

# 1 Photon statistics at beam splitters: an essential tool in quantum information and teleportation

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## 1.1 INTRODUCTION

The statistical behaviour of photons at beam splitters elucidates some of the most fundamental quantum phenomena, such as quantum superposition and randomness. The use of beam splitters was crucial in the development of such early interferometers as the Michelson-Morley interferometer, the Mach-Zehnder interferometer and others for light. The most generally discussed beam splitter is the so-called half-silvered mirror. It was apparently originally considered to be a mirror in which the reflecting metallic layer is so thin that only half of the incident light is reflected, the other half being transmitted, splitting an incident beam into two equal parts. Today beam splitters are no longer constructed in this way, so it might be more appropriate to call them semi-reflecting beam splitters. Unless otherwise noted, we always consider in this paper a beam splitter to be semi-reflecting. While from the point of view of classical physics a beam splitter is a rather simple device and its physical understanding is obvious, its operation becomes highly non-trivial when we consider quantum behaviour. Therefore the questions we ask and discuss in this paper are very simply as follows:

What happens to an individual particle incident on a semi-reflecting beam splitter?

What will be the behaviour of two particle incidents simultaneously on a beam splitter?

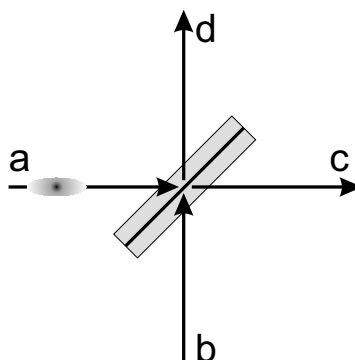
How can the behaviour of one- or two-particle systems in a series of beam splitters like a Mach-Zehnder interferometer be understood?

Rather unexpectedly it has turned out that, in particular, the behaviour of two-particle systems at beam splitters has become the essential element in a number of recent quantum optics experiments, including quantum dense coding, entanglement swapping and quantum teleportation. This is due to the fact that entangled states behave in a rather non-trivial way when incident on beam splitters.

Finally we mention that most of the considerations we present here are not limited to photons but are equally well applicable to massive particles. In the field of matter-wave interferometry, beam splitters have so far been constructed successfully for electrons, neutrons and a number of different atoms and molecules.

## 1.2 BASIC STATISTICS OF ONE AND TWO PHOTON STATES AT BEAM SPLITTERS

### 1.2.1 One particle and a beam splitter



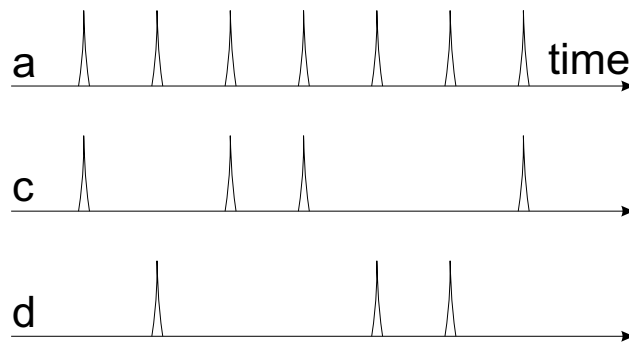
**Fig. 1.1** Action of a simple beam splitter

Consider a 50/50 beam splitter where one particle is incident via beam (a) (Fig. 1.1). Obviously, this particle has a 50% chance of ending up in either in output port (c) or in output port (d). Quantum mechanically we may write the operation of the beam splitter as

$$|a\rangle \rightarrow \frac{1}{\sqrt{2}}(|c\rangle + i|d\rangle) \quad (1.1)$$

Here we have for simplicity assumed that the beam splitter is completely symmetrical. This symmetry implies that a wave experiences a phase shift of  $\pi/2$  upon reflection relative to transmission, as signified by the phase factor

$i$  in Equation 1.1. If we envisage a continuous stream of incoming particles equidistant in time incident from beam port (a) then the detectors positioned in outgoing beams (c) and (d) will each register a random sequence of photons, as each single photon has the same probability to be detected either in (c) or in (d). This is signified in Fig. 1.2 and is a direct consequence of probabilistic interpretation of the quantum state in Equation 1.1. In other words the introduction of a beam splitter into a continuous beam in general introduces new noise in the outgoing beams, should these beams be measured.



**Fig. 1.2** A beam splitter introduces noise into its output ports due to the probabilistic behaviour.

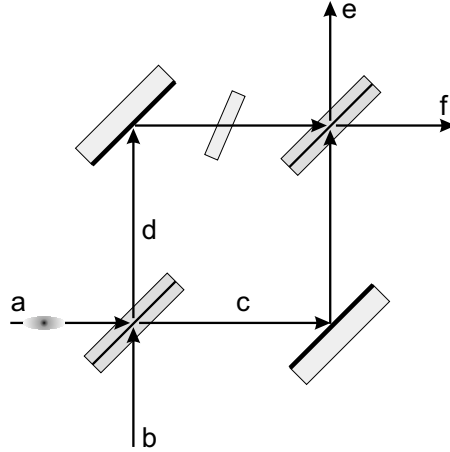
An alternative viewpoint within quantum electrodynamics is to assume that the quantum state describing the input in beam (b) is a vacuum state which is equally well divided by the beam splitter. The noise occurring in beams (c) and (d) can then be seen as arising from this vacuum state which necessarily has zero-point fluctuations.

