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Abstract

A limit theory is developed for mildly explosive autoregression under both weakly and strongly dependent innovation errors. We find that the asymptotic behaviour of the sample moments is affected by the memory of the innovation process both in the in the form of the limiting distribution and, in the case of long range dependence, in the rate of convergence. However, this effect is not present in least squares regression theory as it is cancelled out by the interaction between the sample moments. As a result, the Cauchy regression theory of Phillips and Magdalinos (2007a) is invariant to the dependence structure of the innovation sequence even in the long memory case.

Keywords: Central limit theory, Explosive autoregression, Long memory, Cauchy distribution.

1. Introduction

Autoregressive processes of the form

$$y_t = \rho y_{t-1} + \epsilon_t \quad \epsilon_t =_d NID\left(0, \sigma^2\right)$$

with an explosive root $|\rho| > 1$ were first discussed in early contributions by White (1958) and Anderson (1959). Assuming a zero initial condition for y_t, a Cauchy limit theory was derived for the OLS/ML estimator $\hat{\rho}_n = \left(\sum_{t=1}^n y_{t-1} y_t\right) \left(\sum_{t=1}^n y_{t-1}^2\right)^{-1}$:

$$\frac{\rho^n}{\rho^2 - 1} \left(\hat{\rho}_n - \rho\right) \Rightarrow \mathcal{C} \quad \text{as } n \to \infty, \tag{1}$$

where $\mathcal{C}$ denotes a standard Cauchy variate. It is important to note that the Gaussianity assumption imposed on the innovation sequence $(\epsilon_t)_{t \in \mathbb{N}}$ cannot be relaxed without changing the asymptotic distribution in (1). Anderson (1959) provides examples demonstrating that central limit theory does not apply and that the asymptotic distribution of the least squares estimator is characterised by the distributional assumptions imposed on the innovations. Thus, no general asymptotic inference is possible for purely explosive autoregressions.

The situation becomes more favourable to least squares regression when the explosive root approaches unity as the sample size n tends to infinity. Phillips and Magdalinos (2007a, hereafter PM_a) and Giraitis and Phillips (2006) considered autoregressive processes with root $\rho_n = 1 + c/n^\alpha$, $\alpha \in (0, 1)$. When $c > 0$, such roots are explosive in finite samples and approach unity with rate slower than $O\left(n^{-1}\right)$. The asymptotic behaviour of such "mildly explosive" or "moderately explosive" autoregressions is more regular than that of their purely explosive counterparts. Under the assumption of i.i.d. innovations with finite second moment, PM_a establish central limit theorems for sample moments generated by mildly explosive processes and obtain the following least squares regression theory:

$$\frac{1}{2c} n^\alpha \rho_n^n \left(\hat{\rho}_n - \rho_n\right) \Rightarrow \mathcal{C} \quad \text{as } n \to \infty. \tag{2}$$

This Cauchy limit theory is invariant to both the distribution of the innovations and to the initialization of the mildly explosive process.

The results of PM_a were generalised by Phillips and Magdalinos (2007b, hereafter PM_b) to include a class of weakly dependent innovations. Aue and Horvath (2007) relaxed the moment conditions on the innovations by considering an i.i.d. innovation sequence that belongs to the domain of attraction of a stable law. The limiting distribution in this case takes the form of a ratio of two independent and identically distributed stable random variables, which reduces to a Cauchy distribution when the innovations have finite variance. Multivariate extensions are included in Magdalinos and Phillips (2008).

In this paper, we consider mildly explosive autoregressions generated by a correlated innovation sequence that may exhibit long range dependence. We show that central limit theory continues to apply and that the asymptotic behaviour of the least squares estimator is given by (2). Although the asymptotic behaviour of the sample variance and the sample covariance is affected by long range dependence both in the rate of convergence and in the form of the limiting distribution, their ratio is not affected by the memory of the innovation sequence. Hence, the mildly explosive regression theory of PM_a is invariant to the dependence structure of the innovation sequence even in the long memory case. Our results generalise those in PM_a and PM_b and are complementary to the results in Aue and Horvath (2007).

2. Main results

Consider the mildly explosive process

$$X_t = \rho_n X_{t-1} + u_t, \quad t \in \{1, ..., n\} \tag{3}$$

$$\rho_n = 1 + \frac{c}{n^\alpha}, \quad \alpha \in (0,1)\,,\ c > 0 \tag{4}$$

with innovations $(u_t)_{t\in\mathbb{N}}$ and initialization X_0 that satisfy the following conditions.

Assumption LP. *For each $t \in \mathbb{N}$, u_t has Wold representation*

$$u_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j},$$

where, given the natural filtration $\mathcal{F}_t := \sigma(\varepsilon_t, \varepsilon_{t-1}, ...)$, $(\varepsilon_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ is a martingale difference sequence, $(\varepsilon_t^2)_{t\in\mathbb{Z}}$ is a uniformly integrable sequence with $E_{\mathcal{F}_{t-1}}(\varepsilon_t^2) = \sigma^2$ for all $t \in \mathbb{Z}$, and $(c_j)_{j\geq 0}$ is a sequence of constants satisfying one of the following conditions:

(i) $\sum_{j=0}^{\infty} |c_j| < \infty$.

(ii) *For each $j \in \mathbb{N}$*

$$c_j = L(j)\, j^{-\kappa}, \quad \textit{for some} \quad \kappa \in (1/2, 1)$$

where $L : (0,\infty) \to (0,\infty)$ is a slowly varying function at infinity such that $\varphi(t) := L(t)\, t^{-\kappa}$ is eventually non-increasing (i.e. φ is non-increasing on $[t_0, \infty)$ for some $t_0 > 0$) and

$$\sup_{t\in[0,B]} t^\delta L(t) < \infty \quad \textit{for any } \delta, B > 0. \tag{5}$$

(iii) $c_j = \theta j^{-1}$, $j \in \mathbb{N}$, *for some $\theta \neq 0$.*

Assumption IC. *X_0 can be any fixed constant or a random process $X_0(n)$, independent of $\sigma(u_1, ..., u_n)$, satisfying $X_0(n) = o_p\left(n^{\alpha/2}\right)$ under Assumption LP(i), $X_0(n) = o_p\left(n^{(3/2-\kappa)\alpha} L(n^\alpha)\right)$ under LP(ii) and $X_0(n) = o_p\left(n^{\alpha/2} \log n\right)$ under LP(iii).*

Under Assumption LP, $(u_t)_{t\in\mathbb{N}}$ is a covariance stationary linear process, since $(c_j)_{j\geq 0}$ is square summable and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is an uncorrelated sequence with constant variance. Uniform integrability of $(\varepsilon_t^2)_{t\in\mathbb{Z}}$ controls the the tails of the distribution of each element of $(\varepsilon_t)_{t\in\mathbb{Z}}$ and is equivalent to $\sigma^2 < \infty$ when $(\varepsilon_t)_{t\in\mathbb{Z}}$ is an identically distributed sequence. Thus, the primitive innovations ε_t considered in this paper belong to a more general class than the i.i.d. $(0, \sigma^2)$ family considered in PM$_b$.

Assumption LP(i) ensures absolute summability of the autocovariance function of u_t thereby giving rise to a weakly dependent innovation sequence. Note that LP(i) further extends the class of weakly dependent innovation sequences of PM$_b$ by requiring a weaker summability condition on $(c_j)_{j\geq 0}$ than the condition $\sum_{j=0}^{\infty} j\,|c_j| < \infty$ imposed in PM$_b$.

Assumption LP(ii) implies that $\sum_{j=0}^{\infty} |E(u_j u_1)| = \infty$ and induces strong dependence (or long memory) in the innovation sequence. The parametrisation $c_j = L(j)\, j^{-\kappa}$ is standard for stationary linear processes that exhibit long memory, see e.g. Giraitis, Koul and Surgailis (1996) and Wu and Min (2005). The memory parameter κ can be expressed in standard AFRIMA notation as $\kappa = 1 - d$, $d \in (0, 1/2)$, so Assumption LP(ii) includes stationary AFRIMA processes.

Recall that a function L is slowly varying at ∞ if and only if

$$\lim_{t\to\infty} \frac{L(ut)}{L(t)} = 1 \quad \text{for any } u > 0 \tag{6}$$

(see Bingham Goldie and Teugels (1987) hereafter referred to as BGT). The assumption that $\varphi(t) = L(t)\, t^{-\kappa}$ is eventually non-increasing ensures the validity of an Euler-type approximation (cf. Lemma A4) used in the calculation of the asymptotic variance of various sample moments. The class of functions defined by the above assumption includes differentiable slowly varying functions as a subclass (see BGT, Theorem 1.5.5). Assumption (5) is a standard requirement for the validity of Abelian theorems for integrals involving regularly varying functions in a neighbourhood of the origin (see BGT, Proposition 4.1.2(a) and Lemma A3 below). BGT, Seneta (1976) and Korevaar (2004) offer a detailed discussion of slow and regular variation. See also Phillips (2007) for an application of differentiable slowly varying functions to regression theory.

As in the analysis of Anderson (1959) and PM$_a$, least squares regression theory is driven by the stochastic sequences

$$Y_n(\kappa) := \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} u_{n+1-t} \tag{7}$$

and

$$Z_n(\kappa) := \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} u_t \tag{8}$$

with $\rho_n = 1 + c/n^\alpha$ as defined in (3) and

$$\tau_n(\beta) = \left\lfloor \frac{n^\beta}{2} \right\rfloor \quad \text{for some } \beta \in \left(\alpha, \min\left\{\frac{3\alpha}{2}, 1\right\}\right). \tag{9}$$

For notational convenience, we write $Y_n(1)$ and $Z_n(1)$ for the sequences in (7) and (8) under both Assumption LP(i) and Assumption LP(iii). This convention is justified since formally substituting $\kappa = 1$ in (7) and (8) produces the $n^{\alpha/2}$ normalisation that applies under weak dependence.

By covariance stationarity of the innovation sequence u_t, $Y_n(\kappa)$ and $Z_n(\kappa)$ have identical variance given by

$$\frac{1}{n^{(3-2\kappa)\alpha}} \left[\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-2t} \gamma_u(0) + 2 \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-2t} \sum_{h=1}^{\tau_n(\beta)-t} \rho_n^{-h} \gamma_u(h) \right] \quad \text{for all } n \in \mathbb{N} \tag{10}$$

where $\gamma_u(h) := E(u_t u_{t-h})$ denotes the autocovariance function of u_t. The asymptotic behaviour as $n \to \infty$ of the common variance of $Y_n(\kappa)$ and $Z_n(\kappa)$ depends on the memory properties of the innovation sequence u_t, as the following result shows.

Lemma 1. *Let $Z_n(\kappa)$ be the sequence defined in (8), $\omega^2 = \sigma^2 \left(\sum_{j=0}^\infty c_j\right)^2$ and $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$. Then, as $n \to \infty$:*

(i) *Under Assumption LP(i), $E\left[Z_n(1)^2\right] \to \omega^2/2c$.*

(ii) *Under Assumption LP(ii), $L(n^\alpha)^{-2} E\left[Z_n(\kappa)^2\right] \to V_\kappa$, where*

$$V_\kappa := \sigma^2 c^{2\kappa-3} \Gamma(1-\kappa)^2 \frac{\sin(\pi\kappa)}{\sin\{\pi(2\kappa-1)\}}. \tag{11}$$

(iii) *Under Assumption LP(iii), $(\log n^\alpha)^{-2} E\left[Z_n(1)^2\right] \to \sigma^2\theta^2/2c$.*

The proof of Lemma 1 can be found in Section 3. The argument is facilitated by employing an Abelian theorem and an Euler-type approximation established, respectively, by Lemma A3 and Lemma A4 in the Appendix.

Determining the joint asymptotic behaviour of $Y_n(\kappa)$ and $Z_n(\kappa)$ is the key to establishing a limit theory for explosive and mildly explosive autoregressions. We present the results for the short memory and long memory case separately in the following two lemmas, the proof of which can be found in Section 3.

Lemma 2. *Under Assumption LP(i),*

$$[Y_n(1), Z_n(1)] \Rightarrow [Y_1, Z_1] \quad \text{as } n \to \infty,$$

where Y_1 *and* Z_1 *are independent* $N(0, \omega^2/2c)$ *variates.*

Lemma 2 generalises the corresponding results of PM_a and PM_b by considering a larger class of weakly dependent innovation sequences $(u_t)_{t\in\mathbb{N}}$.

Characterising the joint asymptotic behaviour of $Y_n(\kappa)$ and $Z_n(\kappa)$ for strongly dependent innovations is more challenging and the main result is provided below.

Lemma 3. *Under Assumption LP(ii) we obtain, for each* $\kappa \in (1/2, 1)$,

$$\frac{1}{L(n^\alpha)}[Y_n(\kappa), Z_n(\kappa)] \Rightarrow [Y_\kappa, Z_\kappa] \quad \text{as } n \to \infty,$$

where Y_κ *and* Z_κ *are independent* $N(0, V_\kappa)$ *random variables and* V_κ *is given by* (11).

