpha}\right) \cdot \Gamma\left(\alpha - \frac{2}{\alpha}\right) - \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2 \cdot \Gamma\left(\alpha-\frac{1}{\alpha}\right)^2}{\Gamma(\alpha)} \right)}} \tag{78}$$

$$\gamma = \mu - \beta \cdot \frac{\Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{79}$$

The equations needed to estimate the parameters with L-moments have the following expressions:

$$L_1 = \gamma + \frac{\beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{80}$$

$$L_2 = \beta \cdot \Gamma\left(1+\frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)} - \frac{\Gamma\left(2\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}\right) \tag{81}$$

$$L_3 = \beta \cdot \Gamma\left(1+\frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)} + \frac{2\cdot\Gamma\left(3\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)} - \frac{3\cdot\Gamma\left(2\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}\right) \tag{82}$$

An approximate form can be adopted based on the parameter estimation depending on L-skewness, as follows:

$$\alpha = \exp\left(\begin{array}{l} 0.000006321 - 0.499775727 \cdot \ln(\tau_3) + 0.126690856 \cdot \ln(\tau_3)^2 + \\ 0.067638333 \cdot \ln(\tau_3)^3 + 0.002255123 \cdot \ln(\tau_3)^4 - 0.00279206 \cdot \ln(\tau_3)^5 - \\ 0.000477164 \cdot \ln(\tau_3)^6 - 0.000016444 \cdot \ln(\tau_3)^7 + 0.000001032 \cdot \ln(\tau_3)^8 \end{array}\right) \tag{83}$$

$$\beta = \frac{L_2}{\Gamma\left(1+\frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)} - \frac{\Gamma\left(2\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}\right)} \tag{84}$$

$$\gamma = L_1 - \frac{\beta \cdot \Gamma\left(1+\frac{1}{\alpha}\right) \cdot \Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{85}$$

2.2.12. Inverse Paralogistic (IPR)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + \beta \cdot \frac{\Gamma\left(\alpha+\frac{1}{\alpha}\right) \cdot \Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{86}$$

$$\sigma^2 = \frac{\beta^2}{\Gamma(\alpha)} \cdot \left(\Gamma\left(1-\frac{2}{\alpha}\right) \cdot \Gamma\left(\alpha+\frac{2}{\alpha}\right) - \frac{\Gamma\left(1-\frac{1}{\alpha}\right)^2 \cdot \Gamma\left(\alpha+\frac{1}{\alpha}\right)^2}{\Gamma(\alpha)}\right) \tag{87}$$

Because it is too long, the relationship for estimating skewness is presented in Appendix C.

The parameter α can be obtained approximately depending on the skewness coefficient, using the following rational functions ($1.3 \leq C_s \leq 7$):

$$\alpha = \exp\left(\begin{array}{l} 9.1148085 - 48.5411515 \cdot \ln(C_s) + 178.4282739 \cdot \ln(C_s)^2 - 411.0985176 \cdot \ln(C_s)^3 + \\ 611.4661111 \cdot \ln(C_s)^4 - 596.305336 \cdot \ln(C_s)^5 + 378.4065467 \cdot \ln(C_s)^6 - \\ 150.2898201 \cdot \ln(C_s)^7 + 33.8871674 \cdot \ln(C_s)^9 - 3.3074833 \cdot \ln(C_s)^9 \end{array}\right) \tag{88}$$

$$\beta = \frac{\sigma}{\sqrt{\frac{1}{\Gamma(\alpha)} \cdot \left(\Gamma\left(1-\frac{2}{\alpha}\right) \cdot \Gamma\left(\alpha+\frac{2}{\alpha}\right) - \frac{\Gamma\left(1-\frac{1}{\alpha}\right)^2 \cdot \Gamma\left(\alpha+\frac{1}{\alpha}\right)^2}{\Gamma(\alpha)}\right)}} \tag{89}$$

$$\gamma = \mu - \beta \cdot \frac{\Gamma\left(1-\frac{1}{\alpha}\right) \cdot \Gamma\left(\alpha+\frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{90}$$

The equations needed to estimate the parameters with L-moments have the following expressions:

$$L_1 = \gamma + \frac{\beta \cdot \Gamma\left(\frac{1}{\alpha}+\alpha\right) \cdot \Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{91}$$

$$L_2 = \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\frac{1}{\alpha} + 2 \cdot \alpha\right)}{\Gamma(2 \cdot \alpha)} - \frac{\Gamma\left(\frac{1}{\alpha} + \alpha\right)}{\Gamma(\alpha)}\right) \tag{92}$$

$$L_3 = \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha + \frac{1}{\alpha}\right)}{\Gamma(\alpha)} + \frac{2 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\alpha}\right)}{\Gamma(3 \cdot \alpha)} - \frac{3 \cdot \Gamma\left(2 \cdot \alpha + \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{93}$$

An approximate form can be adopted based on the parameter estimation depending on L-skewness, as follows:

if $0.17 \leq \tau_s < 1/3$:

$$\alpha = \frac{5.07675 \cdot 10^3 - 7.55684 \cdot 10^4 \cdot \tau_3 + 5.36797 \cdot 10^5 \cdot \tau_3^2 - 1.35970 \cdot 10^6 \cdot \tau_3^3 + 1.30233 \cdot 10^6 \cdot \tau_3^4}{1 - 4.59437 \cdot 10^3 \cdot \tau_3 + 2.70167 \cdot 10^4 \cdot \tau_3^2} \tag{94}$$

if $1/3 < \tau_s < 1$:

$$\begin{aligned} \alpha = 5.37952 \cdot 10 - 4.84016 \cdot 10^2 \cdot \tau_3 + 2.02215 \cdot 10^3 \cdot \tau_3^2 - 4.79644 \cdot 10^3 \cdot \tau_3^3 + \\ 6.86370 \cdot 10^3 \cdot \tau_3^4 - 5.88301 \cdot 10^3 \cdot \tau_3^5 + 2.78592 \cdot 10^3 \cdot \tau_3^6 - 5.61108 \cdot 10^2 \cdot \tau_3^7 \end{aligned} \tag{95}$$

$$\beta = \frac{L_2}{\Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\frac{1}{\alpha} + 2 \cdot \alpha\right)}{\Gamma(2 \cdot \alpha)} - \frac{\Gamma\left(\frac{1}{\alpha} + \alpha\right)}{\Gamma(\alpha)}\right)} \tag{96}$$

$$\gamma = L_1 - \frac{\beta \cdot \Gamma\left(\frac{1}{\alpha} + \alpha\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{97}$$

2.2.13. Generalized Beta Exponential (BEG)

The equations needed to estimate the parameters with MOM have the following expressions [26]:

$$\mu = \gamma + \beta \cdot (\psi(\alpha + \lambda) - \psi(\alpha)) \tag{98}$$

$$\sigma^2 = \beta^2 \cdot \left(\frac{d}{d\alpha}\psi(\alpha) - \frac{d}{d\alpha}\psi(\alpha + \lambda)\right) \tag{99}$$

where $\psi(.)$ is the digamma function.