### 1.2.2 One photon and two beam splitters

The most basic case of interferometry arises when a second beam splitter is added (Fig. 1.3), where the beams (c) and (d) emerging from the first beam splitter are then superposed again. One then also has the possibility to insert a phase-shifter into both beams. Since an overall phase is unimportant it is sufficient to consider the phase-shifter inserted in beam (d), as shown in the figure. Then the quantum states evolve following the rules

$$|c\rangle \rightarrow \frac{1}{\sqrt{2}}(|e\rangle + i|f\rangle) \quad (1.2)$$

$$|d\rangle \rightarrow e^{i\varphi}|d\rangle \rightarrow \frac{e^{i\varphi}}{\sqrt{2}}(i|e\rangle + |f\rangle) \quad (1.3)$$



**Fig. 1.3** Two beam splitters in series form a Mach-Zehnder interferometer which has an additional free phase  $\varphi$ .

Combining this with the rule from Equation 1.1 we find that the final outgoing state becomes

$$|a\rangle \rightarrow ie^{i\varphi/2} \left[ -\sin \frac{\varphi}{2} |e\rangle + \cos \frac{\varphi}{2} |f\rangle \right] \quad (1.4)$$

This final result has a number of interesting implications. Firstly we noticed that the probability to find the photon in beams (e) or (f) is

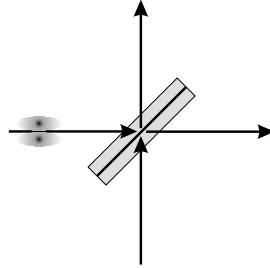
$$p_e = \sin^2 \frac{\varphi}{2} = \frac{1}{2}(1 - \cos \varphi) \quad (1.5)$$

$$p_f = \cos^2 \frac{\varphi}{2} = \frac{1}{2}(1 + \cos \varphi) \quad (1.6)$$

This simply implies that for a phase-shift  $\varphi = n\pi$  the photon will definitely end up in one of the two outputs while the output is maximally uncertain for phase shifts exactly in between. Again for an incoming stream of photons we therefore conclude that the noise in the output beams (e) and (f) can be adjusted by adjusting the phase-shift  $\varphi$  to the appropriate values.

Another interesting consequence is the observation that the Mach-Zehnder interferometer shown in Fig. 1.3 can be viewed as a realisation of a beam splitter where the reflectivity is simply adjusted by proper choice of the phase  $\varphi$ .

### 1.2.3 Two particles and a beam splitter



**Fig. 1.4** Two particles incident on to the same beam splitter input.

Evidently when we consider two particles and a beam splitter we have various possibilities. The first and most simple one is to consider the case where the two particles are incident from the same side, say via beam (a) (Fig. 1.4). When we analyse this situation quantum mechanically we have to be cautious to define each photon in its own Hilbert space. Therefore the incident state is

$$|\psi_{in}\rangle = |a\rangle_1 |a\rangle_2, \quad (1.7)$$

where  $|a\rangle_1$  means that photon 1 is incident in beam (a). We then have to apply the beam splitter rule of Equation 1.1 twice and finally obtain

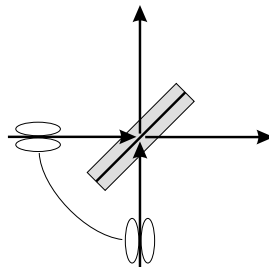
$$|a\rangle_1 |a\rangle_2 \rightarrow \frac{1}{2} (|c\rangle_1 + i|d\rangle_1) (i|c\rangle_2 + i|d\rangle_2) \quad (1.8)$$

$$\frac{1}{2} (|c\rangle_1 |c\rangle_2 + i|c\rangle_1 |d\rangle_2 + i|d\rangle_1 |c\rangle_2 - |d\rangle_1 |d\rangle_2) \quad (1.9)$$

This result simply reflects the fact that each of the two particles behave independently just as classical particles would do. We therefore end up with the following probabilities:

$$\begin{aligned} p(\text{both particles in } c) &= 25\% \\ p(\text{both particles in } d) &= 25\% \\ p(\text{one particle each in } c \text{ and } d) &= 50\% \end{aligned} \quad (1.10)$$

Again, like in the single particle case, the beam splitter adds new noise to the system.



**Fig. 1.5** Two particles can enter the beam splitter in a superposition of two configurations.

So far we have considered the case that both particles are incident from the same input port. While classically this implies that they have to be incident either from input port (a) or from input port (b) quantum mechanically this need not be the case. Indeed we can consider the two photons to be incident in a superposition of both being in (a) or both being in (b). If for simplicity we assume that the amplitudes for the two cases are equal then a quantum state describing such a superposition in general is

$$\frac{1}{\sqrt{2}} (|a\rangle_1 |a\rangle_2 + e^{i2\varphi} |b\rangle_1 |b\rangle_2) \quad (1.11)$$

where we have introduced the phase  $2\varphi$  instead of  $\varphi$  is a very simple one. Suppose we had from some upstream optical devices an incident beam without the phase difference shown in Equation 1.11. Inserting then a phase shifter into beam (b) imposes a phase shift of  $\varphi$  both on the state of particle 1 and on the state of particle 2, resulting in the phase shift  $2\varphi$  as shown in Equation 1.11.