Remark 1. Lemma 3 shows that the introduction of long memory in the innovation sequence affects the components that drive mildly explosive autoregression not only in the form of the limiting distribution but also in the rate of convergence. This contrasts the weakly dependent case (see Lemma 2 and PM_b) where the result differs from the i.i.d. error case of PM_a only in the asymptotic variance.

Remark 2. The asymptotic variance in (11) diverges to ∞ at the boundary values $\kappa = 1/2, 1$. This is expected at the boundary value $\kappa = 1/2$ since u_t has infinite variance for any $\kappa \leq 1/2$. On the other hand, $\kappa = 1$ provides a boundary between short range and long range dependence in the innovation sequence u_t. Lemma 3 then implies that the distribution of $Y_n(\kappa)$ and $Z_n(\kappa)$ does not admit a smooth transition from short to long memory. The underlying reason is that the normalisation $n^{(3/2-\kappa)\alpha}$ cannot distinguish between a short memory linear process and a linear process with harmonic coefficients c_j as in Assumption LP(iii): Lemma 3 would assign the short memory normalisation $n^{\alpha/2}$ to $[Y_n(1), Z_n(1)]$ generated by the latter process, which is not sufficient since the harmonic series diverges with rate $\sum_{j=1}^n j^{-1} \sim \log n$.

As pointed out in Remark 2, a complete discussion of the asymptotic behaviour of $[Y_n(\kappa), Z_n(\kappa)]$ would have to include the case of transition between short and long range dependence in the innovations u_t. This is the aim of the next result.

Lemma 4. *Under Assumption LP(iii)*

$$\frac{1}{\log n^\alpha}[Y_n(1), Z_n(1)] \Rightarrow [Y_1', Z_1'] \quad \text{as } n \to \infty,$$

where Y_1' *and* Z_1' *are independent* $N\left(0, \sigma^2\theta^2/2c\right)$ *random variables.*

Remark 3. The slowly varying function L has been replaced by a constant in Assumption LP(iii) since taking $c_j = L(j) j^{-1}$ would produce a limiting distribution in Lemma 4 that is not invariant to the choice of L. The problem is that the asymptotic variance of $\psi_n^{-1} Z_n(1)$ can be expressed in terms of the integral $I_n := \psi_n^{-1} \int_1^{n^\alpha/c} \frac{L(z)}{z} dz$, where $\psi_n := (\log n^\alpha)^{-1} L(n^\alpha)$. Assume for simplicity that L is differentiable with

$$\frac{L'(t)}{L(t)} = \frac{\varepsilon(t)}{t} \quad \text{for all } t \geq 1 \tag{12}$$

for some function ε which determines L and satisfies $\varepsilon(t) \to 0$ as $t \to \infty$. Equation (12) can be deduced directly from the Karamata representation of L. Using integration by parts and (12) we obtain

$$\int_1^{n^\alpha/c} \frac{L(z)}{z} dz = \log(n^\alpha/c) L(n^\alpha/c) - \int_1^{n^\alpha/c} \frac{L(z)}{z} \varepsilon(z) \log z dz. \tag{13}$$

The value of the integral in (13) depends on the choice of ε and hence on the choice of L. If $\varepsilon(z) = (\log z)^{-2}$ in (12), the second integral in (13) is $O(L(n^\alpha) \log(\log n))$, giving $I_n \to 1$. If $\varepsilon(z) = \lambda / \log z$ for some $\lambda \neq 0$, (13) yields $I_n \to (1+\lambda)^{-1}$. The above observation implies that the asymptotic variance of $\psi_n^{-1} Y_n(1)$ and $\psi_n^{-1} Z_n(1)$ depends on the choice of L.

Once the joint asymptotic behaviour of $Y_n(\kappa)$ and $Z_n(\kappa)$ has been derived, it is easy to obtain the limiting distribution of the sample moments of X_t by employing a standard approximation argument (see Anderson (1959) and PM$_a$) for explosive and mildly explosive processes: roughly, the sample variance and the sample covariance behave like $Z_n(\kappa)^2$ and $Y_n(\kappa) Z_n(\kappa)$ respectively.

Lemma 5. *Let L denote an arbitrary slowly varying function at infinity. Then*

$$\begin{aligned}
\frac{\rho_n^{-2n}}{n^\alpha n^{(3-2\kappa)\alpha} L(n^\alpha)^2} \sum_{t=1}^n X_{t-1}^2 &= \frac{1}{2c} \left[\frac{1}{L(n^\alpha)} Z_n(\kappa) \right]^2 + o_p(1) \\
\frac{\rho_n^{-n}}{n^{(3-2\kappa)\alpha} L(n^\alpha)^2} \sum_{t=1}^n X_{t-1} u_t &= \frac{Y_n(\kappa)}{L(n^\alpha)} \frac{Z_n(\kappa)}{L(n^\alpha)} + o_p(1)
\end{aligned}$$

as $n \to \infty$ where:

(i) *Under Assumption LP(i), $\kappa = 1$ and $L(x) = 1$ for all $x > 0$.*

(ii) *Under Assumption LP(ii), $\kappa \in (1/2, 1)$ and L satisfies LP(ii).*

(iii) *Under Assumption LP(iii), $\kappa = 1$ and $L(x) = \log x$ for all $x > 0$.*

Combining Lemma 5 with Lemmas 2, 3 and 4, we deduce that, under the appropriate normalisation, joint convergence in distribution of $\left(\sum_{t=1}^{n} X_{t-1}u_t, \sum_{t=1}^{n} X_{t-1}^2\right)$ applies under both weak and strong dependence. The asymptotic behaviour of the centered least squares estimator

$$\hat{\rho}_n - \rho_n = \frac{\sum_{t=1}^{n} X_{t-1}u_t}{\sum_{t=1}^{n} X_{t-1}^2}$$

is then an immediate consequence of the continuous mapping theorem and the fact that the limiting random vectors (Y_1, Z_1), (Y_κ, Z_κ) and (Y_1', Z_1') of Lemmas 2, 3 and 4 consist of independent components.

Theorem 1. *For the mildly explosive process generated by (3) under Assumptions LP and IC, the following limit theory applies as* $n \to \infty$:

$$\frac{1}{2c} n^{\alpha} \rho_n^n \left(\hat{\rho}_n - \rho_n\right) \Rightarrow \mathcal{C} \quad \text{as } n \to \infty,$$

where $\mathcal{C}$ *denotes a standard Cauchy variate.*

Remark 4. Theorem 1 shows that the Cauchy regression theory of PM$_a$ is invariant to the dependence structure of the innovation sequence even in the long memory case. The limit theory is independent of the memory parameter κ and the normalisation consists only of the parameters c and α that determine the degree of mild explosion, i.e. the neighbourhood of unity that contains the mildly explosive root ρ_n. At first glance, this result may seem surprising given that the limit theory for both the sample variance $\sum_{t=1}^{n} X_{t-1}^2$ and the sample covariance $\sum_{t=1}^{n} X_{t-1}u_t$ is affected by the presence of long memory in the innovation sequence both in the rate of convergence and in the form of the limiting distribution. The interaction between these two sample moments, however, cancels out this effect: Lemma 5 implies that the asymptotic behaviour of the normalised and centred least squares estimator is driven by the ratio $Y_n(\kappa)/Z_n(\kappa)$ in which the numerator and the denominator have identical rate of convergence and limiting distribution (by Lemmas 2, 3 and 4). Therefore, any increase in the rate of convergence of $Y_n(\kappa)$ is offset by an equal increase in the rate of $Z_n(\kappa)$, leaving least squares regression theory invariant to the degree of persistence of the innovations. This suggests that the least squares estimator retains the rate of convergence of Theorem 1 under more general innovation processes including non-stationary long memory, although such a generalisation would require a different method of proof.

3. Proofs

This section contains the proof of Lemmas 1-5. We begin by establishing some notation. Using the linear process representation of u_t, the process $Z_n(\kappa)$ defined in (8) can be decomposed into the sum of two uncorrelated components:

$$Z_n(\kappa) = Z_n^{(1)}(\kappa) + Z_n^{(2)}(\kappa), \tag{14}$$

where

$$Z_n^{(1)}(\kappa) = \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} \sum_{j=0}^{t} c_j \varepsilon_{t-j}, \tag{15}$$

$$Z_n^{(2)}(\kappa) = \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{j=1}^{\infty} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+j} \right) \varepsilon_{-j} \tag{16}$$

and $\tau_n(\beta)$ is the sequence defined in (9).

The process $Y_n(\kappa)$ defined in (7) can be written as:

$$\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} u_{n+1-t} = \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} \sum_{j=0}^{\infty} c_j \varepsilon_{n+1-t-j} = \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} \sum_{k=t}^{\infty} c_{k-t} \varepsilon_{n+1-k}.$$

Changing the order of summation in the last expression, we obtain the following decomposition of $Y_n(\kappa)$ into the sum of two uncorrelated components:

$$Y_n(\kappa) = Y_n^{(1)}(\kappa) + Y_n^{(2)}(\kappa), \tag{17}$$

where

$$Y_n^{(1)}(\kappa) = \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{k=1}^{\tau_n(\beta)} \left(\sum_{t=1}^{k} \rho_n^{-t} c_{k-t} \right) \varepsilon_{n+1-k} \tag{18}$$

$$Y_n^{(2)}(\kappa) = \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{k>\tau_n(\beta)} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{k-t} \right) \varepsilon_{n+1-k}. \tag{19}$$

Finally, we use $\|\cdot\|$ to denote the Euclidian norm of a vector and $\|\cdot\|_r$ to denote the L_r norm of a random variable: $\|X\|_r = (E|X|^r)^{1/r}$. Given a σ-algebra $\mathcal{F}$, $E_{\mathcal{F}}$ and $P_{\mathcal{F}}$ denote conditional expectation and conditional probability respectively.

3.1. Proof of Lemma 1

Under Assumption LP(i), the result follows immediately from (14) and Propositions 3.2.1 and 3.2.2(ii) below.

Proof under Assumption LP(ii). Under Assumption LP(ii), the autocovariance function of u_t is given by

$$\gamma_u(h) = \sigma^2 \left[L(h) h^{-\kappa} + \sum_{j=1}^{\infty} c_j c_{j+h} \right] \quad h \in \mathbb{N}.$$

Using the Cauchy-Schwarz inequality and the fact the function $\varphi(x) = x^{-\kappa} L(x)$ is eventually non-increasing we obtain, for large enough n,

$$\sum_{j>\tau_n(\beta)} c_j c_{j+h} \leq \sum_{j>\tau_n(\beta)} c_j^2 = \sum_{j>\tau_n(\beta)} \varphi(j)^2 \leq \int_{\tau_n(\beta)}^{\infty} \varphi(x)^2 \, dx = O\left(L\left(n^\beta\right)^2 n^{-(2\kappa-1)\beta} \right)$$

by (9) and Karamata's theorem (BGT, Proposition 1.5.8). Thus,

$$\gamma_u(h) = \sigma^2 \left[L(h) h^{-\kappa} + \sum_{j=1}^{\tau_n(\beta)} c_j c_{j+h} \right] + O\left(L\left(n^\beta\right)^2 n^{-(2\kappa-1)\beta} \right) \tag{20}$$

as $n \to \infty$, uniformly in h. For brevity, let

$$\lambda_n := n^{(1-\kappa)\alpha} L(n^\alpha). \tag{21}$$

Using the fact that

$$\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-2t} \sum_{h=\tau_n(\beta)-t+1}^{\tau_n(\beta)} \rho_n^{-h} \left|\gamma_u(h)\right| = o\left(n^{2\alpha} \rho_n^{-\tau_n(\beta)} \right) \quad \text{and} \quad \frac{1}{n^\alpha} \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-2t} \to \frac{1}{2c},$$