Because they are too long, the relations for estimating skewness and kurtosis are presented in Appendix C.

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

$$L_1 = \gamma + \beta \cdot (\psi(\alpha + \lambda) - \psi(\alpha)) \tag{100}$$

$$L_2 = \int_0^1 (\gamma - \beta \cdot \ln(qbeta(p, \alpha, \lambda))) \cdot (1 - 2 \cdot p) \cdot dp \tag{101}$$

$$L_3 = \int_0^1 (\gamma - \beta \cdot \ln(qbeta(p, \alpha, \lambda))) \cdot \left(6 \cdot p^2 - 6 \cdot p + 1\right) \cdot dp \tag{102}$$

$$L_4 = \int_0^1 (\gamma - \beta \cdot \ln(qbeta(p, \alpha, \lambda))) \cdot \left(1 - 20 \cdot p^3 + 30 \cdot p^2 - 12 \cdot p\right) \cdot dp \tag{103}$$

2.2.14. Exponential Exponentiated (EE)

The equations needed to estimate the parameters with MOM have the following expressions [26]:

$$\mu = \gamma + \beta \cdot (\psi(1+\lambda) + \gamma_e) \tag{104}$$

$$\sigma^2 = \beta^2 \cdot \left(\frac{\pi^2}{6} - \frac{d}{d\lambda}\psi(1+\lambda)\right) \tag{105}$$

$$C_s = -sign(\beta) \cdot \frac{-2 \cdot \zeta_3 - \frac{d^2}{d\lambda^2}\psi(1+\lambda)}{\left(\frac{\pi^2}{6} - \frac{d}{d\lambda}\psi(1+\lambda)\right)^{1.5}} \tag{106}$$

where $\zeta_3 = \frac{12 \cdot \sqrt{6} \cdot \sum_{i=1}^{10^6} \frac{1}{i^3}}{\pi^3} = 1.20206$ represents the Apéry constant; γ_e is the Euler constant.

The parameter λ can be obtained approximately depending on the skewness coefficient, using the following rational functions:

if $1.14 < C_s \le 3.5$:

$$\lambda = \frac{-5.7038508 + 52.1673063 \cdot C_s - 13.12714 \cdot C_s^2 + 1.3289768 \cdot C_s^3}{1 - 38.9615308 \cdot C_s + 33.4202875 \cdot C_s^2} \tag{107}$$

If $3.5 < C_s \le 6$:

$$\lambda = \frac{3.3925215 + 0.846506 \cdot C_s - 0.0917235 \cdot C_s^2 + 0.0039149 \cdot C_s^3}{1 - 2.7850046 \cdot C_s + 2.3792237 \cdot C_s^2} \tag{108}$$

$$\beta = \frac{\sigma}{\sqrt{\frac{\pi^2}{6} - \frac{d}{d\lambda}Psi(1+\lambda)}} \tag{109}$$

$$\gamma = \mu - \beta \cdot (Psi(1+\lambda) + \gamma_e) \tag{110}$$

The parameter estimation with the L-moment method is carried out numerically (definite integrals) based on the equations using the quantile of the function.

An approximate form can be adopted based on the parameter estimation depending on L-skewness, as follows:

if $0.2 < \tau_3 \le 0.5$:

$$\lambda = \exp\begin{pmatrix} -12.004042281 - 47.743332457 \cdot \ln(|\tau_3|) - 85.899138968 \cdot \ln(|\tau_3|)^2 - 80.388606858 \cdot \ln(|\tau_3|)^3 - \\ 37.480240859 \cdot \ln(|\tau_3|)^4 - 7.01919696 \cdot \ln(|\tau_3|)^5 \end{pmatrix} \tag{111}$$

if $0.5 < \tau_3 < 1$:

$$\lambda = \exp\begin{pmatrix} -5.364702604 - 34.021073341 \cdot \ln(|\tau_3|) - 188.611284247 \cdot \ln(|\tau_3|)^2 - 646.938215195 \cdot \ln(|\tau_3|)^3 - \\ 1242.386890647 \cdot \ln(|\tau_3|)^4 - 1234.11526034 \cdot \ln(|\tau_3|)^5 - 492.882068577 \cdot \ln(|\tau_3|)^6 \end{pmatrix} \tag{112}$$

$$\beta = \frac{L_2}{Psi(1+\lambda) + \gamma_e + 2 \cdot z} \tag{113}$$

$$\gamma = L_1 - \beta \cdot (Psi(1+\lambda) + \gamma_e) \tag{114}$$

where $z = \int_0^1 \ln(qbeta(p,1,\lambda)) \cdot p \cdot dp$, which can be approximated with the following equation:

$$z = 0.016876369 - 0.245383036 \cdot \lambda - 0.065754 \cdot \lambda^2 + 0.063041485 \cdot \lambda^3 - 0.018734017 \cdot \lambda^4 + 0.002522947 \cdot \lambda^5 - 0.00012935 \cdot \lambda^6 \tag{115}$$

3. Case Study

The presented case study consists of the determination of maximum annual flows on the Prigor River, Romania, using the proposed probability distributions.

The Prigor River is the left tributary of the Nera River, and it is located in the south-western part of Romania, as shown in Figure 3. The geographical coordinates of the location are 44°55′25.5″ N 22°07′21.7″ E.