Evaluating now the state emerging from Fig. 1.5 we need the additional beam splitter rule

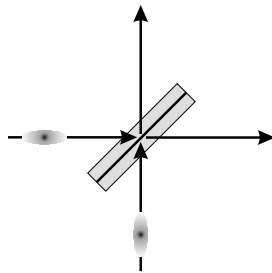
$$|b\rangle \rightarrow \frac{1}{\sqrt{2}} (i|c\rangle + |d\rangle) \quad (1.12)$$

Inserting Equation 1.1 and 1.12 into Equation 1.11 and keeping track of the separate subscripts of the two photons the finally emerging state becomes

$$ie^{i\varphi} [\cos \varphi (|d\rangle_1 |d\rangle_2 - |c\rangle_1 |c\rangle_2) + \sin \varphi (|c\rangle_1 |d\rangle_2 + |d\rangle_1 |c\rangle_2)] \quad (1.13)$$

The implications of this state are quite interesting. It turns out that for specific phases  $\varphi = n\pi$  the two photons again emerge in an equal superposition of being both in beam (c) and being both in beam (d), but now with

the definite phase factor of -1 between the two. This now simply implies that detectors in beams (c) and (d) will register the same quantum noise as detectors in beam (a) and (b) would have. Yet alternatively one can set the phase to odd multiples of  $\pi/2$ , i.e.  $\varphi = (2n + 1)\pi/2$ . In that case the emerging state says that we either find photon 1 in beam (c) and photon 2 in beam (d) or photon 1 in beam (d) and photon 2 in beam (c). In other words we will definitely find one photon in either outgoing beam and noise will be suppressed in the output ports.



**Fig. 1.6** One particle in each input mode.

What would then happen if we consider the final possibility, namely one particle incident from input port (a) and on from (b) in Fig. 1.6. Now a completely new consideration has to be made, namely relating to the statistics obeyed by the particles. If, as we assume, the two particles are indistinguishable, we do not know whether particle 1 is incident from beam (a) and particle 2 is incident from beam (b) or vice versa. Therefore the quantum state has to be a superposition of  $|a\rangle_1|b\rangle_2$  and  $|a\rangle_1|b\rangle_2$ . In this superposition as in any superposition we have to discuss which phase factor to choose between the two terms. In this specific case the phase factor has to follow from the symmetry properties of the particles themselves, that is, do we assume them to be fermions or bosons? In the case of bosons the state has to be symmetrical upon each interchange of the two particles. This means that it becomes

$$|\psi\rangle_{\text{boson}} = \frac{1}{\sqrt{2}} (|a\rangle_1|b\rangle_2 + |a\rangle_1|b\rangle_2) \quad (1.14)$$

In the case of fermions it has to be anti-symmetrical

$$|\psi\rangle_{\text{fermion}} = \frac{1}{\sqrt{2}} (|a\rangle_1|b\rangle_2 - |a\rangle_1|b\rangle_2) \quad (1.15)$$

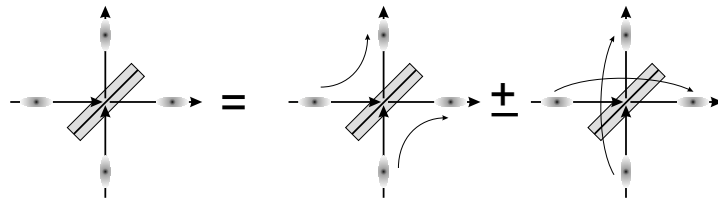
As we will immediately see the two states imply completely different statistical behaviour for the two particles after the beam splitter. Indeed, in the case of fermions the outgoing state becomes

$$|\psi\rangle_{\text{fermion}} = \frac{1}{\sqrt{2}} (|c\rangle_1 |d\rangle_2 - |d\rangle_1 |c\rangle_2) \quad (1.16)$$

while for incident bosons the outgoing state is

$$|\psi\rangle_{\text{boson}} = \frac{i}{\sqrt{2}} (|c\rangle_1 |c\rangle_2 + |d\rangle_1 |d\rangle_2) \quad (1.17)$$

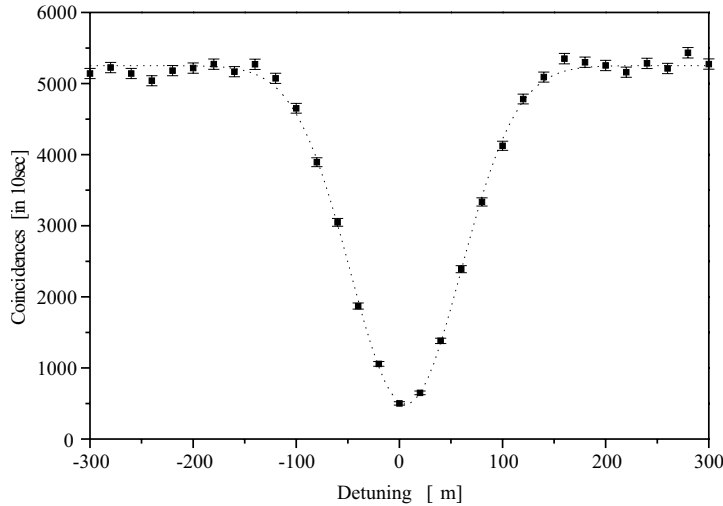
These relations simply imply that for the case of two fermions incident one each from each side of the beam splitter we will again find one particle in each of the outgoing beams. Thus if the incoming beams have no noise, the outgoing beams will also be noise-free. In contrast for bosons we will find both particles either in beam (c) or in beam (d), as above, even if the incident beam is noise-free, we will have the usual partition noise after transmission through the beam splitter.



**Fig. 1.7** Two particles incident on a beam splitter via separate inputs can end up in different outputs either if both are reflected or both are transmitted. The two terms on the right hand side of this “equation” can interfere constructively or destructively depending on their relative phase with which they are superposed. The phase depends on the symmetry of the state and therefore in turn on the statistics of the actual particles.

It is important to notice that the behaviour just discussed is due to quantum interference. The fact that in Equation 1.16 no terms appear where both particles are found in either beam and likewise the fact that in Equation 1.17 no terms appear where one particle is found in each outgoing beam is simply due to destructive or constructive interference of the respective terms after insertion of Equations 1.1 and 1.12 into Equations 1.14 and 1.15 respectively. This interference is illustrated by Fig. 1.7 We should also notice that more generally the anti-symmetrical state of Equation 1.15 is an eigenstate of any beam splitter operation. This can be viewed as another manifestation of the Pauli exclusion principle, since otherwise the beam splitter would be able to operate in such a way as to, at least sometimes, put two fermions into the same outgoing state, a clear impossibility.