(10) implies that the variance of $Z_n(\kappa)$ has the following asymptotic behaviour as $n \to \infty$:

$$\begin{aligned}
\left\| \frac{Z_n(\kappa)}{L(n^\alpha)} \right\|_2^2 &= \frac{1}{c} \frac{1}{\lambda_n^2} \sum_{h=1}^{\tau_n(\beta)} \rho_n^{-h} \gamma_u(h) + O\left(\frac{1}{L(n^\alpha)^2 n^{2(1-\kappa)\alpha}} \right) \\
&= \frac{1}{c} \frac{\sigma^2}{\lambda_n^2} \sum_{h=1}^{\tau_n(\beta)} \rho_n^{-h} \left[L(h) h^{-\kappa} + \sum_{j=1}^{\tau_n(\beta)} c_j c_{j+h} \right] + O\left(\frac{L\left(n^\beta\right)^2}{L(n^\alpha)^2} \frac{1}{n^{(2\kappa-1)(\beta-\alpha)}} \right) \\
&= \frac{1}{c} \frac{\sigma^2}{\lambda_n^2} \sum_{h=1}^{\tau_n(\beta)} \rho_n^{-h} \sum_{j=1}^{\tau_n(\beta)} c_j c_{j+h} + O\left(\frac{1}{n^{(1-\kappa)\alpha}} \right) \\
&= \frac{1}{c} \frac{\sigma^2}{\lambda_n^2} \sum_{h=1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha} h} \sum_{j=1}^{\tau_n(\beta)} c_j c_{j+h} + o(1) \qquad (22)
\end{aligned}$$

where the second line follows from (20), the third line follows from Lemma A3 in the Appendix and the final line follows since Lemma A1 in the Appendix and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}
\sup_{1\leq h\leq \tau_n(\beta)} \left|\rho_n^{-h} - e^{-\frac{c}{n^\alpha}h}\right| \frac{1}{\lambda_n^2}\sum_{h=1}^{\tau_n(\beta)}\sum_{j=1}^{\tau_n(\beta)} c_j c_{j+h} &\leq \frac{1}{\lambda_n^2}\left(\sum_{j=1}^{\tau_n(\beta)} c_j\right)^2 O\left(\frac{1}{n^{\alpha/2}}\right) \\
&= O\left[\frac{1}{n^{\alpha/2}\lambda_n^2}\left(\int_1^{\tau_n(\beta)} \varphi\left(x\right)dx\right)^2\right] \\
&= O\left(\left[\frac{L\left(n^\beta\right)}{L\left(n^\alpha\right)}\right]^2 \frac{n^{2(1-\kappa)(\beta-\alpha)}}{n^{\alpha/2}}\right) = o\left(1\right)
\end{aligned}$$

for all $\beta \in (\alpha, 3\alpha/2)$ by Karamata's theorem. Applying the Euler approximation of Lemma A4 in the Appendix to (22) and letting $\psi := c\left(\lfloor t_0\rfloor + 1\right)$, we obtain

$$\begin{aligned}
\left\|\frac{Z_n\left(\kappa\right)}{L\left(n^\alpha\right)}\right\|_2^2 &= \frac{1}{c}\frac{\sigma^2}{\lambda_n^2}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}x}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} L\left(y\right)L\left(y+x\right)y^{-\kappa}\left(y+x\right)^{-\kappa}dydx + o\left(1\right) \\
&= \frac{\sigma^2 c^{2\kappa-3}}{L\left(n^\alpha\right)^2}\int_{\frac{\psi}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} e^{-u}\int_{\frac{\psi}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} L\left(\frac{n^\alpha z}{c}\right)L\left(\frac{n^\alpha\left(z+u\right)}{c}\right)\left[z\left(u+z\right)\right]^{-\kappa}dzdu \\
&= \sigma^2 c^{2\kappa-3}\int_{\frac{\psi}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} e^{-u}\left[I_{n1}\left(u\right)+I_{n2}\left(u\right)\right]du \qquad (23)
\end{aligned}$$

where

$$\begin{aligned}
I_{n1}\left(u\right) &= \frac{1}{L\left(n^\alpha\right)^2}\int_{\frac{\psi}{n^\alpha}}^{1} L\left(\frac{n^\alpha}{c}z\right)L\left[\frac{n^\alpha}{c}\left(z+u\right)\right]z^{-\kappa}\left(u+z\right)^{-\kappa}dz \\
I_{n2}\left(u\right) &= \frac{1}{L\left(n^\alpha\right)^2}\int_{1}^{\frac{c\tau_n(\beta)}{n^\alpha}} L\left(\frac{n^\alpha}{c}z\right)L\left[\frac{n^\alpha}{c}\left(z+u\right)\right]z^{-\kappa}\left(u+z\right)^{-\kappa}dz.
\end{aligned}$$

For some $\delta \in \left(0, \min\left\{1-\kappa, \left(2\kappa-1\right)/2\right\}\right)$ define the regularly varying functions

$$R\left(x\right) = x^\delta L\left(x\right) \quad \text{and} \quad r\left(x\right) = x^{-\delta}L\left(x\right).$$

By the uniform convergence theorem for regularly varying functions with negative index (BGT, Theorem 1.5.2)

$$\sup_{z\in[\gamma,\infty)}\left|\frac{r\left(n^\alpha z\right)}{r\left(n^\alpha\right)} - z^{-\delta}\right| \to 0 \quad \text{as } n\to\infty \qquad (24)$$

for any fixed $\gamma > 0$. In this notation, $I_{n2}(u)$ can be written as

$$\begin{aligned} I_{n2}(u) &= \frac{c^{-2\delta}}{r(n^\alpha)^2}\int_1^{\frac{c\tau_n(\beta)}{n^\alpha}} r\left(\frac{n^\alpha}{c}z\right) r\left(\frac{n^\alpha}{c}(z+u)\right) z^{-(\kappa-\delta)}(u+z)^{-(\kappa-\delta)}\,dz \\ &= \frac{c^{-2\delta}}{r(n^\alpha)}\int_1^{\frac{c\tau_n(\beta)}{n^\alpha}} \left\{\frac{r\left(\frac{n^\alpha}{c}z\right)}{r(n^\alpha)} - \left(\frac{z}{c}\right)^{-\delta}\right\} r\left(\frac{n^\alpha}{c}(z+u)\right)[z(u+z)]^{-(\kappa-\delta)}\,dz \\ &\quad + \frac{c^{-\delta}}{r(n^\alpha)}\int_1^{\frac{c\tau_n(\beta)}{n^\alpha}} r\left(\frac{n^\alpha}{c}(z+u)\right) z^{-\kappa}(u+z)^{-(\kappa-\delta)}\,dz. \end{aligned} \tag{25}$$

Since $\int_1^\infty z^{-2(\kappa-\delta)}dz < \infty$ for $\delta \in (0, (2\kappa-1)/2)$ the first term of (25) is bounded by

$$\sup_{z\in[1/c,\infty)}\left|\frac{r(n^\alpha z)}{r(n^\alpha)} - z^{-\delta}\right| \sup_{x\in[1/c,\infty)}\left|\frac{r(n^\alpha x)}{r(n^\alpha)}\right| c^{-2\delta}\int_1^\infty z^{-2(\kappa-\delta)}dz = o(1)$$

uniformly in $u \in (0,\infty)$, by (24). Thus, as $n \to \infty$,

$$\sup_{u>0}\left|I_{n2}(u) - \frac{c^{-\delta}}{r(n^\alpha)}\int_1^{\frac{c\tau_n(\beta)}{n^\alpha}} r\left(\frac{n^\alpha}{c}(z+u)\right) z^{-\kappa}(u+z)^{-(\kappa-\delta)}\,dz\right| = o(1).$$

Adding and subtracting $[(z+u)/c]^{-\delta}$ in the above integral and using (24) in a similar way for the estimation of the remainder term, we obtain

$$\sup_{u>0}\left|I_{n2}(u) - \int_1^{\frac{c\tau_n(\beta)}{n^\alpha}} z^{-\kappa}(u+z)^{-\kappa}\,dz\right| \to 0 \quad \text{as } n \to \infty.$$

Thus, (9) and the dominated convergence theorem yield, as $n \to \infty$,

$$\begin{aligned} \int_{\frac{c}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} e^{-u} I_{n2}(u)\,du &\to \int_0^\infty e^{-u}\int_1^\infty z^{-\kappa}(u+z)^{-\kappa}\,dz du \\ &= \int_1^\infty e^z z^{-\kappa}\int_z^\infty e^{-x}x^{-\kappa}dxdz. \end{aligned} \tag{26}$$

For the first term of (23), using the substitution $x = z+u$ we obtain

$$\begin{aligned} \int_{\frac{\psi}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} e^{-u} I_{n1}(u)\,du &= \int_{\frac{\psi}{n^\alpha}}^1 \frac{L\left(\frac{n^\alpha z}{c}\right)}{L(n^\alpha)} e^z z^{-\kappa}\int_{z+\frac{\psi}{n^\alpha}}^{z+\frac{c\tau_n(\beta)}{n^\alpha}} \frac{L\left(\frac{n^\alpha x}{c}\right)}{L(n^\alpha)} e^{-x}x^{-\kappa}dxdz \\ &= c^\delta\int_{\frac{\psi}{n^\alpha}}^1 \frac{R\left(\frac{n^\alpha}{c}z\right)}{R(n^\alpha)} e^z z^{-\kappa-\delta}\int_{z+\frac{\psi}{n^\alpha}}^{z+\frac{c\tau_n(\beta)}{n^\alpha}} \frac{L\left(\frac{n^\alpha}{c}x\right)}{L(n^\alpha)} e^{-x}x^{-\kappa}dxdz \\ &= \int_{\frac{\psi}{n^\alpha}}^1 e^z z^{-\kappa}\left(\int_{z+\frac{\psi}{n^\alpha}}^{z+\frac{c\tau_n(\beta)}{n^\alpha}} \frac{L\left(\frac{n^\alpha}{c}x\right)}{L(n^\alpha)} e^{-x}x^{-\kappa}dx\right) dz + o(1) \end{aligned} \tag{27}$$

as $n \to \infty$ because, since $\int_0^1 z^{-\kappa-\delta} dz < \infty$ for all $\delta \in (0, 1-\kappa)$,

$$\begin{aligned}
&\int_{\frac{\psi}{n^\alpha}}^{1} \left| \frac{R\left(\frac{n^\alpha}{c} z\right)}{R\left(n^\alpha\right)} - \left(\frac{z}{c}\right)^{\delta} \right| e^z z^{-\kappa-\delta} \int_{z+\frac{\psi}{n^\alpha}}^{z+\frac{c\tau_n(\beta)}{n^\alpha}} \frac{L\left(\frac{n^\alpha x}{c}\right)}{L\left(n^\alpha\right)} e^{-x} x^{-\kappa} dx dz \\
\leq & \sup_{z \in (0,1/c]} \left| \frac{R\left(n^\alpha z\right)}{R\left(n^\alpha\right)} - z^\delta \right| \left(\int_{\frac{\psi}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}+1} \frac{L\left(\frac{n^\alpha x}{c}\right)}{L\left(n^\alpha\right)} e^{-x} x^{-\kappa} dx \right) \left(\int_0^1 z^{-\kappa-\delta} dz \right) \\
= & \sup_{z \in (0,1/c]} \left| \frac{R\left(n^\alpha z\right)}{R\left(n^\alpha\right)} - z^\delta \right| O(1) = o(1)
\end{aligned}$$

by the uniform convergence theorem for regularly varying functions with positive index (BGT, Theorem 1.5.2) and Lemma A3 in the Appendix. Now the integrand in (27) is bounded by $J_n(\kappa) e^z z^{-\kappa}$, where $J_n(\kappa)$ is defined (50). By (51), $J_n(\kappa) e^z z^{-\kappa}$ is integrable on $[0,1]$ and hence the dominated convergence theorem, (6) and (9) yield

$$\begin{aligned}
\int_{\frac{\psi}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} e^{-u} I_{n1}(u) du &= \int_0^1 \mathbf{1}_{\left[\frac{\psi}{n^\alpha}, 1\right]}(z) e^z z^{-\kappa} \left(\int_{z+\frac{\psi}{n^\alpha}}^{z+\frac{c\tau_n(\beta)}{n^\alpha}} e^{-x} x^{-\kappa} \frac{L\left(\frac{n^\alpha}{c} x\right)}{L\left(n^\alpha\right)} dx \right) dz \\
&\to \int_0^1 e^z z^{-\kappa} \int_z^\infty e^{-x} x^{-\kappa} dx dz \quad \text{as } n \to \infty. \qquad (28)
\end{aligned}$$

Combining (23), (26) and (28) we obtain

$$\left\| \frac{Z_n(\kappa)}{L\left(n^\alpha\right)} \right\|_2^2 \to \sigma^2 c^{2\kappa-3} \int_0^\infty e^z z^{-\kappa} \Gamma(1-\kappa, z) dz \quad \text{as } n \to \infty,$$

where $\Gamma(x,z) = \int_z^\infty u^{x-1} e^{-u} du$ denotes the "complementary" incomplete gamma function. The integral on the right can be evaluated as follows:

$$\begin{aligned}
\int_0^\infty e^z z^{-\kappa} \Gamma(1-\kappa, z) dz &= \sum_{j=0}^{\infty} \frac{1}{j!} \int_0^\infty z^{j-\kappa} \Gamma(1-\kappa, z) dz \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma(j+2-2\kappa)}{j+1-\kappa} \\
&= \frac{\pi^2}{\sin(\pi\kappa) \sin\{\pi(2\kappa-1)\} \Gamma(\kappa)^2} \\
&= \Gamma(1-\kappa)^2 \frac{\sin(\pi\kappa)}{\sin\{\pi(2\kappa-1)\}},
\end{aligned}$$

where the integral on the second line is calculated by 6.5.37 of Abramowitz and Stegun (1972) and the last line is obtained by using the duplication formula for the gamma function.