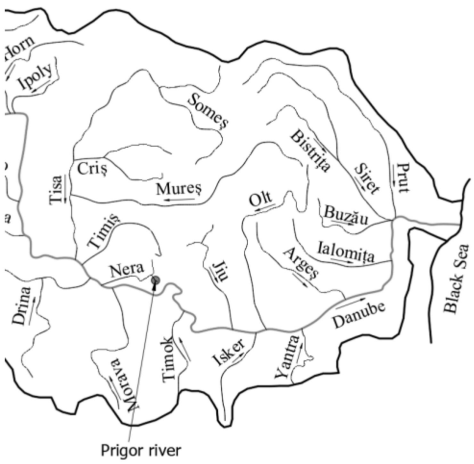


Figure 3. The location of Prigor River and Prigor hydrometric station.

The main morphometric characteristics of the river are presented in Table 3 [29].

Table 3. The morphometric characteristics.

Length [km]	Average Stream Slope [‰]	Sinuosity Coefficient [-]	Average Altitude, [m]	Watershed Area, [km^2]
33	22	1.83	713	153

In the section of the hydrometric station, the watershed area is 141 km^2 and the average altitude is 729 m. There are 31 annual observed data values with their values presented in Table A1 from Appendix B.

The main statistical indicators of the observed date are presented in Table 4.

Table 4. The statistical indicators of the data series.

Prigor	Statistical Indicators											
	μ	σ	C_v	C_s	C_k	L_1	L_2	L_3	L_4	τ_2	τ_3	τ_4
	[m^3/s]	[m^3/s]	[-]	[-]	[-]	[m^3/s]	[m^3/s]	[m^3/s]	[m^3/s]	[-]	[-]	[-]
	27.6	21.1	0.762	1.66	5.17	27.6	10.7	4.26	2.43	0.386	0.399	0.228

where $\mu, \sigma, C_v, C_s, C_k, L_1, L_2, L_3, L_4, \tau_2, \tau_3, \tau_4$ represent the mean, the standard deviation, the coefficient of variation, the skewness, the kurtosis, the four L-moments, the L-coefficient of variation, the L-skewness, and the L-kurtosis, respectively. For parameter estimation with L-moments, the data series must be in ascending order for the calculation of natural estimators, respectively L-moments [3,8,11].

4. Results

The proposed distributions were applied to perform a flood frequency analysis using the maximum annual flows (AM) on the Prigor River.

MOM and L-moments were used to estimate the parameters of the distributions. For the MOM, the skewness coefficient was chosen depending on the origin of the flows according to Romanian regulations [2], based on some multiplication coefficients for C_v. For the Prigor River, the multiplication coefficient of 3 applied to the coefficient of variation of the observed data was used, resulting in a skewness of 2.29 compared to 1.66 of the observed values.

Tables 5 and 6 present the results values of quantile distributions, for some of the most common exceedance probabilities in flood frequency analysis.

Table 5. Quantile results of the analyzed distributions for MOM.

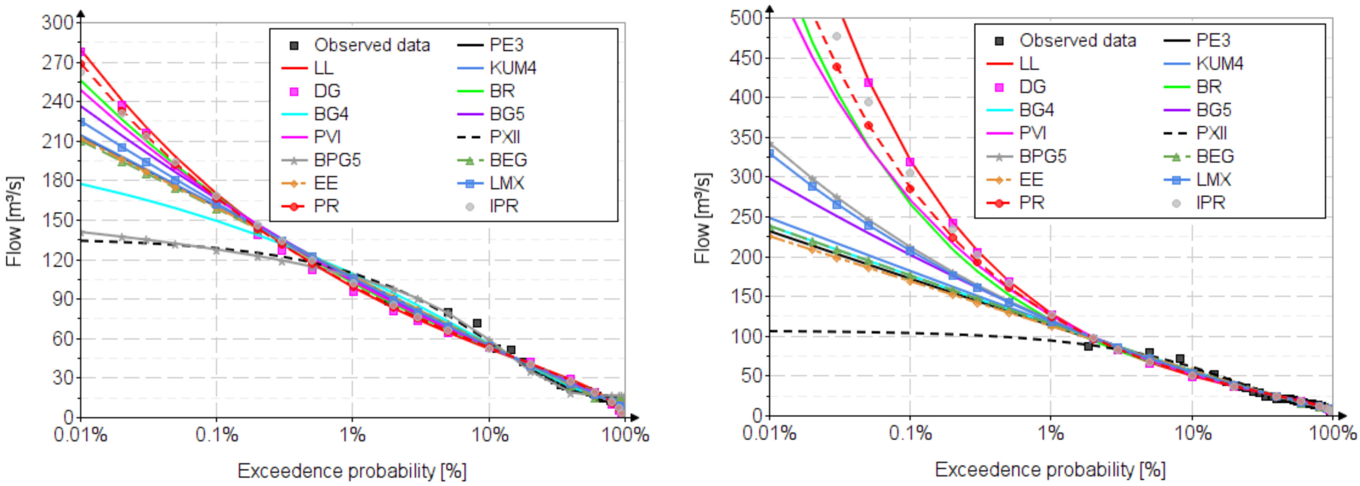
Method of Ordinary Moments (MOM)												
Exceedance Probability [%]												
Distr.	0.01	0.1	0.5	1	2	3	5	10	20	40	50	80
PE3	214	160	122	106	90.7	81.5	69.9	54.5	39.4	24.9	20.5	12
BG5	236	166	122	105	88.4	79.2	68	53.5	39.7	26.3	21.9	11.9
BG4	177	149	122	109	94.1	84.9	72.7	55.6	38.4	22.9	18.8	13.3
KUM4	214	160	123	107	90.6	81.4	69.8	54.4	39.4	25	20.6	12
PXII	134	128	118	109	98.1	89.8	77.4	57.3	36.1	20	17.2	15.4
BPG5	141	127	114	107	97.3	90.2	79.1	58.9	35.1	18.8	17	16.2
PVI	248	166	121	103	87.1	78.1	67.3	53.4	40	26.8	22.3	11.7
LMX	225	162	122	106	89.6	80.4	69.1	54.2	39.6	25.5	21.1	11.8
LL	279	169	117	98.7	82.7	74.2	64.3	51.9	40.4	28.6	24.3	12
DG	278	163	113	95.6	81	73.3	64.3	53	41.9	29.2	24.1	10
BR4	256	167	120	102	85.8	77	66.4	52.9	40.1	27.4	23	11.8
PR	269	167	117	99.6	84	75.6	65.6	52.9	40.7	28	23.5	11.4
IPR	262	169	120	102	85.4	76.5	66	52.7	40.1	27.5	23.2	11.9
BEG	210	158	122	107	91.2	82.1	70.7	55.1	39.6	24	19.1	12.7
EE	212	159	122	107	90.9	81.7	70.2	54.6	39.4	24.7	20.3	12.1

Table 6. Quantile results of the analyzed distributions for L-moments.