**Fig. 1.8** Coincidence rate at detectors in the two output beams of a beam splitter of Fig. 1.7 as a function of the path length difference of the incident beams from the source. A decrease in the coincidence rate at zero path difference is observed for a two particle state with bosonic symmetry at the input.

The observation of the interference phenomena described above is usually done using an interferometer invented in Ref. [8]. In this interferometer (see also Fig. 1.10) simultaneous pairs of photons are produced by parametric down-conversion. The two photons are then directed to the two input ports of the beam splitter. Coincidence counts after the beam splitter indicate that the two photons left the beam splitter via different output modes. The path difference for the two photons from their production to the beam splitter may be scanned. If the two photons are indistinguishable in all other respects for path differences smaller than their coherence length an increase or decrease in the coincidence rate will be observed depending on whether we look at states with fermionic or bosonic symmetry respectively.

The different statistical behaviour of symmetrical and anti-symmetrical states of beam splitters will be of utmost importance in the various applications of beam splitters in identifying entangled quantum states, as we will discuss in the next chapters.

#### 1.2.4 Polarization and beam splitters

So far we have only discussed the spatial part of the particle's wavefunctions and the results above are true for all kinds of particles. In the case when the

particles in question also have spin, there is an additional degree of freedom. This additional degree of freedom is used in most of the applications that are discussed in the following sections. Here we will only concentrate on the case of photons where we have to include the two-state space of polarization in addition to the spatial degree of freedom.

The spatial transformations (Eqs. 1.1 and 1.12) performed by a beam splitter with reflection and transmission amplitudes  $t$  and  $r$  can be summarised in a  $2 \times 2$  matrix

$$\begin{pmatrix} t & ir \\ ir & t \end{pmatrix} \quad (1.18)$$

if we describe the input and output states as vectors in the two input modes ((a) and (b)) and the two output modes ((c) and (d)) respectively. This matrix becomes

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad (1.19)$$

for the symmetric case.<sup>1</sup> If we now add the polarisation degree of freedom we have to use four modes. A vector in this mode space shall consist of the components

$$\begin{pmatrix} a_H \\ b_H \\ a_V \\ b_V \end{pmatrix}, \quad (1.20)$$

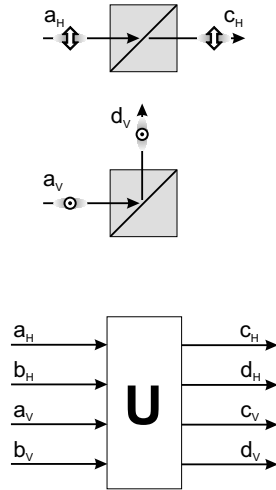
where (a) and (b) again denote the two possible spatial modes and H and V denote the two possible values of the polarization. A general polarisation dependent beam splitter is now a device with the following matrix:

$$\begin{pmatrix} t_H & ir_H & 0 & 0 \\ ir_H & t_H & 0 & 0 \\ 0 & 0 & t_V & ir_V \\ 0 & 0 & ir_V & t_V \end{pmatrix} \quad (1.21)$$

Now there are two limiting cases: 1. the non-polarising beam splitter with  $t_H = t_V$ ,  $r_H = r_V$  and 2. the (totally) polarizing beam splitter (Fig. 1.9) with  $t_H = 1$  and  $r_V = 1$ . Thus the matrix of the latter reads

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (1.22)$$

<sup>1</sup>In quantum information the beam splitter transformation is called Hadamard transformation and is one of the basic building blocks in many quantum computational circuits and algorithms.



**Fig. 1.9** (top) The operation of a totally polarising beam splitter is shown in the upper part. Usually light having polarisation parallel (here “V”) to the splitting surface is reflected and orthogonally polarised (here “H”) light is transmitted. (bottom) The polarisation dependent beam splitter is equivalent to a unitary eight-port device “U”.

It is interesting that this matrix is exactly identical to the matrix of the CNOT-gate which is a universal gate in quantum computation. The two qubits of the CNOT-gate are here represented by the spatial and polarization degrees of freedom respectively. This fact has led to the idea that some quantum computing algorithms could be realized using photons and only linear optics. The problem here is that this CNOT-gate cannot operate on two particles but only on the different degrees of freedom of one particle. Thus no spatially separable entangled states can be produced with this simple technique.

The polarizing beam splitter is used in the production of GHZ-states as shown in section 1.4.4 and of course also for polarization analysis in various other experiments.

### 1.3 ENTANGLED PHOTON STATES AT BEAM SPLITTERS

The polarization degree of freedom opens new interesting possibilities for photonic experiments using beam splitters. As photons are bosonic we know that their total quantum state should always be symmetric under permutation of the particles. Now, as we have the spatial and polarisation components of the

state there are two ways by which it is possible to achieve the required symmetry. Either both parts are symmetric under permutation or both parts are antisymmetric. If we only view the spatial part the photons behave as if they had an antisymmetric wavefunction. In Ref. [10] and more detailed in Ref. [12] it was shown that indeed both the “bosonic” interference (Eq. 1.17) and the “fermionic” type (Eq. 1.16) can be realized by engineering the polarisation state.