Proof under Assumption LP(iii). Under Assumption LP(iii), an identical argument to that leading to (22) yields

$$\left\|\frac{Z_n(1)}{\log n^\alpha}\right\|_2^2 = \frac{1}{c}\frac{\theta^2\sigma^2}{(\log n^\alpha)^2}\sum_{h=1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}h}\sum_{j=1}^{\tau_n(\beta)} j^{-1}(j+h)^{-1} + o(1) \quad \text{as } n\to\infty.$$

Approximating the above sums by integrals using Lemma A4 yields

$$\begin{aligned}\left\|\frac{Z_n(1)}{\log n^\alpha}\right\|_2^2 &= \frac{1}{c}\frac{\theta^2\sigma^2}{(\log n^\alpha)^2}\int_1^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}x}\int_1^{\tau_n(\beta)} y^{-1}(y+x)^{-1}\,dydx + o(1)\\ &= \frac{1}{c}\frac{\theta^2\sigma^2}{(\log n^\alpha)^2}\int_{\frac{c}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} z^{-1}\int_{\frac{c}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} e^{-u}(z+u)^{-1}\,dudz\\ &= \frac{1}{c}\frac{\theta^2\sigma^2}{(\log n^\alpha)^2}\int_{\frac{c}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} z^{-1}e^{z}E_1\left(z+\frac{c}{n^\alpha}\right)dz + O\left(e^{-\frac{c}{2}n^{\beta-\alpha}}\right)\end{aligned}$$

where

$$E_1(z) := \int_z^\infty x^{-1}e^{-x}\,dx \tag{29}$$

denotes the exponential integral. Using the Cauchy-Schwarz inequality, $E_1(z) \le e^{-z}z^{-1/2}$ so $\int_1^\infty e^z z^{-1}E_1(z)\,dz \le \int_1^\infty z^{-3/2}dz < \infty$. Thus,

$$\left\|\frac{Z_n(1)}{\log n^\alpha}\right\|_2^2 = \frac{1}{c}\frac{\theta^2\sigma^2}{(\log n^\alpha)^2}\int_{\frac{c}{n^\alpha}}^{1} z^{-1}e^{z}E_1\left(z+\frac{c}{n^\alpha}\right)dz + O\left(\frac{1}{(\log n)^2}\right).$$

The asymptotic expansion of E_1 (see 5.1.11 in Abramowitz and Stegun, 1972) implies that $\sup_{z\in(0,1]}|E_1(z)+\log z| < \infty$. Hence, approximating $E_1(z)$ by $-\log z$ and using the power series for the exponential function yields

$$\begin{aligned}\left\|\frac{Z_n(1)}{\log n^\alpha}\right\|_2^2 &= -\frac{1}{c}\frac{\theta^2\sigma^2}{(\log n^\alpha)^2}\int_{\frac{c}{n^\alpha}}^{1} z^{-1}e^{z}\log z\,dz + O\left(\frac{1}{\log n}\right)\\ &= -\frac{1}{c}\frac{\theta^2\sigma^2}{(\log n^\alpha)^2}\int_{\frac{c}{n^\alpha}}^{1} z^{-1}\log z\,dz + O\left(\frac{1}{n^\alpha\log n}\right)\\ &\to \frac{\theta^2\sigma^2}{2c}.\end{aligned}$$

3.2. Proof of Lemma 2

We maintain Assumption LP(i) throughout this subsection.

Proposition 3.2.1. *As* $n\to\infty$, $Z_n^{(2)}(1)\to_{L_2} 0$ *and* $Y_n^{(2)}(1)\to_{L_2} 0$.

Proof. Since $\rho_n^{-t} \leq 1$ for all t and $\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} = O\left(n^{\alpha}\right)$ we obtain

$$\begin{aligned}
\left\| Z_n^{(2)}(1) - \frac{1}{n^{\alpha/2}} \sum_{k > \left\lfloor n^{\alpha/2} \right\rfloor} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+k} \right) \varepsilon_{-k} \right\|_2^2 &= \frac{\sigma^2}{n^{\alpha}} \sum_{k=1}^{\left\lfloor n^{\alpha/2} \right\rfloor} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+k} \right)^2 \\
&\leq \frac{\sigma^2 \left\lfloor n^{\alpha/2} \right\rfloor}{n^{\alpha}} \left(\sum_{t=1}^{\infty} \left| c_t \right| \right)^2 \to 0,
\end{aligned}$$

$$\begin{aligned}
\left\| \frac{1}{n^{\alpha/2}} \sum_{k > \left\lfloor n^{\alpha/2} \right\rfloor} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+k} \right) \varepsilon_{-k} \right\|_2^2 &= \frac{\sigma^2}{n^{\alpha}} \sum_{k > \left\lfloor n^{\alpha/2} \right\rfloor} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+k} \right)^2 \\
&\leq \frac{\sigma^2}{n^{\alpha}} \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} \sum_{k > \left\lfloor n^{\alpha/2} \right\rfloor} \left| c_{t+k} \right| \sum_{s=1}^{\tau_n(\beta)} \rho_n^{-s} \left| c_{s+k} \right| \\
&\leq \left(\sum_{s=1}^{\infty} \left| c_s \right| \right) \frac{\sigma^2}{n^{\alpha}} \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} \sum_{k > \left\lfloor n^{\alpha/2} \right\rfloor} \left| c_{t+k} \right| \\
&\leq O(1) \sum_{k > \left\lfloor n^{\alpha/2} \right\rfloor} \left| c_k \right| \to 0
\end{aligned}$$

by Assumption LP(i). This establishes the result for $Z_n^{(2)}(1)$. For $Y_n^{(2)}(1)$,

$$\begin{aligned}
\left\| Y_n^{(2)}(1) \right\|_2^2 &= \frac{\sigma^2}{n^{\alpha}} \sum_{k > \tau_n(\beta)} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{k-t} \right) \left(\sum_{s=1}^{\tau_n(\beta)} \rho_n^{-s} c_{k-s} \right) \\
&\leq \frac{\sigma^2}{n^{\alpha}} \left(\sum_{s=1}^{\infty} \left| c_s \right| \right) \sum_{k > \tau_n(\beta)} \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} \left| c_{k-t} \right| \\
&= O(1) \frac{1}{n^{\alpha}} \left\{ \sum_{t=1}^{\lfloor \tau_n(\beta)/2 \rfloor} \rho_n^{-t} \sum_{k > \tau_n(\beta)} \left| c_{k-t} \right| + \sum_{t = \lfloor \tau_n(\beta)/2 \rfloor + 1}^{\tau_n(\beta)} \rho_n^{-t} \sum_{k > \tau_n(\beta)} \left| c_{k-t} \right| \right\} \\
&= O\left(\sum_{k > \lfloor \tau_n(\beta)/2 \rfloor} \left| c_k \right| \right) + O\left(\rho_n^{-\tau_n(\beta)/2} \right) = o(1)
\end{aligned}$$

as $n \to \infty$ by Assumption LP(i) and (9).

Proposition 3.2.2.

(i) *The following approximation is valid under both Assumption LP(ii) with* $\kappa \in (1/2, 1)$ *and under Assumptions LP(i) and LP(iii) with* $\kappa = 1$*: As* $n \to \infty$,

$$\left\| Z_n^{(1)}(\kappa) - \left(\frac{1}{n^{(1-\kappa)\alpha}} \sum_{j=0}^{\tau_n(\beta)} c_j \rho_n^{-j} \right) \left(\frac{1}{n^{\alpha/2}} \sum_{k=0}^{\tau_n(\beta)} \rho_n^{-k} \varepsilon_k \right) \right\|_2 \to 0. \tag{30}$$

(ii) *Under Assumption LP(i),*

$$\left\| Z_n^{(1)}(1) \right\|_2^2, \left\| Y_n^{(1)}(1) \right\|_2^2 \to \frac{\omega^2}{2c} \quad \textit{as } n \to \infty.$$

Proof. For part (i), we can write

$$\begin{aligned}
Z_n^{(1)}(\kappa) &= \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{t=0}^{\tau_n(\beta)} \rho_n^{-t} \sum_{j=0}^{t} c_j \varepsilon_{t-j} \\
&= \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{j=0}^{\tau_n(\beta)} c_j \sum_{t=j}^{\tau_n(\beta)} \rho_n^{-t} \varepsilon_{t-j} \\
&= \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{j=0}^{\tau_n(\beta)} c_j \rho_n^{-j} \sum_{t=j}^{\tau_n(\beta)} \rho_n^{-(t-j)} \varepsilon_{t-j} \\
&= \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{j=0}^{\tau_n(\beta)} c_j \rho_n^{-j} \sum_{k=0}^{\tau_n(\beta)-j} \rho_n^{-k} \varepsilon_k,
\end{aligned}$$

so, using the inequality $\left(\sum_{j=0}^{r-1} x_j \right)^2 \leq r \sum_{j=0}^{r} x_j^2$, the remainder term of (30) can be estimated by

$$\begin{aligned}
\frac{1}{n^{(3-2\kappa)\alpha}} \left\| \sum_{j=0}^{\tau_n(\beta)} c_j \rho_n^{-j} \sum_{k=\tau_n(\beta)-j+1}^{\tau_n(\beta)} \rho_n^{-k} \varepsilon_k \right\|_2^2 &\leq \frac{n^\beta + 1}{n^{(3-2\kappa)\alpha}} \sum_{j=0}^{\tau_n(\beta)} c_j^2 \rho_n^{-2j} \left\| \sum_{k=\tau_n(\beta)-j+1}^{\tau_n(\beta)} \rho_n^{-k} \varepsilon_k \right\|_2^2 \\
&\leq \frac{\left(n^\beta + 1\right) \sigma^2}{n^{(3-2\kappa)\alpha}} \sum_{j=0}^{\infty} c_j^2 \rho_n^{-2j} \sum_{k=\tau_n(\beta)-j+1}^{\tau_n(\beta)} \rho_n^{-2k} \\
&= O\left(e^{-cn^{\beta-\alpha}} n^{\beta-(1-\kappa)\alpha} \right) \sum_{j=0}^{\infty} c_j^2 \to 0.
\end{aligned}$$

Note that this approximation only requires square summability of the sequence $(c_j)_{j \geq 0}$.

For part (ii), $\left\| n^{-\alpha/2} \sum_{k=0}^{\tau_n(\beta)} \rho_n^{-k} \varepsilon_k \right\|_2^2 \to \sigma^2/2c$, so, using (30), the asymptotic variance of $Z_n^{(1)}(1)$ will have the required form provided that

$$\sum_{j=0}^{\tau_n(\beta)} c_j \rho_n^{-j} \to \sum_{j=0}^{\infty} c_j \quad \text{as } n \to \infty. \tag{31}$$

Unlike (30), (31) is valid only for absolutely summable sequences $(c_j)_{j\geq 0}$. Since

$$\sum_{j=0}^{\tau_n(\beta)} c_j \rho_n^{-j} = \sum_{j=0}^{\infty} c_j \rho_n^{-j} \mathbf{1}\left\{ j \leq \tau_n(\beta) \right\} \quad \text{and} \quad \sum_{j=0}^{\infty} \left| c_j \rho_n^{-j} \mathbf{1}\left\{ j \leq \tau_n(\beta) \right\} \right| \leq \sum_{j=0}^{\infty} |c_j|$$

absolute summability of $(c_j)_{j\geq 0}$ implies that (31) follows by dominated convergence.

The asymptotic variance of $Y_n^{(1)}(1)$ can be shown to be identical to that of $Z_n^{(1)}(1)$ by using the fact that $Y_n(1)$ and $Z_n(1)$ have the same variance for all n (given by (10)). The triangle inequality for L_2 spaces yields

$$\left| E\left(X^2\right) - E\left(Y^2\right) \right| \leq \|X - Y\|_2 \left(\|X\|_2 + \|Y\|_2 \right) \tag{32}$$

for all $X, Y \in L_2$. Proposition 3.2.1, (14) and the fact that $Z_n^{(1)}(1)$ has finite asymptotic variance imply that $\left\| Z_n(1) - Z_n^{(1)}(1) \right\|_2 \to 0$ and $\sup_{n\in\mathbb{N}} \left\| Z_n^{(1)}(1) \right\|_2 < \infty$. Hence, $\sup_{n\in\mathbb{N}} \|Z_n(1)\|_2 < \infty$ and (32) yields

$$\left| \|Z_n(1)\|_2^2 - \left\| Z_n^{(1)}(1) \right\|_2^2 \right| \to 0 \quad \text{as } n \to \infty.$$

By (17) and Proposition 3.2.1 $\left\| Y_n(1) - Y_n^{(1)}(1) \right\|_2 \to 0$. Since $\sup_{n\in\mathbb{N}} \|Z_n(1)\|_2 < \infty$, (10) ensures that both $\sup_{n\in\mathbb{N}} \|Y_n(1)\|_2$ and $\sup_{n\in\mathbb{N}} \left\| Y_n^{(1)}(1) \right\|_2$ are finite, and consequently (32) implies that

$$\left| \|Y_n(1)\|_2^2 - \left\| Y_n^{(1)}(1) \right\|_2^2 \right| \to 0 \quad \text{as } n \to \infty.$$

Since $\|Y_n(1)\|_2^2 = \|Z_n(1)\|_2^2$ for all n, the triangle inequality for real numbers yields

$$\left| \left\| Y_n^{(1)}(1) \right\|_2^2 - \left\| Z_n^{(1)}(1) \right\|_2^2 \right| \leq \left| \|Y_n(1)\|_2^2 - \left\| Y_n^{(1)}(1) \right\|_2^2 \right| + \left| \|Z_n(1)\|_2^2 - \left\| Z_n^{(1)}(1) \right\|_2^2 \right| \to 0$$

showing that $Y_n^{(1)}(1)$ and $Z_n^{(1)}(1)$ have the same asymptotic variance.