Method of Linear Moments (L-Moments)												
Exceedance Probability [%]												
Distr.	0.01	0.1	0.5	1	2	3	5	10	20	40	50	80
PE3	231	172	130	113	95.4	85.3	72.7	55.9	39.7	24.4	19.8	11.4
BG5	298	202	143	119	97.9	86	71.9	54.2	38.4	24.4	20.2	11.8
BG4	238	176	133	115	96.7	86.3	73.3	56	39.4	24	19.5	11.5
KUM4	248	181	136	116	97.4	86.5	73.1	55.6	39.1	24	19.6	11.6
PXII	106	103	98.2	94	87.8	82.9	75.1	61.1	43.1	23.6	18.2	11.2
BPG5	342	211	144	119	97.1	85.2	71.3	54	38.4	24.4	20.2	11.8
PVI	559	270	159	125	97.8	84	68.9	51.5	37	24.6	20.8	12.3
LMX	329	207	142	118	96.8	85.1	71.4	54.2	38.6	24.5	20.2	11.7
LL	799	320	169	128	96.7	82.1	66.6	49.6	36.2	24.7	21.2	12.5
DG	792	319	168	128	96.8	82.2	66.7	49.7	36.2	24.7	21.2	12.6
BR4	600	265	151	119	93.7	81.3	67.7	52.1	38.4	25.1	20.6	11.3
PR	647	286	161	125	96.7	82.9	67.9	51	36.9	24.7	20.9	12.2
IPR	719	305	166	128	97.4	82.9	67.4	50.2	36.3	24.6	21	12.6
BEG	238	176	133	115	96.7	86.3	73.3	56	39.4	24	19.5	11.5
EE	225	169	129	112	95.1	85.2	72.8	56.2	39.8	24.3	19.7	11.4

The results are presented comparatively, both for the method of ordinary moments and for the method of linear moments, with the Pearson III distribution which is the "parent" distribution in Romania for flood frequency analysis.

Figure 4 shows the fitting distributions for annual maximum flow for the Prigor River. For plotting positions, the Nguyen formula was used [30,31].



(**a**) MOM (**b**) L-moments

Figure 4. The fitting distributions.

Table 7 shows the values of the distributions' parameters for the two methods of estimation.

Table 7. Estimated parameter values.

Distr.	Methods of Parameter Estimation													
	MOM							L-Moments						
	α	β	γ	λ	δ	a	c	α	β	γ	λ	δ	a	c
PE3	0.766	24.1	9.2	-	-	-	-	0.694	26.9	9.0	-	-	-	-
BG5	55.4	17,088	1.1675	0.1375	83.9	-	-	30.49	3558	6.51	0.1193	28.73	-	-
BG4	0.404	5.59	-	-	-	12.7	235	0.645	3937	-	-	-	9.43	11,125
KUM4	0.88	59.4	8.72	1853	-	-	-	0.799	42.3	9.03	1853	-	-	-
PXII	0.203	121	15.38	-	-	-	-	0.354	95.9	10.7	-	-	-	-
BPG5	0.0227	141	16.2	6.3042	11.1	-	-	0.0632	103	−102	−104.68	560.3	-	-
PVI	2.7164	75.8	-	8.446	-	-	-	10.31	6.29	-	3.34	-	-	-
LMX	-	454.42	7.49	23.556	-	-	-	-	120.9	7.95	7.137	-	-	-
LL	5.230	52.8	−28.5	-	-	-	-	2.51	20.3	0.90	-	-	-	-
DG	4.34	46.1	0.24	-	-	-	-	2.53	19.84	1.13	-	-	-	-
BR4	57.95	7.641	−83.5	59.7	-	-	-	0.226	2.760	8.54	36.1	-	-	-
PR	2.2531	44.95	−5.11	-	-	-	-	1.6775	24.39	4.545	-	-	-	-
IPR	6.9615	66.76	−69.1	-	-	-	-	2.7342	17.22	−6.092	-	-	-	-
BEG	0.0648	1.45	12.7	0.1236	-	-	-	28.17	790	9.43	0.645	-	-	-
EE	-	22.8	9.68	0.703	-	-	-	-	24.5	9.164	0.662	-	-	-

The performance of the analyzed distribution was evaluated using the relative mean error (RME) and relative absolute error (RAE) criteria [22,32,33]. For the L-moments method, the selection criterion is represented by the values and the L-skewness–L-kurtosis diagram.

$$RME = \frac{1}{n} \cdot \sqrt{\sum_{i=1}^{n} \left(\frac{x_i - x(p)}{x_i} \right)^2} \tag{116}$$

$$RAE = \frac{1}{n} \cdot \sum_{i=1}^{n} \left| \frac{x_i - x(p)}{x_i} \right| \tag{117}$$

where $n, x_i, x(p)$ represent sample size, observed value, and estimated value for a given probability, respectively.

The distributions' performance values are presented in Table 8.

Table 8. Distributions performance values.

Distr.	Statistical Measures							
	Methods of Parameter Estimation						Observed Data	
	MOM		L-Moments					
	RME	RAE	RME	RAE	τ_3	τ_4	τ_3	τ_4
PE3	0.0231	0.0841	0.0219	0.0885	0.399	0.192	0.399	0.228
BG5	0.0186	0.0871	0.0165	0.0712		0.228		
BG4	0.0432	0.1198	0.0237	0.0912		0.228		
KUM4	0.0214	0.0804	0.022	0.0843		0.228		
PXII	0.0660	0.2092	0.0329	0.1338		0.118		
BPG5	0.0742	0.2481	0.0171	0.0734		0.228		
PVI	0.0233	0.1098	0.0149	0.0639		0.271		
LMX	0.0184	0.0784	0.0181	0.0765		0.221		
LL	0.0537	0.2060	0.0165	0.0715		0.299		
DG	0.0480	0.2141	0.0164	0.0714		0.300		
BR4	0.0330	0.1424	0.0214	0.0910		0.228		
PR	0.0398	0.1696	0.0155	0.0672		0.272		
IPR	0.0357	0.1506	0.0158	0.0678		0.295		
BEG	0.0441	0.1399	0.0237	0.0912		0.228		
EE	0.0250	0.0872	0.0224	0.0907		0.188		