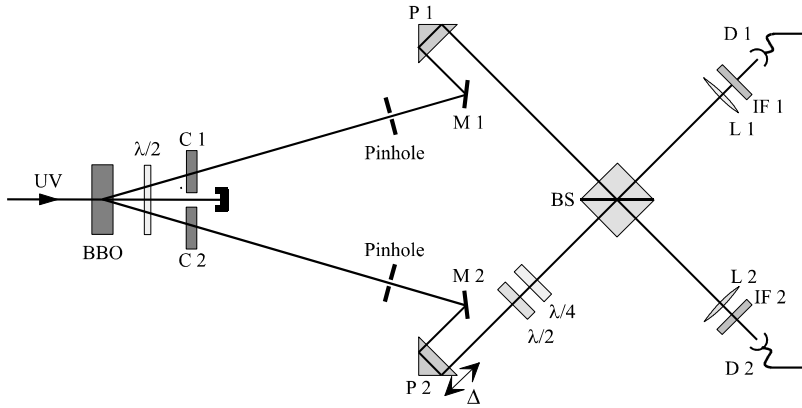
If we combine two two-state systems we enter a 4-dimensional Hilbert space. One possible basis for this space is the so-called Bell-basis which consists of the following four states:

$$\begin{aligned}
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2) \\
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2 - |V\rangle_1|V\rangle_2)
\end{aligned} \tag{1.23}$$

In contrast to the canonical basis which consists of product states only this basis has just maximally entangled states. Of these  $|\Psi^-\rangle$  is the only one, which is antisymmetric under particle exchange, all others are symmetric. If two photons in an experiment are indistinguishable otherwise we know that we have to associate with them a totally symmetric wavefunction. For two photons, one in each of two spatial modes (a) and (b), and polarization modes H and V this yields

$$\begin{aligned}
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2)(|a\rangle_1|b\rangle_2 + |b\rangle_1|a\rangle_2) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2)(|a\rangle_1|b\rangle_2 - |b\rangle_1|a\rangle_2) \\
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2)(|a\rangle_1|b\rangle_2 + |b\rangle_1|a\rangle_2) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2 - |V\rangle_1|V\rangle_2)(|a\rangle_1|b\rangle_2 + |b\rangle_1|a\rangle_2).
\end{aligned} \tag{1.24}$$

We immediately see that we can discriminate the  $|\Psi^-\rangle$  polarisation state from the others by the symmetry of the accompanying spatial quantum state. Combining this with results of the previous sections we find that if we detect two photons in coincidence in distinct outputs of a symmetric beam splitter we know that the spatial input state must have been antisymmetric and therefore also the polarisation state must have been antisymmetric ( $|\Psi^-\rangle$ ). In the case



**Fig. 1.10** Schematic of the experimental setup to demonstrate photon statistics at beam splitters.

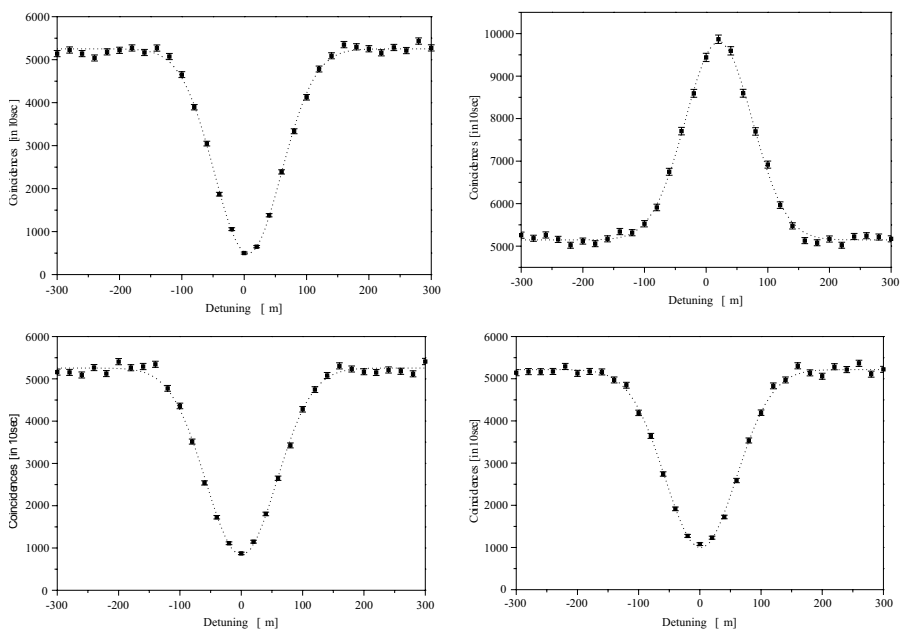
when there is no coincidence – the two photons have left the beam splitter via different output ports – we know that they are in one of the three symmetric Bell-states.

To confirm this phenomenon an experiment was conducted by M. Oberparleiter [12] using the apparatus shown in Fig. 1.10. In this experiment all four Bell-states were produced in a non-collinear type-II spontaneous parametric down-conversion arrangement together with two birefringent waveplates. Depending on the setting of the waveplates any of the four states could be produced. Fig. 1.11 shows the corresponding interferograms for all four cases.

With this setup one can only discriminate between two cases. In section 1.4.1 we will see that by adding two polarising beam splitters we can discriminate between three different cases. Unfortunately with only linear optical elements it is not possible to do a complete Bell-state analysis because this corresponds to a disentangling operation which requires interaction between the constituents. However, many of the applications which rely on Bell-state analysis can still be demonstrated with incomplete Bell-state analysis albeit with a lower efficiency.

The generalization of a beam splitter to devices with more inputs and outputs (multiports) yields a wealth of interesting possibilities [9, 11].<sup>2</sup> The application of these devices makes it possible to probe higher-dimensional Hilbert spaces. This includes fundamental research testing quantum nonlocality [15] and also demonstrations of quantum information processing circuits [5].

<sup>2</sup>Any higher dimensional device (any complicated unitary operator) can be constructed from only a few basic elements of which the most important one is the beam splitter[16].