Proof of Lemma 2. By Propositions 3.2.1, 3.2.2 and (31) we obtain that

$$\begin{bmatrix} Y_n(1) \\ Z_n(1) \end{bmatrix} = \sum_{k=1}^{\tau_n(\beta)} \zeta_{nk} + o_p(1)$$

where

$$\zeta_{nk} := \frac{1}{n^{\alpha/2}} \begin{bmatrix} \left(\sum_{t=1}^{k} \rho_n^{-t} c_{k-t}\right) \varepsilon_{n+1-k} \\ \left(\sum_{j=0}^{\infty} c_j\right) \rho_n^{-k} \varepsilon_k \end{bmatrix} \tag{33}$$

is a martingale difference array with respect to $\mathcal{F}_k = \sigma(\varepsilon_k, \varepsilon_{k-1}, ...)$ since, by (9), $2\tau_n(\beta) \leq n^{\beta} < n+1$ implying that $n+1-k > k$ for all $k \in \{1, ..., \tau_n(\beta)\}$. Therefore, $\mathcal{F}_{k-1} \subseteq \mathcal{F}_{n-k}$ so $E_{\mathcal{F}_{k-1}}(\varepsilon_{n+1-k}) = 0$, $E_{\mathcal{F}_{k-1}}\left(\varepsilon_{n+1-k}^2\right) = \sigma^2$ and $E_{\mathcal{F}_{k-1}}(\varepsilon_k \varepsilon_{n+1-k}) = 0$, all the above equalities holding almost surely by the chain rule for iterated conditional expectations (Kallenberg, 2002, Theorem 6.1(vii)).

We now apply a standard martingale CLT on $\sum_{k=1}^{\tau_n(\beta)} \zeta_{nk}$ (Corollary 3.1 of Hall and Heyde (1980) or Proposition A1 of Magdalinos and Phillips, (2008)). By Proposition 3.2.2, the conditional variance of $\sum_{k=1}^{\tau_n(\beta)} \zeta_{nk}$ is given by

$$\sum_{k=1}^{\tau_n(\beta)} E_{\mathcal{F}_{k-1}} \zeta_{nk} \zeta_{nk}' = \operatorname{diag}\left(\left\|Y_n^{(1)}(1)\right\|_2^2, \left\|Z_n^{(1)}(1)\right\|_2^2\right) \to \frac{\omega^2}{2c} I_2$$

as $n \to \infty$, where I_2 denotes the 2×2 identity matrix. Therefore, provided that the Lindeberg condition

$$\sum_{k=1}^{\tau_n(\beta)} E_{\mathcal{F}_{k-1}}\left(\|\zeta_{nk}\|^2 \mathbf{1}\{\|\zeta_{nk}\| > \delta\}\right) \to_p 0 \quad \delta > 0 \tag{34}$$

holds, Lemma 1 follows from the aforementioned martingale CLT. To establish (34), let $\eta := \delta / \left(\sum_{j=0}^{\infty} |c_j|\right)$ and note that

$$\begin{aligned} \mathbf{1}\{\|\zeta_{nk}\| > \delta\} &\leq \mathbf{1}\left\{\left(\sum_{t=1}^{k} |c_{k-t}|\right)^2 \varepsilon_{n+1-k}^2 + \left(\sum_{j=0}^{\infty} |c_j|\right)^2 \varepsilon_k^2 > n^{\alpha}\delta^2\right\} \\ &\leq \mathbf{1}\left\{\varepsilon_{n+1-k}^2 + \varepsilon_k^2 > n^{\alpha}\eta^2\right\} \\ &\leq \mathbf{1}\left\{\varepsilon_{n+1-k}^2 > n^{\alpha}\eta^2/2\right\} + \mathbf{1}\left\{\varepsilon_k^2 > n^{\alpha}\eta^2/2\right\}. \end{aligned}$$

Thus, expanding the left side of (34) and noting that, as $n \to \infty$,

$$\frac{1}{n^{\alpha}} \sum_{k=1}^{\tau_n(\beta)} \left\{\rho_n^{-2k} + \left(\sum_{t=1}^{k} \rho_n^{-t} c_{k-t}\right)^2\right\} = O(1) + \frac{1}{\sigma^2} \left\|Y_n^{(1)}(1)\right\|_2^2 = O(1)$$

we obtain that the following condition is sufficient for (34):

$$\sup_{1\leq k\leq\tau_n(\beta)}\max_{r,s\in S_k}\left\|E_{\mathcal{F}_{k-1}}\left(\varepsilon_r^2\mathbf{1}\left\{\varepsilon_s^2>n^\alpha\eta^2/2\right\}\right)\right\|_1\to 0, \tag{35}$$

where $S_k := \{k, n+1-k\}$. When $r = s$, the left side of (35) is bounded by

$$\sup_{1\leq j\leq n} E\left(\varepsilon_j^2\mathbf{1}\left\{\varepsilon_j^2>n^\alpha\eta^2/2\right\}\right)\to 0$$

as $n\to\infty$ by uniform integrability of $\left(\varepsilon_j^2\right)_{j\in\mathbb{Z}}$. When $r < s$, the fact that $\mathcal{F}_{k-1}\subseteq\mathcal{F}_{n-k}$ for all $k\in\{1,...,\tau_n(\beta)\}$ and the conditional Markov inequality yield

$$\begin{aligned}
E_{\mathcal{F}_{k-1}}\left(\varepsilon_r^2\mathbf{1}\left\{\varepsilon_s^2>n^\alpha\eta^2/2\right\}\right) &= E_{\mathcal{F}_{k-1}}\left(\varepsilon_k^2\mathbf{1}\left\{\varepsilon_{n+1-k}^2>n^\alpha\eta^2/2\right\}\right)\\
&= E_{\mathcal{F}_{k-1}}\left\{\varepsilon_k^2E_{\mathcal{F}_{n-k}}\left(\mathbf{1}\left\{\varepsilon_{n+1-k}^2>n^\alpha\eta^2/2\right\}\right)\right\}\\
&= E_{\mathcal{F}_{k-1}}\left(\varepsilon_k^2P_{\mathcal{F}_{n-k}}\left\{\varepsilon_{n+1-k}^2>n^\alpha\eta^2/2\right\}\right)\\
&\leq \frac{2}{n^\alpha\eta^2}E_{\mathcal{F}_{k-1}}\left(\varepsilon_k^2E_{\mathcal{F}_{n-k}}\varepsilon_{n+1-k}^2\right)=\frac{2\sigma^4}{n^\alpha\eta^2}
\end{aligned}$$

establishing (35). Since $E_{\mathcal{F}_{k-1}}\left(\varepsilon_{n+1-k}^2\mathbf{1}\left\{\varepsilon_k^2>n^\alpha\eta^2/2\right\}\right)=\sigma^2P_{\mathcal{F}_{k-1}}\left\{\varepsilon_k^2>n^\alpha\eta^2/2\right\}$, an identical argument shows (35) for $r > s$.

3.3. Proof of Lemma 3 and Lemma 4

We begin by deriving the asymptotic variance of $Z_n(\kappa)$. We show that, unlike the weakly dependent case, both components in (14) will contribute to the limiting distribution. We consider each component separately.

Proposition 3.3.1. *Under Assumption LP(ii), we obtain, for each* $\kappa\in(1/2,1)$

$$\left\|\frac{1}{L(n^\alpha)}Z_n^{(1)}(\kappa)-c^{\kappa-1}\Gamma(1-\kappa)\left(\frac{1}{n^{\alpha/2}}\sum_{k=0}^{\tau_n(\beta)}\rho_n^{-k}\varepsilon_k\right)\right\|_2\to 0\quad \text{as } n\to\infty \tag{36}$$

where $\tau_n(\beta)$ *is the sequence defined in (9), and*

$$\left\|\frac{1}{L(n^\alpha)}Z_n^{(1)}(\kappa)\right\|_2^2\to\frac{\sigma^2}{2}c^{2\kappa-3}\Gamma(1-\kappa)^2\quad \text{as } n\to\infty. \tag{37}$$

Proof. Lemma A3 in the Appendix shows that

$$\frac{1}{L(n^\alpha)}\frac{1}{n^{(1-\kappa)\alpha}}\sum_{j=0}^{\tau_n(\beta)}c_j\rho_n^{-j}\to c^{\kappa-1}\Gamma(1-\kappa)\quad\text{as } n\to\infty.$$

Combining the above with (30) and the fact that $\left\|n^{-\alpha/2}\sum_{k=0}^{\tau_n(\beta)}\rho_n^{-k}\varepsilon_k\right\|_2^2\to\sigma^2/2c$ proves both (36) and (37).

Proposition 3.3.2. *Under Assumption LP(ii):*

(i) *For each* $\kappa \in (1/2, 1)$

$$\left\| \frac{1}{L(n^\alpha)} Z_n^{(2)}(\kappa) - \frac{1}{L(n^\alpha)} \frac{1}{n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{j=1}^{\tau_n(\beta)} \left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+j} \right) \varepsilon_{-j} \right\|_2 \to 0, \tag{38}$$

as $n \to \infty$*, where* $\tau_n(\beta)$ *is the sequence defined in (9).*

(ii) *For each* $\kappa \in (1/2, 1)$

$$\left\| \frac{1}{L(n^\alpha)} Z_n^{(2)}(\kappa) \right\|_2^2 \to \sigma^2 c^{2\kappa-3} \Gamma(1-\kappa)^2 \left(\frac{\sin \pi\kappa}{\sin \pi(2\kappa-1)} - \frac{1}{2} \right). \tag{39}$$

Proof. The remainder of (38) can be estimated as follows:

$$\begin{aligned}
\left\| \frac{1}{L(n^\alpha) n^{\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{j>\tau_n(\beta)} \sum_{t=1}^{n} \rho_n^{-t} c_{t+j} \varepsilon_{-j} \right\|_2^2 &= \frac{\sigma^2}{L(n^\alpha)^2 n^{2\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{j>\tau_n(\beta)} \left(\sum_{t=1}^{n} \rho_n^{-t} c_{t+j} \right)^2 \\
&= \frac{\sigma^2}{L(n^\alpha)^2 n^{2\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{t,s=1}^{n} \rho_n^{-t-s} \sum_{j>\tau_n(\beta)} c_{t+j} c_{s+j} \\
&\leq \frac{\sigma^2}{L(n^\alpha)^2 n^{2\left(\frac{3}{2}-\kappa\right)\alpha}} \sum_{t,s=1}^{n} \rho_n^{-t-s} \left(\sum_{j>\tau_n(\beta)} c_j^2 \right) \\
&= O\left(\frac{1}{L(n^\alpha)^2} \frac{n^{2\alpha}}{n^{2\left(\frac{3}{2}-\kappa\right)\alpha}} \right) \sum_{j>\tau_n(\beta)} j^{-2\kappa} L(j)^2 \\
&= O\left(\frac{1}{n^{(2\kappa-1)(\beta-\alpha)}} \right) \left[\frac{L(n^\beta)}{L(n^\alpha)} \right]^2 \to 0
\end{aligned}$$

because $\sum_{j>\tau_n(\beta)} j^{-2\kappa} L(j)^2 = O\left(n^{\beta(1-2\kappa)} L(n^\beta)^2 \right)$ by Karamata's theorem. Since $Z_n^{(1)}(\kappa)$ and $Z_n^{(2)}(\kappa)$ are uncorrelated, part (ii) follows immediately from (14), Lemma 1 and (37).

We now turn our attention to the asymptotic variance of $Y_n(\kappa)$.

Proposition 3.3.3. *Under Assumption LP(ii):*

(i) $L(n^\alpha)^{-1} Y_n^{(2)}(\kappa) \to_{L_2} 0$.