5. Discussions

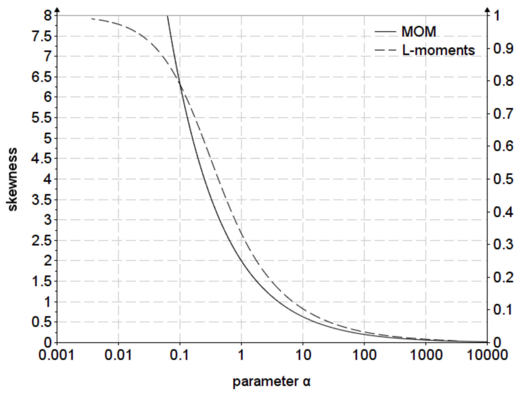
Flood frequency analysis is necessary to determine the maximum flows with certain exceeding probabilities necessary for the design of hydrotechnical constructions and establishing the bankfull discharge. The choice of distributions and the methods of estimating the parameters of these distributions have an important role in the correct performance of such an analysis.

In Romania, the regulations [34] do not provide sufficient rigorous mathematical criteria. The normative approach is deficient, analyzing in a random way distributions with two or three parameters, and only the method of ordinary moments is treated as the method of parameter estimation. Thus, taking into account the modern approaches of using the L-moments method in estimating the parameters of the distributions in the frequency analysis of extreme events, this manuscript presents 14 statistical distributions, from three families of Beta distributions, with applicability in flood frequency analysis, which use the method of ordinary moments and L-moments for parameter estimation. An important criterion for choosing the distributions in the analysis with L-moments is the τ_3-τ_4 variation; it is recommended to use distributions that have the values of these indicators very close to those of the observed data. Figure A1 from Appendix A shows the graph and τ_3-τ_4 variation relations for some of the most used distributions in hydrology, including some of the distributions analyzed in this manuscript.

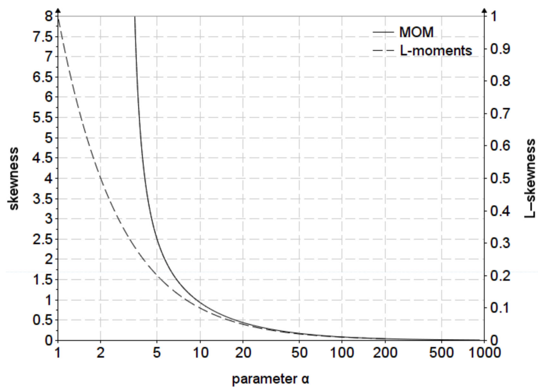
All the results obtained in the case study are presented compared to the Pearson III distribution, which is considered the parent distribution in Romania.

From the obtained results it can be seen that for the distributions with three parameters, the calibration is satisfactory for MOM, the skewness being chosen according to the genesis of the flows, as is the hydrological practice in Romania. It should be mentioned that for the MOM estimation, both for the three-parameter distributions and for the four- and five-parameter distributions, the resulting values are characterized by a degree of uncertainty due to the fact that a proper calibration of the kurtosis cannot be done.

For the L-moments method, the results obtained using the three-parameter distributions are generally unsatisfactory, generating unrealistic values in the area of small exceedance probabilities (left hand), and do not achieve a proper calibration of the high-order linear moments. The PIII and EE distributions are an exception, but this is due to the fact that the variation of the shape parameter for the two estimation methods does not differ much. For example, Figure 5 shows the variation graph of the shape parameter for the Pearson III and Log–logistic distributions, for both estimation methods. Both skewness and L-skewness depend only on the shape coefficient α.



(**a**) Pearson III



(**b**) Log–logistic

Figure 5. The variation of parameter α.

It can be seen that the Pearson III distribution is more stable. A significant difference in the variation of the shape coefficient was registered for a narrow area, namely for $\tau_3 > 0.8$, and $C_s > 6.5$, values that are not usually found in flood frequency analysis.

In the case of the Log–logistic distribution, the variation of the parameter differs greatly for the two methods, over almost the entire range of values of skewness, and L-skewness, an aspect also observed in the results presented in the graph of Figure 4.

Among the analyzed distributions with four or five parameters, including following the research carried out within the Faculty of Hydrotechnics, as well as the results obtained following the case study in this manuscript, the distributions from the Beta families recommended in flood frequency analysis using the L-moments method to estimate parameters, are the distributions KUM4, BEG, BG4, and BG5, which showed a stability related to both the length of the observed data and the presence of outliers. It can also be observed that these distributions exactly approximate all the indicators obtained based on linear moments.

This criterion of calibration τ_3-τ_4 is the most important criterion in the correct selection of distributions in analysis with L-moments. The results obtained with the RAE and RME performance indicators, both for MOM and for L-moments, provide relevant information only in the area of the probabilities of the observed values; outside this area (left-hand, upper part of the graph) it loses its relevance.

The manuscript does not exclude the applicability of other distributions from other families (Gamma, GEV, Pareto) in frequency analysis of extreme values, especially since these families were also analyzed within the research carried out in the Faculty of Hydrotechnics and presented in other materials [3,19,24].

In general, the L-moments method is a much more stable method than other estimation methods, being less influenced by the length of observed data as well as by the presence of extreme values (outliers), [8,9,17,18]. However, in certain situations (small length of data, n < 20), the statistical indicators obtained with this method require some correction, which can be achieved using the least squares method [35].

In the Supplementary Materials, the graphs of the analyzed distributions are presented, highlighting the confidence intervals determined based on the simplified Chow relationship [11] for MOM, along with those based on the simplified relationship for L-moments presented in [19], both using the frequency factors specific to the distributions.

6. Conclusions

This manuscript presents 14 statistical distributions of three, four and five parameters, from three families of Beta distributions. Some of these distributions have received limited attention for frequency analysis of extreme values, especially flood frequency analysis.

These families of distributions, along with other families of distributions, were analyzed in the research carried out in the Faculty of Hydrotechnics regarding the improvement of the existing legislation for the determination of extreme events, respectively the elaboration of a norm regarding frequency analysis in hydrology [3,19,27].

To estimate the parameters of the analyzed distributions, the method of ordinary moments and the method of linear moments were used, two of the most used methods for estimating parameters in hydrology.