**Fig. 1.11** Two-photon interference for the four possible Bell-states of a photon pair in the two inputs of a beam splitter. The coincidence rate as detected in the two outputs is shown as a function of the path difference for the two photons to the beam splitter.

## 1.4 PHOTON-STATISTICS APPLICATIONS

### 1.4.1 Dense coding

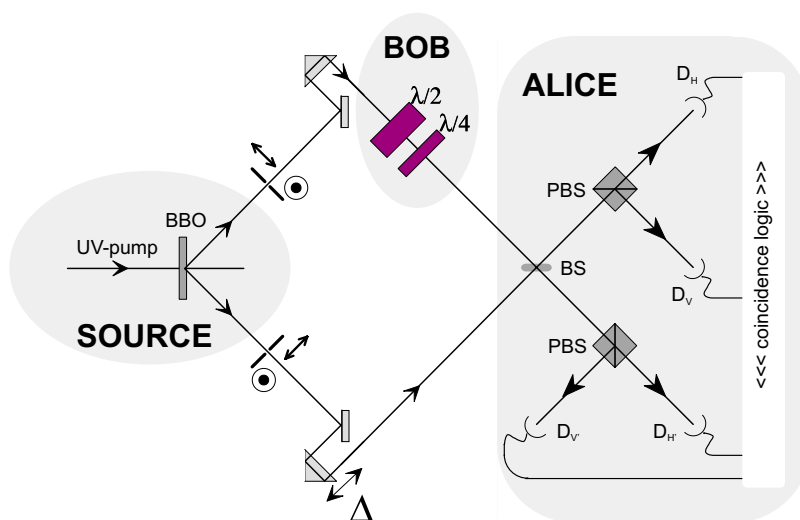
The transition from the theoretical concept of entangled states and statistics to the almost practical issues of quantum information processing occurs when we replace our labels ‘H’ and ‘V’ by the binary digits ‘0’ and ‘1’. This is viewing the two-state space of a polarized photon as representing information. A two level system carries 1 qubit of quantum information. Necessarily a two photon system can carry 2 qubits. In a product state description the basis states could be enumerated as

$$\begin{aligned}
 &|0\rangle |0\rangle \\
 &|0\rangle |1\rangle \\
 &|1\rangle |0\rangle
 \end{aligned}
 \tag{1.25}$$

$$|1\rangle|1\rangle.$$

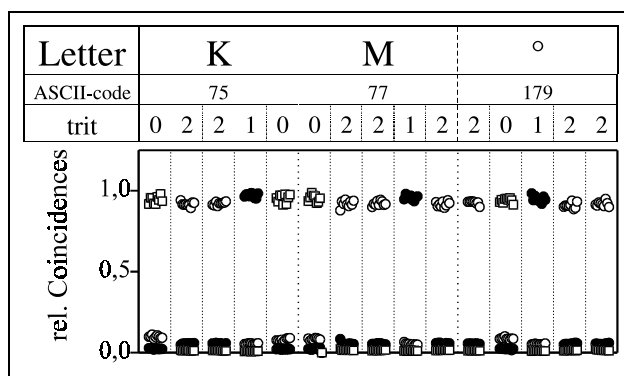
The special choice of a basis cannot change the possible information content of a physical system. Therefore it is clear that using only entangled basis states we can still encode two bits of information, that is we have four different possibilities, but now this encoding is done in such a way that none of the particles carries any well-defined information on its own. All information is encoded into relational properties of the two qubits. It thus follows immediately that in order to read out the information one has to have access to both qubits. The corresponding measurement is the Bell-state measurement that has been described in the previous section. This is to be compared with the classical case where access to one qubit is simply enough to determine the answer to one *yes/no* question. In contrast, in the case of the maximally entangled basis access to an individual qubit does not provide any information.

Whenever two parties A (Alice) and B (Bob) wish to communicate with each other they have to agree first on a coding procedure, that is they have to agree which symbol means what. In classical coding the situation is very simple. Restricting ourselves to binary information, that is to bits, we need some information carrier which has two states. In quantum physics again we can have information encoding in a novel way using entangled states and thus encode information into joint properties of elementary systems. Then the elementary systems themselves do not carry any information. A first elementary case where this is clearly demonstrated is quantum dense coding.



**Fig. 1.12** Experimental apparatus to demonstrate the dense coding application of entangled photon statistics at beam splitters.

The maximally entangled Bell basis of equation 1.24 has a very important and interesting property which was exploited by Bennett and Wiesner [1] in their proposal for quantum dense coding. This is the property that in order to switch from any one of the four Bell states to all other four it is sufficient to manipulate one of the two qubits only while in the classical case one has to manipulate both. Thus, the sender Bob (see Fig. 1.12) can actually encode two bits of information into the whole entangled system by just manipulating one of the two qubits. Let us, for example, assume that we start from the state  $|\Psi^+\rangle$  then we can obtain  $|\Psi^-\rangle$  by just introducing a phase shift of  $\pi$  onto, say, the second qubit,  $|\Phi^+\rangle$  is obtained by flipping the second qubit and the last state  $|\Phi^-\rangle$  is obtained by a combination of both.



**Fig. 1.13** A sample transmission of three symbols using the dense coding technique.

In order to read out this information the receiver, Alice, needs to be able to identify the four Bell states, that is she needs a Bell-state analyzer. In order to identify all four Bell states, one needs some non-linear interaction between the two qubits as already stated above. The experimental realisation [10] is shown in Fig. 1.12. The setup is a modification of the one shown in Fig. 1.10 by adding one polarising beam splitter to each output of the analysing beam splitter and ideally also another beam splitter into each output port of the two polarising beam splitters (not shown).