(ii) *For each* $\kappa \in (1/2, 1)$, $L(n^\alpha)^{-1} Y_n^{(1)}(\kappa)$ *and* $L(n^\alpha)^{-1} Z_n(\kappa)$ *have the same asymptotic variance as* $n \to \infty$*, given by (11).*

Proof. For part (i), since $\rho_n^{-i} = O\left(e^{-\frac{c}{n^\alpha}i}\right)$ as $n \to \infty$ for all $i \in \{1, ..., n\}$ and $\sup_{i\geq 1} i^{-\delta} L(i) < \infty$ for any $\delta > 0$, there exists $C \in (0, \infty)$ such that

$$\begin{aligned}\left\|\frac{Y_n^{(2)}(\kappa)}{L(n^\alpha)}\right\|_2^2 &= \frac{1}{L(n^\alpha)^2}\frac{\rho_n^{-2}}{n^{(3-2\kappa)\alpha}}\left\|\sum_{j=0}^{\infty}\sum_{i=j+1}^{n+j}\rho_n^{-(n+j-i)}c_i\varepsilon_{-j}\right\|_2^2 \\ &\leq \frac{Ce^{-2cn^{1-\alpha}}}{L(n^\alpha)^2 n^{(3-2\kappa)\alpha}}\sum_{j=1}^{\infty}e^{-2\frac{c}{n^\alpha}j}\left(\sum_{i=j+1}^{n+j}e^{\frac{c}{n^\alpha}i}i^{-(\kappa-\delta)}\right)^2 + O\left(\frac{L(n^\alpha)^{-2}}{n^{2(1-\kappa)\alpha}}\right),\end{aligned}$$

where δ can be chosen as follows:

$$\delta \in \left(0, \frac{(1-\alpha)(2\kappa-1)}{2}\right). \tag{40}$$

We now make use of the fact that, for any decreasing function f on $[0, \infty)$, $\sum_{j=k}^{N} f(j) \leq \int_{k-1}^{N} f(x)\, dx$ for all $k, N \in \mathbb{N}$. Since, for large enough n, $e^{\frac{c}{n^\alpha}i}i^{-(\kappa-\delta)}$ is decreasing in i we obtain $\sum_{i=j+1}^{n+j} e^{\frac{c}{n^\alpha}i}i^{-(\kappa-\delta)} \leq \int_{j}^{n+j} e^{\frac{c}{n^\alpha}x}x^{-(\kappa-\delta)}dx$. Also, since the function

$$g(y) = e^{-2\frac{c}{n^\alpha}y}\left(\int_{y}^{n+y} e^{\frac{c}{n^\alpha}x}x^{-(\kappa-\delta)}dx\right)^2 = \left(\int_{0}^{n} e^{\frac{c}{n^\alpha}z}(z+y)^{-(\kappa-\delta)}dz\right)^2$$

is decreasing, $\sum_{j=1}^{\infty} g(j) \leq \int_0^\infty g(y)\, dy$. Denoting by C a fixed finite constant that may take different values and using the bound $e^{-x}\int_0^x e^u u^{-(\kappa-\delta)}du \leq Cx^{-(\kappa-\delta)}$ for all x bounded away from the origin, we obtain, for large enough n,

$$\begin{aligned}\left\|\frac{Y_n^{(2)}(\kappa)}{L(n^\alpha)}\right\|_2^2 &\leq \frac{C}{L(n^\alpha)^2}\frac{e^{-2cn^{1-\alpha}}}{n^{(3-2\kappa)\alpha}}\int_0^\infty e^{-2\frac{c}{n^\alpha}y}\left(\int_y^{n+y} e^{\frac{c}{n^\alpha}x}x^{-(\kappa-\delta)}dx\right)^2 dy + o(1) \\ &= \frac{Cn^{2\alpha\delta}}{L(n^\alpha)^2}\int_0^\infty\left[e^{-\left(cn^{1-\alpha}+z\right)}\int_z^{cn^{1-\alpha}+z}e^u u^{-(\kappa-\delta)}du\right]^2 dz \\ &\leq \frac{Cn^{2\alpha\delta}}{L(n^\alpha)^2}\int_0^\infty\left(cn^{1-\alpha}+z\right)^{-2(\kappa-\delta)}dz \\ &= O\left(\frac{1}{L(n^\alpha)^2}\frac{1}{n^{(1-\alpha)(2\kappa-1)-2\delta}}\right) = o(1)\end{aligned}$$

as $n \to \infty$ by the choice of δ in (40).

For part (ii), the fact that the asymptotic variance of $Z_n(\kappa)$ is given by (11) may be obtained directly from (14), (37) and (39), since $Z_n^{(1)}(\kappa)$ and $Z_n^{(2)}(\kappa)$ are uncorrelated. The result for $Y_n^{(1)}(\kappa)$ can be shown by using a similar argument to that used in the proof of Proposition 3.2.2(ii). The triangle inequality gives

$$\left|\left\|\frac{Y_n^{(1)}(\kappa)}{L(n^\alpha)}\right\|_2^2 - \left\|\frac{Z_n(\kappa)}{L(n^\alpha)}\right\|_2^2\right| \leq \left|\left\|\frac{Y_n^{(1)}(\kappa)}{L(n^\alpha)}\right\|_2^2 - \left\|\frac{Y_n(\kappa)}{L(n^\alpha)}\right\|_2^2\right| + \left|\left\|\frac{Y_n(\kappa)}{L(n^\alpha)}\right\|_2^2 - \left\|\frac{Z_n(\kappa)}{L(n^\alpha)}\right\|_2^2\right|.$$

The second term on the right is identically 0 since $Y_n(\kappa)$ and $Z_n(\kappa)$ have the same variance for all n, see (10). The first term on the right tends to 0 as $n \to \infty$ since $\left\| \frac{Y_n^{(1)}(\kappa)}{L(n^\alpha)} - \frac{Y_n(\kappa)}{L(n^\alpha)} \right\|_2 \to 0$ by (17) and part (i).

Proposition 3.3.4. *Under Assumption LP(ii), we obtain, for each* $\kappa \in (1/2, 1)$

$$L(n^\alpha)^{-1}\left[Z_n^{(1)}(\kappa), Z_n^{(2)}(\kappa), Y_n^{(1)}(\kappa)\right] \Rightarrow \left[Z^{(1)}(\kappa), Z^{(2)}(\kappa), Y(\kappa)\right] \quad \text{as } n \to \infty \tag{41}$$

where $Z^{(1)}(\kappa), Z^{(2)}(\kappa)$ *and* $Y(\kappa)$ *are independent zero-mean Gaussian random variables with variances given by (37), (39) and (11) respectively.*

Proof. By (36) and (38) we obtain that

$$\frac{1}{L(n^\alpha)}\begin{bmatrix} Z_n^{(1)}(\kappa) \\ Z_n^{(2)}(\kappa) \\ Y_n^{(1)}(\kappa) \end{bmatrix} = \sum_{k=1}^{\tau_n(\beta)} \xi_{nk} + o_p(1) \tag{42}$$

where

$$\xi_{nk} := \begin{bmatrix} c^{\kappa-1}\Gamma(1-\kappa)\, n^{-\alpha/2}\rho_n^{-k}\varepsilon_k \\ L(n^\alpha)^{-1} n^{-\left(\frac{3}{2}-\kappa\right)\alpha}\left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+k}\right)\varepsilon_{-k} \\ L(n^\alpha)^{-1} n^{-\left(\frac{3}{2}-\kappa\right)\alpha}\left(\sum_{t=1}^{k} \rho_n^{-t} c_{k-t}\right)\varepsilon_{n+1-k} \end{bmatrix}$$

is a martingale difference array with respect to $\mathcal{F}_{-k} = \sigma(\varepsilon_{-k}, \varepsilon_{-k-1}, ...)$ since, by (9), $n+1-k > k > -k$ for all $k \in \{1, ..., \tau_n(\beta)\}$, so $\mathcal{F}_{-k-1} \subseteq \mathcal{F}_{k-1} \subseteq \mathcal{F}_{n-k}$. Given the set $\Lambda_k = \{k, -k, n+1-k\}$, the above inclusions imply that $E_{\mathcal{F}_{-k-1}}(\varepsilon_r\varepsilon_s) = 0$ *a.s.* for all $r \neq s$, $r, s \in \Lambda_k$.

We now apply the martingale CLT used in the proof of Lemma 1 (Hall and Heyde, 1980) on $\sum_{k=1}^{\tau_n(\beta)} \xi_{nk}$:

$$\begin{aligned} \sum_{k=1}^{\tau_n(\beta)} E_{\mathcal{F}_{-k-1}}\xi_{nk}\xi_{nk}' &= \frac{1}{L(n^\alpha)^2}\operatorname{diag}\left[\left\|Z_n^{(1)}(\kappa)\right\|_2^2, \left\|Z_n^{(2)}(\kappa)\right\|_2^2, \left\|Y_n^{(1)}(\kappa)\right\|_2^2\right] \\ &\to \sigma^2 c^{2\kappa-3}\Gamma(1-\kappa)^2 \operatorname{diag}\left[\frac{1}{2}, \frac{\sin\pi\kappa}{\sin\pi(2\kappa-1)} - \frac{1}{2}, \frac{\sin\pi\kappa}{\sin\pi(2\kappa-1)}\right] \end{aligned}$$

by (37), (39) and Proposition 3.3.3(ii). Since the limit of the conditional variance of $\sum_{k=1}^{\tau_n(\beta)} \xi_{nk}$ is a diagonal matrix, the limit random vector in (41) consists of uncorrelated components. It remains to verify the Lindeberg condition

$$\sum_{k=1}^{\tau_n(\beta)} E_{\mathcal{F}_{-k-1}}\left(\|\xi_{nk}\|^2 \mathbf{1}\{\|\xi_{nk}\| > \delta\}\right) \to_p 0 \quad \delta > 0. \tag{43}$$

By the Cauchy-Schwarz inequality both $\left(\sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} c_{t+k}\right)^2$ and $\left(\sum_{t=1}^{k} \rho_n^{-t} c_{k-t}\right)^2$ are bounded by $\sum_{j=0}^{\infty} c_j^2 O\left(n^{\alpha}\right)$ uniformly in k. Therefore, by square summability of $\left(c_j\right)_{j\geq 0}$ for each $\kappa \in (1/2, 1)$, there exist finite constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}
\mathbf{1}\left\{\|\xi_{nk}\| > \delta\right\} &\leq \mathbf{1}\left\{\varepsilon_k^2 > n^{\alpha} C_1 \delta^2/3\right\} + \mathbf{1}\left\{\varepsilon_{-k}^2 > n^{2(1-\kappa)\alpha} C_2 \delta^2/3\right\} \\
&\quad + \mathbf{1}\left\{\varepsilon_{n+1-k}^2 > n^{2(1-\kappa)\alpha} C_3 \delta^2/3\right\} \\
&\leq \mathbf{1}\left\{\varepsilon_k^2 > n^{2(1-\kappa)\alpha}\eta\right\} + \mathbf{1}\left\{\varepsilon_{-k}^2 > n^{2(1-\kappa)\alpha}\eta\right\} + \mathbf{1}\left\{\varepsilon_{n+1-k}^2 > n^{2(1-\kappa)\alpha}\eta\right\}
\end{aligned}$$

where $\eta := \min\left\{C_1, C_2, C_3\right\} \delta^2/3$. Using the above inequality we obtain

$$\sum_{k=1}^{\tau_n(\beta)} \left\| E_{\mathcal{F}_{-k-1}}\left(\|\xi_{nk}\|^2 \mathbf{1}\left\{\|\xi_{nk}\| > \delta\right\}\right)\right\|_1 \leq S \sup_{1\leq k\leq \tau_n(\beta)} \max_{r,s\in\Lambda_k} \left\| E_{\mathcal{F}_{-k-1}}\left(\varepsilon_r^2 \mathbf{1}\left\{\varepsilon_s^2 > n^{2(1-\kappa)\alpha}\eta\right\}\right)\right\|_1$$

where $S = \sup_{n\in\mathbb{N}} \left\{ n^{-\alpha} \sum_{k=1}^{\tau_n(\beta)} \rho_n^{-2k} + \sigma^{-2} \left\| Y_n^{(1)}(\kappa)\right\|_2^2 + \sigma^{-2} \left\| Z_n^{(2)}(\kappa)\right\|_2^2 \right\} < \infty$. Hence,

$$\sup_{1\leq k\leq \tau_n(\beta)} \max_{r,s\in\Lambda_k} \left\| E_{\mathcal{F}_{-k-1}}\left(\varepsilon_r^2 \mathbf{1}\left\{\varepsilon_s^2 > n^{2(1-\kappa)\alpha}\eta\right\}\right)\right\|_1 \to 0 \qquad (44)$$

is sufficient for (43). When $r = s$,

$$\sup_{1\leq k\leq \tau_n(\beta)} \max_{r\in\Lambda_k} \left\| E_{\mathcal{F}_{-k-1}}\left(\varepsilon_r^2 \mathbf{1}\left\{\varepsilon_r^2 > n^{2(1-\kappa)\alpha}\eta\right\}\right)\right\|_1 \leq \sup_{1\leq j\leq n} E\left(\varepsilon_j^2 \mathbf{1}\left\{\varepsilon_j^2 > n^{2(1-\kappa)\alpha}\eta\right\}\right) \to 0$$

by uniform integrability of $\left(\varepsilon_j^2\right)_{j\in\mathbb{Z}}$. Next, we know by (9) that $\min(r, s) > -k$ for all $r, s \in \Lambda_k$. Therefore, when $r > s$, the conditional Markov inequality yields

$$\begin{aligned}
E_{\mathcal{F}_{-k-1}}\left(\varepsilon_r^2 \mathbf{1}\left\{\varepsilon_s^2 > n^{2(1-\kappa)\alpha}\eta\right\}\right) &= E_{\mathcal{F}_{-k-1}}\left[\mathbf{1}\left\{\varepsilon_s^2 > n^{2(1-\kappa)\alpha}\eta\right\} E_{\mathcal{F}_{r-1}}\left(\varepsilon_r^2\right)\right] \\
&= \sigma^2 P_{\mathcal{F}_{-k-1}}\left\{\varepsilon_s^2 > n^{2(1-\kappa)\alpha}\eta\right\} \\
&\leq \frac{\sigma^2}{n^{2(1-\kappa)\alpha}\eta} E_{\mathcal{F}_{-k-1}} \varepsilon_s^2 = \frac{\sigma^4}{n^{2(1-\kappa)\alpha}\eta},
\end{aligned}$$

showing (44) for $r > s$. An identical argument shows (44) for $r < s$:

$$\begin{aligned}
E_{\mathcal{F}_{-k-1}}\left(\varepsilon_r^2 \mathbf{1}\left\{\varepsilon_s^2 > n^{2(1-\kappa)\alpha}\eta\right\}\right) &= E_{\mathcal{F}_{-k-1}}\left[\varepsilon_r^2 P_{\mathcal{F}_{s-1}}\left\{\varepsilon_s^2 > n^{2(1-\kappa)\alpha}\eta\right\}\right] \\
&\leq \frac{\sigma^4}{n^{2(1-\kappa)\alpha}\eta}.
\end{aligned}$$

This completes the proof of (43) and the proposition.