All the necessary elements for their use are presented, including the probability density functions, the complementary cumulative distribution functions, the quantile functions and the exact and approximate relations for estimating parameters. Approximation relationships of distribution parameters eliminates the need for iterative numerical calculation; in many cases this was an inconvenience in the application of certain probability distributions.

A flood frequency analysis case study was carried out for the Prigor River to verify the performances of the proposed distributions. The performance of these distributions was evaluated using relative mean error and relative absolute error [23]. Performance indicators are only valid for the range of recorded values, thus, additional selection criteria are required., The selection criterion for parameter estimation with the L-moments method is the τ_3-τ_4 diagram, because it is also valid outside the range of recorded values (low exceedance probabilities) [3,21]. In Romania, short series of data are available, and so the L-moments method is recommended because it also eliminates the often arbitrary criteria for choosing the skewness as practiced with MOM. The L-moments method is a more stable method than MOM, and is generally less influenced by relatively small lengths of data [8,10,18,19,36,37].

Among the distributions from the analyzed Beta families, for flood frequency analysis and the L-moments estimation method, good candidates are the KUM4, BEG, BG4 and BG5 distributions, which presented a stability related to both the length of the observed data and the presence of outliers.

The future scope of the research is to establish the necessary guidelines for a robust, clear and concise norm regarding the determination of extreme events using the L-moment estimation method, using distributions from a wide range of families (Gamma, Beta, Generalized Extreme Value, Generalized Pareto, etc.).

The methods and the new elements presented in the manuscript will be used to create computer applications specialized in flood frequency analysis, which will be open source, to facilitate the application of the new standard and a proper transition from MOM to L-moments method.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/app13074640/s1, Figure S1: The fitting distributions with Confidence Intervals.

Author Contributions: Conceptualization, C.I. and C.G.A.; methodology, C.I. and C.G.A.; software, C.I. and C.G.A.; validation, C.I. and C.G.A.; formal analysis, C.I. and C.G.A.; investigation, C.I. and C.G.A.; resources, C.I. and C.G.A.; data curation, C.I. and C.G.A.; writing—original draft preparation, C.I. and C.G.A.; writing—review and editing, C.I. and C.G.A.; visualization, C.I. and C.G.A.; supervision, C.I. and C.G.A.; project administration, C.I. and C.G.A.; funding acquisition, C.I. and C.G.A. All authors have read and agreed to the published version of the manuscript.

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Abbreviations

MOM	the method of ordinary moments
L-moments	the method of linear moments
μ	expected value; arithmetic mean
σ	standard deviation
C_v	coefficient of variation
C_s	coefficient of skewness; skewness
C_k	coefficient of kurtosis; kurtosis
L_1, L_2, L_3	linear moments
τ_2, LC_v	coefficient of variation based on the L-moments method
τ_3, LC_s	coefficient of skewness based on the L-moments method
τ_4, LC_k	coefficient of kurtosis based on the L-moments method
m_1, m_2, m_3, m_4	central moments (with MOM)
g_r, B_r	represents the function that generates characteristic moments
Distr.	Distributions
RME	relative mean error
RAE	relative absolute error
$\mathrm{x_i}$	observed values

Appendix A. The Variation of L-kurtosis–L-skewness

In the next section, we present the variation of L-kurtosis, depending on the positive L-skewness, obtained with the L-moments method, for certain theoretical distributions often used in hydrology and in this manuscript [3,11,19].

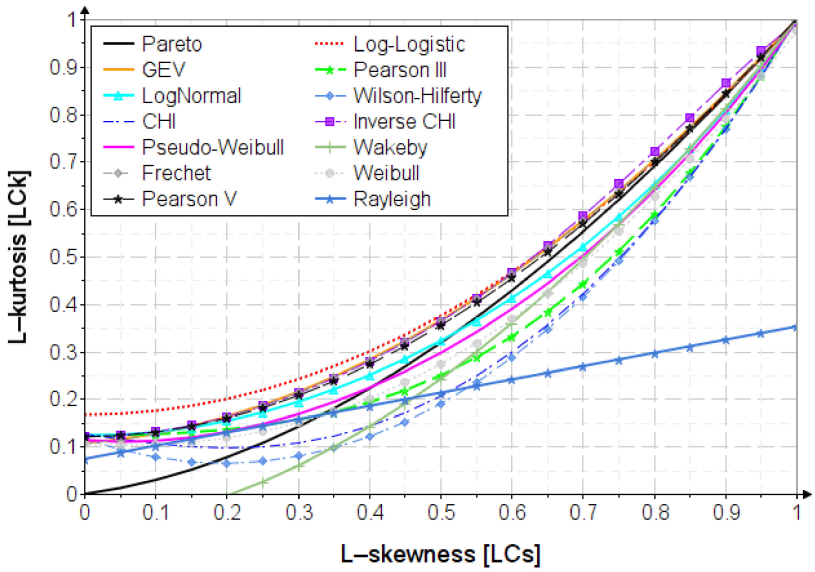

Figure A1. The variation diagram of $LC_s - LC_k$.

Pearson III:

$$\tau_4 = 0.1217175 + 0.030285 \cdot \tau_3 + 0.0266125 \cdot \tau_3^2 + 0.8774691 \cdot \tau_3^3 - 0.0564795 \cdot \tau_3^4$$

Log–logistic:

$$\tau_4 = \frac{1 + 5 \cdot \tau_3^2}{6} \simeq 0.16667 + 0.83333 \cdot \tau_3^2$$

Paralogistic:

$$\tau_4 = 0.1262814 + 0.0078207 \cdot \tau_3 + 0.9179335 \cdot \tau_3^2 - 0.0328508 \cdot \tau_3^3 - 0.0190348 \cdot \tau_3^4$$

Inverse-Paralogistic:

$$\tau_4 = 0.0577651 + 0.5568896 \cdot \tau_3 - 0.2198157 \cdot \tau_3^2 + 0.9069583 \cdot \tau_3^3 - 0.3025029 \cdot \tau_3^4$$

Appendix B. The Annual Maximum Observed Data from the Prigor River

The annual maximum observed data are presented in Table A1.

Table A1. The observed data from the Prigor hydrometric station.