Whenever there is a  $|\Psi^+\rangle$  state (cf. Eq. 1.24) incident onto the analyser the two photons will leave the first beam splitter through the same output port as discussed in the previous section. As they have different polarisation they will split up at one of the polarising beam splitters and can be detected in coincidence afterwards giving a clear signal for the measurement of a  $|\Psi^+\rangle$  state.



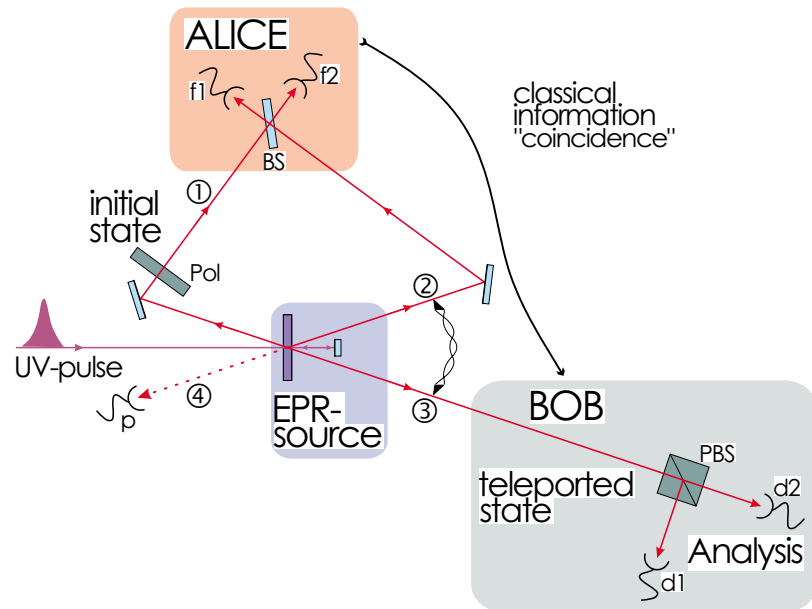
In turn, for a  $|\Psi^-\rangle$  state we expect a different behaviour, namely that the two photons will separate into different outputs of the first splitter and therefore give a coincidence signal between two detectors on either side downstream of the first beam splitter.

The third case allows us to also identify the presence of  $|\Phi^\pm\rangle$  albeit with lower probability than for the other two states.  $|\Phi^\pm\rangle$  will both always contain two photons of identical polarisation. The two photons may now split up at an additional beam splitter inserted after the polarising ones which allows to find a positive signal for the  $|\Phi^\pm\rangle$  states in half of the cases. If efficient detectors would exist that could perfectly discriminate between one and two photons this could be an even better alternative way. To sum up, we find that with only linear elements one can identify three of the four Bell states. With the setup shown in Fig. 1.12 it was thus possible to encode and identify  $\log_2 3 = 1,58$  bits of information per photon manipulated. A sample transmission is shown in Fig. 1.13.

### 1.4.2 Teleportation

A most remarkable application of the concept of beam splitter statistics and entanglement is quantum teleportation. Consider first the problem. Suppose that Alice has an object which Bob, who could be anywhere, might need at a certain time. In classical physics what she can do is perform many precise measurements on the object and send the information to Bob who then can reconstitute the object within classical physics. The accuracy of this is only limited by the precision with which Alice can measure and by the technical abilities of Bob. Yet, we know that in the end any measurement will run into the limitations imposed by quantum mechanics. It is evident that no measurement whatsoever performed by Alice can reveal the full quantum state of the object. We therefore ask which strategy Alice can pursue in order for Bob to obtain the object in its full quantum state when he needs if we imagine a situation where it is not possible to transfer the object – and the mass or energy associated with it – itself, which might even be unsuitable for transportation in the first place.

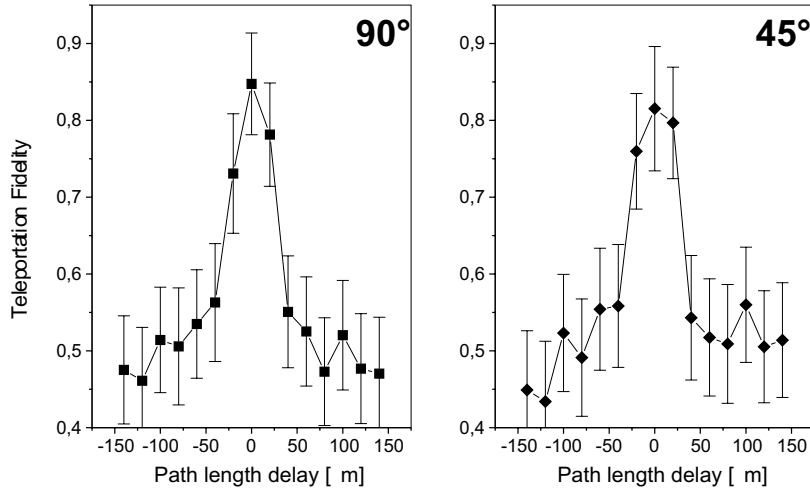
A strategy proposed by Bennet et al. [2] uses exactly the information-theoretic features of entanglement mentioned above. Let us consider for simplicity that the object to be teleported is simply a two-state system, a qubit. Then Alice and Bob share from the beginning an ancillary entangled pair which for convenience we again consider to be in the state  $|\Psi^-\rangle$ . Subsequently Alice performs a Bell-state measurement jointly on her qubit and on one of the two ancillaries. In the photonic case this can again be implemented by Bell-state analysis employing the beam splitter statistics of two photons. Alice will obtain each one of the four possible answers with equal probability, that is her original qubit and her qubit from the ancillary pair will be projected onto any one of the four Bell-states each with probability 25%. We note again that this measurement does not reveal any information, neither about



**Fig. 1.14** Experimental scheme for the teleportation of an arbitrary quantum state of a photon. All the photons are produced by parametric down-conversion of a pulsed laser.

the properties of the original qubit nor about the properties of the ancillary pair. So Alice obtains one of four possible results, “ $\Psi^+$ ”, “ $\Psi^-$ ”, “ $\Phi^+$ ”, or “ $\Phi^-$ ”. She then broadcasts this information, that is two classical bits, such that Bob can receive them. By now Bob is in possession of a specific state as a consequence of Alice’s Bell-state measurement. Performing one of four unitary transformations depending on Alice’s specific result Bob can transform his particle into the original qubit. We also note that the original qubit disappears during the Bell-state measurement, it loses its identity, and thus Bob’s qubit is not a copy but really a teleported reappearance of the original.