Proof of Lemma 3. Lemma 3 follows by Proposition 3.3.4, Proposition 3.3.3(i) and the continuous mapping theorem.

Proof of Lemma 4. Denote by $E_1(\cdot)$ the exponential integral in (29). Using (30), Lemma A1 and the Euler summation formula we obtain

$$\begin{aligned}\left\|\frac{1}{\log n^{\alpha}}Z_n^{(1)}(1)\right\|_2^2 &= \frac{\sigma^2\theta^2}{2c}\left(\frac{1}{\log n^{\alpha}}\sum_{j=1}^{\tau_n(\beta)}j^{-1}e^{-j\frac{c}{n^{\alpha}}}\right)^2+o(1)\\ &= \frac{\sigma^2\theta^2}{2c}\left(\frac{1}{\log n^{\alpha}}\int_1^{\tau_n(\beta)}x^{-1}e^{-x\frac{c}{n^{\alpha}}}dx\right)^2+O\left(\frac{1}{\log n}\right)\\ &= \frac{\sigma^2\theta^2}{2c}\left[\frac{1}{\log n^{\alpha}}E_1\left(\frac{c}{n^{\alpha}}\right)\right]^2+o(1)\to\frac{\sigma^2\theta^2}{2c}\end{aligned}$$

as $n\to\infty$ since $E_1(z)\sim-\log z$ as $z\to 0$. Hence, by Lemma 1, the asymptotic variance of $(\log n^{\alpha})^{-1}Z_n^{(1)}(1)$ coincides with that of $(\log n^{\alpha})^{-1}Z_n(1)$ which implies that $\left\|(\log n^{\alpha})^{-1}Z_n^{(2)}(1)\right\|_2^2\to 0$. Moreover, an identical argument to the proof of Proposition 3.3.3(i) with $\kappa=1$, $\delta=0$ and $L(n^{\alpha})=\log n^{\alpha}$ yields $\left\|(\log n^{\alpha})^{-1}Y_n^{(2)}(1)\right\|_2^2\to 0$. Therefore,

$$\frac{1}{\log n^{\alpha}}\left[Y_n(1),Z_n(1)\right]=\frac{1}{\log n^{\alpha}}\left[Y_n^{(1)}(1),Z_n^{(1)}(1)\right]+o_p(1)$$

and Lemma 4 follows by applying an identical martingale CLT to that used in the proof of Lemma 2 (replacing $n^{\alpha/2}$ by $n^{\alpha/2}\log n^{\alpha}$ in the definition of the martingale difference array ζ_{nk} in (33)).

3.4. Proof of Lemma 5

Proposition 3.4.1. *For $\tau_n(\beta)$ as defined in (9), we obtain, as $n\to\infty$,*

(i) $\sum_{t=1}^{n}\rho_n^{-t}u_t=\sum_{t=1}^{\tau_n(\beta)}\rho_n^{-t}u_t+o_p(1)$,

(ii) $\sum_{t=1}^{n}\rho_n^{-(n-t)-1}u_t=\sum_{t=1}^{\tau_n(\beta)}\rho_n^{-t}u_{n+1-t}+o_p(1)$.

Proof. For part (i), using covariance stationarity of $(u_t)_{t\in\mathbb{N}}$ we obtain

$$\begin{aligned}\left\|\sum_{t=1}^{n}\rho_n^{-t}u_t-\sum_{t=1}^{\tau_n(\beta)}\rho_n^{-t}u_t\right\|_1 &= \left\|\sum_{t=\tau_n(\beta)+1}^{n}\rho_n^{-t}u_t\right\|_1\le E|u_1|\sum_{t=\tau_n(\beta)+1}^{n}\rho_n^{-t}\\ &= O\left(n^{\alpha}\rho_n^{-\tau_n(\beta)}\right)=o(1)\end{aligned}$$

as $n \to \infty$. For part (ii), since $\sum_{t=1}^{n} \rho_n^{-(n-t)-1} u_t = \sum_{t=1}^{n} \rho_n^{-t} u_{n+1-t}$

$$\left\| \sum_{t=1}^{n} \rho_n^{-(n-t)-1} u_t - \sum_{t=1}^{\tau_n(\beta)} \rho_n^{-t} u_{n+1-t} \right\|_1 = \left\| \sum_{t=\tau_n(\beta)+1}^{n} \rho_n^{-t} u_{n+1-t} \right\|_1 = O\left(n^{\alpha} \rho_n^{-\tau_n(\beta)} \right)$$

using the same bound as in part (i).

Proof of Lemma 5. The derivations of this subsection are not affected by the memory of the innovation sequence in any way other than the additional normalisation required in the long memory case. Therefore, it is enough to present the argument for part (ii) of the lemma. An identical argument is valid for part (i) and part (iii) by making the appropriate adjustment for the normalisation.

We start by analysing the sample covariance. For λ_n as in (21), the initialization of the mildly explosive process satisfies $X_0 = o_p\left(n^{\alpha/2}\lambda_n\right)$ by Assumption IC. Using the identity $X_{t-1} = X_0 \rho_n^{t-1} + \sum_{j=1}^{t-1} \rho_n^{t-j-1} u_j$ and Proposition 3.4.1(ii) we obtain

$$\begin{aligned} \frac{\rho_n^{-n}}{n^{\alpha}\lambda_n^2} \sum_{t=1}^{n} X_{t-1} u_t &= \frac{1}{n^{\alpha}\lambda_n^2} \left\{ X_0 \sum_{t=1}^{n} \rho_n^{-(n-t)-1} u_t + \rho_n^{-n} \sum_{t=1}^{n} \sum_{j=1}^{t-1} \rho_n^{t-j-1} u_j u_t \right\} \\ &= \frac{X_0}{n^{\alpha/2}\lambda_n} \frac{Y_n(\kappa)}{L(n^{\alpha})} + \frac{\rho_n^{-n}}{n^{\alpha}\lambda_n^2} \sum_{t=1}^{n} \sum_{j=1}^{t-1} \rho_n^{t-j-1} u_j u_t + o_p(1) \\ &= \frac{\rho_n^{-n}}{n^{\alpha}\lambda_n^2} \sum_{t=1}^{n} \sum_{j=1}^{t-1} \rho_n^{t-j-1} u_j u_t + o_p(1), \end{aligned} \tag{45}$$

since $Y_n(\kappa)/L(n^{\alpha}) = O_p(1)$ by Lemma 3. Now the Cauchy-Schwarz inequality gives

$$\begin{aligned} \left\| \frac{\rho_n^{-n}}{n^{\alpha}\lambda_n^2} \sum_{t=1}^{n} \sum_{j=t}^{n} \rho_n^{t-j-1} u_j u_t \right\|_1 &\leq \frac{\rho_n^{-n}}{n^{\alpha}\lambda_n^2} \sum_{t=1}^{n} \sum_{j=t}^{n} \rho_n^{t-j-1} E\left|u_j u_t\right| \\ &\leq \frac{E\left(u_1^2\right)\rho_n^{-n}}{n^{\alpha}\lambda_n^2} \sum_{t=1}^{n} \rho_n^{t-1} \sum_{j=t}^{n} \rho_n^{-j} \\ &= O\left(\frac{\rho_n^{-n} n^{1+\alpha}}{n^{(3-2\kappa)\alpha} L(n^{\alpha})^2} \right) = o(1). \end{aligned}$$

Thus, (45) and Proposition 3.4.1 yield

$$\begin{aligned} \frac{\rho_n^{-n}}{n^{\alpha}\lambda_n^2} \sum_{t=1}^{n} X_{t-1} u_t &= \left(\frac{1}{n^{\alpha/2}\lambda_n} \sum_{t=1}^{n} \rho_n^{-(n-t)-1} u_t \right) \left(\frac{1}{n^{\alpha/2}\lambda_n} \sum_{j=1}^{n} \rho_n^{-j} u_j \right) + o_p(1) \\ &= \frac{Y_n(\kappa)}{L(n^{\alpha})} \frac{Z_n(\kappa)}{L(n^{\alpha})} + o_p(1), \end{aligned} \tag{46}$$

as required. For the sample variance, by taking the square of (3) and summing over $t \in \{1, ..., n\}$ we obtain

$$\begin{aligned}
\sum_{t=1}^{n} X_{t-1}^2 &= \frac{1}{\rho_n^2 - 1}\left\{X_n^2 - X_0^2 - 2\rho_n \sum_{t=1}^{n} X_{t-1}u_t - \sum_{t=1}^{n} u_t^2\right\} \\
&= \frac{1}{\rho_n^2 - 1}\left[X_n^2 + O\left(\rho_n^n \lambda_n^2\right)\right]
\end{aligned}$$

by (46). Thus, since $\rho_n^{-n} X_n = \sum_{t=1}^{n} \rho_n^{-t} u_t$ and $\rho_n^2 - 1 = 2c/n^\alpha + O\left(n^{-2\alpha}\right)$, we obtain

$$\begin{aligned}
\frac{\rho_n^{-2n}}{n^{2\alpha}\lambda_n^2} \sum_{t=1}^{n} X_{t-1}^2 &= \frac{1}{2c}\left[\frac{1}{n^{\alpha/2}\lambda_n} \sum_{t=1}^{n} \rho_n^{-t} u_t\right]^2 + o_p(1) \\
&= \frac{1}{2c}\left[\frac{1}{L(n^\alpha)} Z_n(\kappa)\right]^2 + o_p(1)
\end{aligned}$$

by Proposition 3.4.1(i). This completes the proof of Lemma 5.

4. Appendix

This section contains some asymptotic results for sums and integrals that are used in the proof of Lemmas 1, 3 and 4. We begin by showing that the asymptotic equivalence $\rho_n^{-t} \sim e^{-\frac{c}{n^\alpha}t}$ as $n \to \infty$ is valid uniformly in $t \in \{1, ..., \tau_n(\beta)\}$.

Lemma A1. *Let $\tau_n(\beta)$ be the sequence defined in (9). Then*

$$\sup_{1 \le t \le \tau_n(\beta)} \left|\rho_n^{-t} - e^{-\frac{c}{n^\alpha}t}\right| = O\left(\frac{1}{n^{\alpha/2}}\right) \quad \text{as } n \to \infty. \tag{47}$$

Proof. Using the expansion $\log(1+x) = x + O(x^2)$ as $x \to 0$, we obtain, as $n \to \infty$

$$\begin{aligned}
\rho_n^{-t} &= \exp\left\{-t \log\left(1 + \frac{c}{n^\alpha}\right)\right\} = \exp\left\{-t\left[\frac{c}{n^\alpha} + O\left(\frac{1}{n^{2\alpha}}\right)\right]\right\} \\
&= e^{-\frac{c}{n^\alpha}t}\left[1 - \left(\exp\left\{O\left(\frac{t}{n^{2\alpha}}\right)\right\} - 1\right)\right].
\end{aligned}$$

Using the mean value theorem and monotonicity of the exponential function we obtain the following elementary inequality: $|e^x - 1| \le xe^x$ for all $x \ge 0$. Application of this inequality yields

$$\begin{aligned}\sup_{1\le t\le\tau_n(\beta)}\left|\rho_n^{-t}-e^{-\frac{c}{n^\alpha}t}\right| &\le \sup_{1\le t\le\tau_n(\beta)}\left|\exp\left\{O\left(\frac{t}{n^{2\alpha}}\right)\right\}-1\right| \\ &\le \sup_{1\le t\le\tau_n(\beta)} O\left(\frac{t}{n^{2\alpha}}\right)\exp\left\{O\left(\frac{t}{n^{2\alpha}}\right)\right\} \\ &= O\left(\frac{n^{3\alpha/2}}{n^{2\alpha}}\right)\exp\left\{O\left(\frac{n^{3\alpha/2}}{n^{2\alpha}}\right)\right\}=O\left(\frac{1}{n^{\alpha/2}}\right)\end{aligned}$$

as required.