Annual Maximum Flows												
		1990	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
Flow	[m^3/s]	9.96	15	10.1	14.8	7.30	21.2	18.2	21.4	13.1	14.5	35
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
Flow	[m^3/s]	19.9	22.1	11.8	80.3	88	51.6	72.2	16.2	42.6	28.5	12.8
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
Flow	[m^3/s]	31.2	24.1	52.2	21.1	18.9	6.40	24.9	15.1	36.6		

Appendix C. The Relationships for Estimating Skewness and Kurtosis

Five-parameter Generalized Beta distribution (BG5):

$$C_s = \frac{\frac{\Gamma\left(\alpha+\frac{3}{\lambda}\right)}{\Gamma\left(\alpha+\frac{3}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} - 3 \cdot \frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)}{\Gamma\left(\alpha+\frac{2}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)^2}{\Gamma(\alpha)^2} \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)}{\Gamma\left(\alpha+\frac{1}{\lambda}+\delta\right)} + 2 \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^3}{\Gamma\left(\alpha+\frac{1}{\lambda}+\delta\right)^3} \cdot \frac{\Gamma(\alpha+\delta)^3}{\Gamma(\alpha)^3}}{\left(\frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)}{\Gamma\left(\alpha+\frac{2}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} - \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^2}{\Gamma\left(\alpha+\frac{1}{\lambda}+\delta\right)^2} \cdot \frac{\Gamma(\alpha+\delta)^2}{\Gamma(\alpha)^2}\right)^{3/2}}$$

$$C_k = \frac{\begin{array}{c}\frac{\Gamma\left(\alpha+\frac{4}{\lambda}\right)}{\Gamma\left(\alpha+\frac{4}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} - 4 \cdot \frac{\Gamma\left(\alpha+\frac{3}{\lambda}\right)}{\Gamma\left(\alpha+\frac{3}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)}{\Gamma\left(\alpha+\frac{1}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} + \\ 6 \cdot \frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)}{\Gamma\left(\alpha+\frac{2}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^2}{\Gamma\left(\alpha+\frac{1}{\lambda}+\delta\right)^2} \cdot \frac{\Gamma(\alpha+\delta)^2}{\Gamma(\alpha)^2} - 3 \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^4}{\Gamma\left(\alpha+\frac{1}{\lambda}+\delta\right)^4} \cdot \frac{\Gamma(\alpha+\delta)^4}{\Gamma(\alpha)^4}\end{array}}{\left(\frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)}{\Gamma\left(\alpha+\frac{2}{\lambda}+\delta\right)} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} - \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^2}{\Gamma\left(\alpha+\frac{1}{\lambda}+\delta\right)^2} \cdot \frac{\Gamma(\alpha+\delta)^2}{\Gamma(\alpha)^2}\right)^2}$$

The Kumaraswamy distribution (KUM4):

$$C_s = \frac{\begin{array}{c}(\lambda-\gamma)^3 \cdot \beta \cdot \frac{\Gamma\left(1+\frac{3}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{3}{\alpha}\right)} + 2 \cdot (\lambda-\gamma)^3 \cdot \beta^3 \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^3\cdot\Gamma(\beta)^3}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^3} - \\ 3 \cdot (\lambda-\gamma)^3 \cdot \beta^2 \cdot \frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma(\beta)^2}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)}\end{array}}{\left((\lambda-\gamma)^2 \cdot \beta \cdot \frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} - (\lambda-\gamma)^2 \cdot \beta^2 \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta)^2}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2}\right)^{3/2}}$$

$$C_k = \frac{\begin{array}{c}(\lambda-\gamma)^4 \cdot \beta \cdot \frac{\Gamma\left(1+\frac{4}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{4}{\alpha}\right)} - 4 \cdot (\lambda-\gamma)^3 \cdot \beta \cdot \frac{\Gamma\left(1+\frac{3}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{3}{\alpha}\right)} \cdot (\lambda-\gamma) \cdot \beta \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} + \\ 6 \cdot (\lambda-\gamma)^2 \cdot \beta \cdot \frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} \cdot (\lambda-\gamma)^2 \cdot \beta^2 \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta)^2}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2} - 3 \cdot (\lambda-\gamma)^4 \cdot \beta^4 \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^4\cdot\Gamma(\beta)^4}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^4}\end{array}}{\left((\lambda-\gamma)^2 \cdot \beta \cdot \frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} - (\lambda-\gamma)^2 \cdot \beta^2 \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta)^2}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2}\right)^2}$$

Five-parameter Beta Prime Generalized distribution (BPG5):

$$C_s = \frac{\frac{\Gamma\left(\alpha+\frac{3}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{3}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} - 3 \cdot \frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{2}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{1}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} + 2 \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^3\cdot\Gamma\left(\delta-\frac{1}{\lambda}\right)^3}{\Gamma(\alpha)^3\cdot\Gamma(\delta)^3}}{\left(\frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{2}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} - \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^2\cdot\Gamma\left(\delta-\frac{1}{\lambda}\right)^2}{\Gamma(\alpha)^2\cdot\Gamma(\delta)^2}\right)^{3/2}}$$

$$C_k = \frac{\begin{array}{c}\frac{\Gamma\left(\alpha+\frac{4}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{4}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} - 4 \cdot \frac{\Gamma\left(\alpha+\frac{3}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{3}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{1}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} + \\ 6 \cdot \frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{2}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^2\cdot\Gamma\left(\delta-\frac{1}{\lambda}\right)^2}{\Gamma(\alpha)^2\cdot\Gamma(\delta)^2} - 3 \cdot \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^4\cdot\Gamma\left(\delta-\frac{1}{\lambda}\right)^4}{\Gamma(\alpha)^4\cdot\Gamma(\delta)^4}\end{array}}{\left(\frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)\cdot\Gamma\left(\delta-\frac{2}{\lambda}\right)}{\Gamma(\alpha)\cdot\Gamma(\delta)} - \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)^2\cdot\Gamma\left(\delta-\frac{1}{\lambda}\right)^2}{\Gamma(\alpha)^2\cdot\Gamma(\delta)^2}\right)^2}$$

The Log–logistic distribution (LL):

$$C_s = \frac{\Gamma\left(1+\frac{3}{\alpha}\right)\cdot\Gamma\left(1-\frac{3}{\alpha}\right)+2\cdot\Gamma\left(1+\frac{1}{\alpha}\right)^3\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^3-3\cdot\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)\cdot\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\left(\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^2\right)^{1.5}}$$