In the remainder of this section we establish two types of asymptotic results that have been used in Section 3: An Abelian theorem for integrals involving regularly varying functions and approximation of sums by integrals by means of the Euler summation formula. In the spirit of Apostol (1957) p.202 we derive the following form of the Euler-Maclaurin formula: Let $m, M\in\mathbb{N}$. If f has finite variation $V_f(m,M)$ on $[m,M]$ then applying the integration by parts formula for Stieltjes integrals on $\int_m^M f(x)\,d(x-\lfloor x\rfloor)$ yields

$$\sum_{j=m}^M f(j)-\int_m^M f(x)\,dx=f(m)+\int_m^M(x-\lfloor x\rfloor)\,df(x)\,.$$

Since $x-\lfloor x\rfloor<1$, the above formula implies the following approximation:

$$\left|\sum_{j=m}^M f(j)-\int_m^M f(x)\,dx\right|\le|f(m)|+V_f(m,M)\quad m,M\in\mathbb{N}. \tag{48}$$

Lemma A2 is a standard Abelian theorem, see Korevaar (2004, Proposition 5.4).

Lemma A2. *Given a slowly varying function L, let $\phi(x)=x^\rho L(x)$. If k is real function satisfying $\int_0^\infty x^\rho|k(x)|\,dx<\infty$ for some $\rho\ne0$ and*

$$\frac{1}{x}\int_0^B k\left(\frac{t}{x}\right)\phi(t)\,dt=o(\phi(x))\quad\text{as } x\to\infty \tag{49}$$

for any $B>0$, then

$$\int_0^\infty k(t)\,\phi(xt)\,dt=[1+o(1)]\,\phi(x)\int_0^\infty k(t)\,t^\rho dt\quad\text{as } x\to\infty.$$

Lemma A3. *Under Assumption LP(ii), let* t_0 *be a positive constant such that* $\varphi(t) = L(t)\,t^{-\kappa}$ *is non-increasing on* $[t_0, \infty)$,

$$I_n(\kappa, \psi) = \frac{1}{L(n^\alpha)} \int_{\frac{\psi}{n^\alpha}}^{\frac{c\tau_n(\beta)}{n^\alpha}} e^{-y} y^{-\kappa} L\left(\frac{n^\alpha y}{c}\right) dy \quad \psi > 0$$

and $\lambda_n = L(n^\alpha)\,n^{(1-\kappa)\alpha}$. *Then, the following hold as* $n \to \infty$:

(i) $\lambda_n^{-1} \sum_{t=1}^{\tau_n(\beta)} L(t)\,t^{-\kappa} \left|\rho_n^{-t} - e^{-\frac{c}{n^\alpha}t}\right| \to 0.$

(ii) $\left|\lambda_n^{-1} \sum_{t=1}^{\tau_n(\beta)} L(t)\,t^{-\kappa} e^{-\frac{c}{n^\alpha}t} - c^{\kappa-1} I_n(\kappa, \lfloor t_0 \rfloor + 1)\right| = O\left(\lambda_n^{-1}\right).$

(iii) $I_n(\kappa, \psi) \to \Gamma(1-\kappa)$ *for all* $\psi > 0$.

Proof. For part (i), Lemma A1 yields that there exists $C \in (0, \infty)$ such that

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{t=1}^{\tau_n(\beta)} L(t)\,t^{-\kappa} \left|\rho_n^{-t} - e^{-\frac{c}{n^\alpha}t}\right| &\leq \frac{C}{\lambda_n n^{\alpha/2}} \sum_{t=1}^{\tau_n(\beta)} t^{-\kappa} L(t) \\ &= O\left(\frac{L\left(n^\beta\right) n^{(1-\kappa)\beta}}{L(n^\alpha)\, n^{\alpha/2} n^{(1-\kappa)\alpha}}\right) = o(1) \end{aligned}$$

by Karamata's theorem, since (9) implies that $\alpha/2 - (1-\kappa)(\beta - \alpha) > 0$.

For part (ii), let $f_n(t) := L(t)\,t^{-\kappa} e^{-\frac{c}{n^\alpha}t}$. Since $\varphi(t)$ non-increasing on $[t_0, \infty)$ so is $f_n(t)$ and

$$\lambda_n^{-1} \sum_{t=1}^{\tau_n(\beta)} f_n(t) = \lambda_n^{-1} \sum_{t=\lfloor t_0 \rfloor + 1}^{\tau_n(\beta)} f_n(t) + O\left(\lambda_n^{-1}\right).$$

We now approximate the sum on the right by the corresponding integral using (48). Since $f_n(\cdot)$ is non-increasing on $[t_0, \infty)$,

$$\sup_{n \in \mathbb{N}} V_{f_n}[t_0, \infty) = \sup_{n \in \mathbb{N}} f_n(t_0) \leq \sup_{t \geq t_0} L(t)\,t^{-\kappa} < \infty$$

so (48) implies that

$$\left|\sum_{j=1}^{\tau_n(\beta)} f_n(t) - \int_{\lfloor t_0 \rfloor + 1}^{\tau_n(\beta)} f_n(t)\,dt\right| = O\left(\lambda_n^{-1}\right) \quad \text{as } n \to \infty.$$

The result follows since $\int_{\lfloor t_0 \rfloor + 1}^{\tau_n(\beta)} f_n(t)\,dt = c^{\kappa-1} I_n(\kappa, \lfloor t_0 \rfloor + 1)$.

For part (iii), we apply Lemma A2 on

$$\begin{aligned} J_n(\kappa) &= \frac{1}{L(n^\alpha)}\int_0^\infty e^{-y}y^{-\kappa}L\left(\frac{n^\alpha y}{c}\right)dy \qquad (50)\\ &= \frac{c^{-\kappa+1}}{\varphi(n^\alpha)}\int_0^\infty e^{-cy}\varphi(n^\alpha y)\,dy. \end{aligned}$$

Since $\int_0^\infty e^{-cy}y^{-\kappa}dy<\infty$, the integrability condition of Lemma A2 is satisfied. To verify (49) write, for any $B>0$ and any $\delta\in(0,1-\kappa)$,

$$\begin{aligned} \frac{1}{n^\alpha}\int_0^B e^{-\frac{c}{n^\alpha}y}\varphi(y)\,dy &\leq \frac{1}{n^\alpha}\int_0^B\varphi(y)\,dy\leq\sup_{t\in(0,B]}t^\delta L(t)\frac{1}{n^\alpha}\int_0^B y^{-\kappa-\delta}dy\\ &= \sup_{t\in[0,B]}t^\delta L(t)\frac{B^{1-\kappa-\delta}}{n^\alpha}=O\left(\frac{1}{n^\alpha}\right)=o\left(\varphi(n^\alpha)\right) \end{aligned}$$

by (5). Thus, using Lemma A2 we obtain

$$J_n(\kappa)\to c^{-\kappa+1}\int_0^\infty e^{-cy}y^{-\kappa}dy=\Gamma(1-\kappa)\quad\text{as } n\to\infty. \qquad (51)$$

It remains to show that $I_n(\kappa,\psi)$ and $J_n(\kappa)$ are asymptotically equivalent:

$$|I_n(\kappa,\psi)-J_n(\kappa)|=\frac{1}{L(n^\alpha)}\left\{\int_0^{\frac{\psi}{n^\alpha}}+\int_{\frac{c\tau_n(\beta)}{n^\alpha}}^\infty\right\}e^{-y}y^{-\kappa}L\left(\frac{n^\alpha y}{c}\right)dy.$$

Choosing $\delta\in(0,1-\kappa)$ and using (5), we obtain that the first integral is bounded by

$$\sup_{y\in[0,1]}y^\delta L(y)\frac{c^\delta}{n^{\alpha\delta}L(n^\alpha)}\int_0^{\frac{\psi}{n^\alpha}}y^{-\kappa-\delta}dy=O\left(\frac{1}{L(n^\alpha)\,n^{\alpha(1-\kappa)}}\right).$$

For the second integral, using the property $\sup_{x\geq u}x^{-\kappa}L(x)\sim u^{-\kappa}L(u)$ as $u\to\infty$ (see Seneta, 1976, p.65) and (9), we obtain the following bound:

$$\frac{n^{\alpha\kappa}}{c^\kappa L(n^\alpha)}\sup_{y\geq\tau_n(\beta)}y^{-\kappa}L(y)\int_{\frac{c\tau_n(\beta)}{n^\alpha}}^\infty e^{-y}dy=o\left(e^{-\frac{c}{2}n^{\beta-\alpha}}\right).$$

Thus, $|I_n(\kappa,\psi)-J_n(\kappa)|\to0$ as $n\to\infty$ and part (ii) follows by (51).

Lemma A4. *Let $f(t,y):=L(t+y)(t+y)^{-\kappa}L(y)y^{-\kappa}$, where $\kappa\in(1/2,1)$ and $L(t)$ is a slowly varying function such that $\varphi(t)=t^{-\kappa}L(t)$ is eventually non-increasing on $[t_0,\infty)$. Then, as $n\to\infty$,*

$$\frac{1}{\lambda_n^2}\left[\sum_{t=1}^{\tau_n(\beta)}e^{-\frac{c}{n^\alpha}t}\sum_{j=1}^{\tau_n(\beta)}f(t,j)-\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)}e^{-\frac{c}{n^\alpha}t}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)}f(t,y)\,dydt\right]\to0$$

where λ_n is the sequence defined in (21). Under Assumption LP(iii), the above formula applies with $f(t,y)=(t+y)^{-1}y^{-1}$, $\lambda_n=\log n$ and $t_0=0$.

Proof. Choosing $\delta \in (0, 1-\kappa)$ and $\eta = \min(\delta, \kappa)$ we obtain

$$\begin{aligned}
\frac{1}{\lambda_n^2}\sum_{t=1}^{\tau_n(\beta)} e^{-\frac{ct}{n^\alpha}}\left[\sum_{j=1}^{\tau_n(\beta)} f(t,j) - \sum_{j=\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(t,j)\right] &= \frac{1}{\lambda_n^2}\sum_{t=1}^{\tau_n(\beta)} e^{-\frac{ct}{n^\alpha}} \sum_{j=1}^{\lfloor t_0\rfloor} f(t,j) \\
&\leq \left[\sup_{t\geq 1} t^{-\eta} L(t)\right]^2 \frac{t_0}{\lambda_n^2}\sum_{t=1}^{\tau_n(\beta)} e^{-\frac{ct}{n^\alpha}} t^{-(\kappa-\delta)} \\
&\leq O(1)\frac{1}{\lambda_n^2}\int_1^\infty e^{-\frac{ct}{n^\alpha}} t^{-(\kappa-\delta)} dt \\
&= O\left(n^{-(1-\kappa-\delta)\alpha}\right)\Gamma(1-\kappa+\delta) \to 0.
\end{aligned}$$

The above calculation shows that

$$\frac{1}{\lambda_n^2}\left[\sum_{t=1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}t}\sum_{j=1}^{\tau_n(\beta)} f(t,j) - \sum_{t=\lfloor t_0\rfloor+1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}t}\sum_{j=\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(t,j)\right] \to 0 \quad \text{as } n\to\infty$$

so we only need to apply the Euler approximation (48) to the second sum, where $f(t,j)$ is non-increasing in both its arguments. We first show that

$$\frac{1}{\lambda_n^2}\sum_{t=\lfloor t_0\rfloor+1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}t}\left|\sum_{j=\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(t,j) - \int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(t,y)\,dy\right| \to 0 \quad \text{as } n\to\infty. \tag{52}$$

Fixing t and regarding $f(t,y)$ as a function of y, f is non-increasing on $[t_0,\infty)$ so $V_f[t_0,\infty) = f(t,t_0)$. Using (48), the left side of (52) is bounded by

$$\begin{aligned}
\frac{2}{\lambda_n^2}\sum_{t=1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}t} f(t,t_0) &= \frac{2L(t_0)\,t_0^{-\kappa}}{\lambda_n^2}\sum_{t=1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}t} L(t+t_0)(t+t_0)^{-\kappa} \\
&= O\left(\frac{1}{\lambda_n^2}\sum_{t=1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}t} L(t)\,t^{-\kappa}\right) = O\left(\frac{1}{\lambda_n}\right)
\end{aligned}$$

by Lemma A3. This shows (52). The lemma will follow from combining (52) and

$$\frac{1}{\lambda_n^2}\left|\sum_{t=\lfloor t_0\rfloor+1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}t}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(t,y)\,dy - \int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} e^{-\frac{c}{n^\alpha}x}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(x,y)\,dydx\right| \to 0. \tag{53}$$

Since the function $g_n(t) = e^{-\frac{c}{n^\alpha}t}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(t,y)\,dy$ is non-increasing on $[t_0,\infty)$, $V_{g_n}[t_0,\infty) = g_n(t_0)$. By (48), the left side of (53) is bounded by

$$\frac{2}{\lambda_n^2} g_n(t_0) \leq \frac{2}{\lambda_n^2}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} f(t_0,y)\,dy \leq \frac{2}{\lambda_n^2}\int_{\lfloor t_0\rfloor+1}^{\tau_n(\beta)} L(y)^2 y^{-2\kappa} dy = O\left(\frac{1}{\lambda_n^2}\right).$$

This shows the lemma under Assumption LP(ii).

Under Assumption LP(iii), the same argument applies and the estimation error is again $O\left(\lambda_n^{-1}\right)$ with $\lambda_n = \log n$. Note that the absence of the slowly varying component implies that $f(t,j)$ and $g_n(t)$ are non-increasing for all $t, j \geq 1$ so we may take $t_0 = 0$.

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