The Dagum distribution (DG):

$$C_s = \frac{\frac{\Gamma\left(\gamma+\frac{3}{\alpha}\right)\cdot\Gamma\left(1-\frac{3}{\alpha}\right)}{\Gamma(\gamma)}-3\cdot\frac{\Gamma\left(\gamma+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma(\gamma)}\cdot\frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(\gamma)}+2\cdot\frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)^3\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^3}{\Gamma(\gamma)^3}}{\left(\frac{\Gamma\left(\gamma+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma(\gamma)}-\frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^2}{\Gamma(\gamma)^2}\right)^{3/2}}$$

$$C_k = \frac{\frac{\Gamma\left(\gamma+\frac{4}{\alpha}\right)\cdot\Gamma\left(1-\frac{4}{\alpha}\right)}{\Gamma(\gamma)}-4\cdot\frac{\Gamma\left(\gamma+\frac{3}{\alpha}\right)\cdot\Gamma\left(1-\frac{3}{\alpha}\right)}{\Gamma(\gamma)}\cdot\frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(\gamma)}+6\cdot\frac{\Gamma\left(\gamma+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma(\gamma)}\cdot\frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^2}{\Gamma(\gamma)^2}-3\cdot\frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)^4\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^4}{\Gamma(\gamma)^4}}{\left(\frac{\Gamma\left(\gamma+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma(\gamma)}-\frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^2}{\Gamma(\gamma)^2}\right)^{2}}$$

Four-parameter Burr distribution (BR4):

$$C_s = \frac{\frac{\Gamma\left(1-\frac{3}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma(\alpha)}-3\cdot\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}+2\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)^3\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)^3}{\Gamma(\alpha)^3}}{\left(\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}-\frac{\Gamma\left(1-\frac{1}{\beta}\right)^2\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)^2}{\Gamma(\alpha)^2}\right)^{3/2}}$$

$$C_k = \frac{\frac{\Gamma\left(1-\frac{4}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{4}{\beta}\right)}{\Gamma(\alpha)}-4\cdot\frac{\Gamma\left(1-\frac{3}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma(\alpha)}\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}+6\cdot\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)^2\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)^2}{\Gamma(\alpha)^2}-3\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)^4\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)^4}{\Gamma(\alpha)^4}}{\left(\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}-\frac{\Gamma\left(1-\frac{1}{\beta}\right)^2\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)^2}{\Gamma(\alpha)^2}\right)^{2}}$$

The Paralogistic distribution (PR):

$$C_s == \frac{3\cdot\Gamma(\alpha)\cdot\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(\alpha-\frac{2}{\alpha}\right)\cdot\Gamma\left(\alpha-\frac{1}{\alpha}\right)-2\cdot\Gamma\left(1+\frac{1}{\alpha}\right)^3\cdot\Gamma\left(\alpha-\frac{1}{\alpha}\right)^3-\Gamma(\alpha)^2\cdot\Gamma\left(1+\frac{3}{\alpha}\right)\cdot\Gamma\left(\alpha-\frac{3}{\alpha}\right)}{\Gamma(\alpha)\cdot\left[\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(\alpha-\frac{1}{\alpha}\right)^2-\Gamma(\alpha)\cdot\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(\alpha-\frac{2}{\alpha}\right)\right]\cdot\sqrt{\frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(\alpha-\frac{2}{\alpha}\right)}{\Gamma(\alpha)}-\frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(\alpha-\frac{1}{\alpha}\right)^2}{\Gamma(\alpha)^2}}}$$

The Inverse Paralogistic distribution (IPR):

$$C_s = \frac{\frac{\Gamma\left(\alpha+\frac{3}{\alpha}\right)\cdot\Gamma\left(1-\frac{3}{\alpha}\right)}{\Gamma(\alpha)}+2\cdot\frac{\Gamma\left(1-\frac{1}{\alpha}\right)^3\cdot\Gamma\left(\alpha+\frac{1}{\alpha}\right)^3}{\Gamma(\alpha)^3}-3\cdot\frac{\Gamma\left(\frac{2}{\alpha}+\alpha\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)\cdot\Gamma\left(\alpha+\frac{1}{\alpha}\right)\cdot\Gamma\left(\alpha+\frac{1}{\alpha}\right)}{\Gamma(\alpha)^2}}{\left(\frac{\beta^2}{\Gamma(\alpha)}\cdot\left(\Gamma\left(1-\frac{2}{\alpha}\right)\cdot\Gamma\left(\alpha+\frac{2}{\alpha}\right)-\frac{\Gamma\left(1-\frac{1}{\alpha}\right)^2\cdot\Gamma\left(\alpha+\frac{1}{\alpha}\right)^2}{\Gamma(\alpha)}\right)\right)^{1.5}}$$

The Generalized Beta Exponential distribution (BEG):

$$C_s = -sign(\beta)\cdot\frac{\frac{d^2}{d\alpha^2}\psi(\alpha)-\frac{d^2}{d\alpha^2}\psi(\alpha+\lambda)}{\left(\frac{d}{d\alpha}\psi(\alpha)-\frac{d}{d\alpha}\psi(\alpha+\lambda)\right)^{1.5}}$$

$$C_k = \frac{\frac{d^3}{d\alpha^3}\psi(\alpha) - \frac{d^3}{d\alpha^3}\psi(\alpha+\lambda) + 3\cdot\left(\frac{d}{d\alpha}\psi(\alpha)\right)^2 - 6\cdot\frac{d}{d\alpha}\psi(\alpha)\cdot\frac{d}{d\alpha}\psi(\alpha+\lambda) + 3\cdot\left(\frac{d}{d\alpha}\psi(\alpha+\lambda)\right)^2}{\left(\frac{d}{d\alpha}\psi(\alpha) - \frac{d}{d\alpha}\psi(\alpha+\lambda)\right)^2}$$

Appendix D. Built-In Function in Mathcad and Excel

$\Gamma(x)$—returns the value of the Euler gamma function of x;
$\Gamma(\alpha, x)$—returns the value of the incomplete gamma function of x with parameter a;
$ibeta(a, x, y)$—incomplete Beta, returns the value of the incomplete beta function of x and y with parameter a;
$qbeta(p, s_1, s_2)$—returns the inverse cumulative probability distribution for probability p, for beta distribution. This can also be found in other dedicated programs (BETA.INV function in Excel).
$\psi(.)$—the digamma function; returns the derivative of the natural logarithm of the gamma function $\Gamma(z)$.

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