The major problem in the experimental verification of quantum teleportation [4] was the Bell-state measurement of two independently created quantum systems. This is even more difficult than the interferometers we have looked at in the previous sections. Here, one has to perform a second order interference measurement on particles originating from independent sources. This means that these two qubits have to be measured such that their identity is lost, that it is not possible to infer which detection event refers to which source. One condition that emerges from this criterion is that the coincidence window should be significantly smaller than the coherence time of the photons



**Fig. 1.15** Sample data of the experimental verification of the teleportation procedure. The teleportation fidelity is plotted as a function of the path length difference between the paths of the photon which is to be teleported and the one from the entangled pair respectively. Random photons would yield a fidelity of 0.5. The results clearly surpass the limit of  $2/3$  for a teleportation procedure relying on classical communication only.

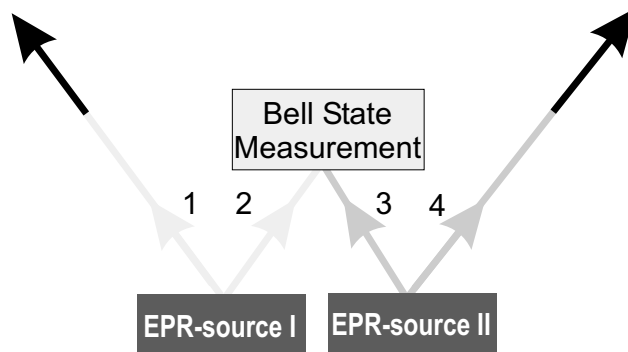
(“ultracoincident”). It turns out that in the experiment it is rather tricky to achieve this situation. As can be seen in Ref. [19], it involves an elaborate application of a quantum erasure technique, using mode-locked pulsed lasers to make the coincidence window independent of any electronic limitations while still maintaining reasonable count rates.

### 1.4.3 Entanglement swapping

Entanglement used to be considered as a consequence of the fact that the entangled particles interacted in their past or that they came from a common source. That this is too restricted a view is witnessed by the concept of entanglement swapping [18]. In the simplest implementation of entanglement swapping we take two entangled pairs (Fig. 1.16) and subject two particles, one from each source, to a Bell-state measurement. Then the other two particles which have never interacted in the past and also did not come from a common source are projected onto an entangled state. Remembering that we can perform projective Bell-state measurements using a beam splitter and single-photon detectors we find that we can construct entanglement using only beam splitters.

Let us, for simplicity of discussion, just consider the case where we have sources that produce our two qubits in the anti-symmetric state  $|\Psi^-\rangle$ . This state has the unique feature that it is anti-symmetric in any basis. Thus, in terms of its information content, the statement is that we know the two qubits are different whatever basis we choose<sup>3</sup>. We thus know simply by the choice of preparation that in each of the two entangled pairs to be used in entanglement swapping the two qubits are completely different.

We now have to discuss the information content of Bell-state analysis. At first we note that, whichever states we would produce at the sources, a fair Bell-state analyser will return any of the four possible answers with equal probability of 25%. That is, the action of the Bell-state analyser is such that it projects the two photons onto an entangled state and, since in our case the two qubits are themselves members of maximally entangled pairs and therefore carry no information, this has to happen with equal probability for all four Bell states measured. In fact, the Bell-state measurement does not reveal any information about any of the qubits emitted by one of the two sources nor any joint information about each source. Yet, what we gain is joint (or relative) information about the two sources.



**Fig. 1.16** Block diagram of the entanglement swapping procedure. Two particles from independent EPR sources become entangled by a projective Bell-state measurement.

Suppose, specifically, that in a certain experimental run we obtain the result  $|\Psi^-\rangle$  for the Bell-state measurement. We then know that qubits 2 and

<sup>3</sup>We remark that for the three symmetric states we can make analogous statements but the situation is slightly more complicated. It turns out in the end that, since the Hilbert-space of the four Bell-states is four-dimensional, we can encode two independent bits of information into these four states. Therefore the four states can be characterised by *yes/no* answers to two distinct questions. These two questions are questions about the identity of the two qubits in two different bases conjugate to each other.

3 have been projected by the measurement onto a state which is characterized by the fact that these two qubits are different in whatever basis. Interestingly, and again most remarkably, this statement is even true as none of the two qubits themselves are yet well-defined in any basis and have no properties by themselves.

Now we are in a position to complete our chain of reasoning. By the properties of source I we know that qubit 1 and qubit 2 are different. By the result of the specific Bell-state measurement we know that qubit 2 and 3 are different and, finally, from the property of source II we know that qubit 3 and 4 are different. Therefore, since our qubits are defined in a Hilbert-space of dimension 2 only, we conclude that qubit 1 and 4 also have to be different in any basis. Therefore they emerge in the anti-symmetric state  $|\Psi^-\rangle$ . Analogous reasoning can be built up for the other three possible Bell-state measurement results.

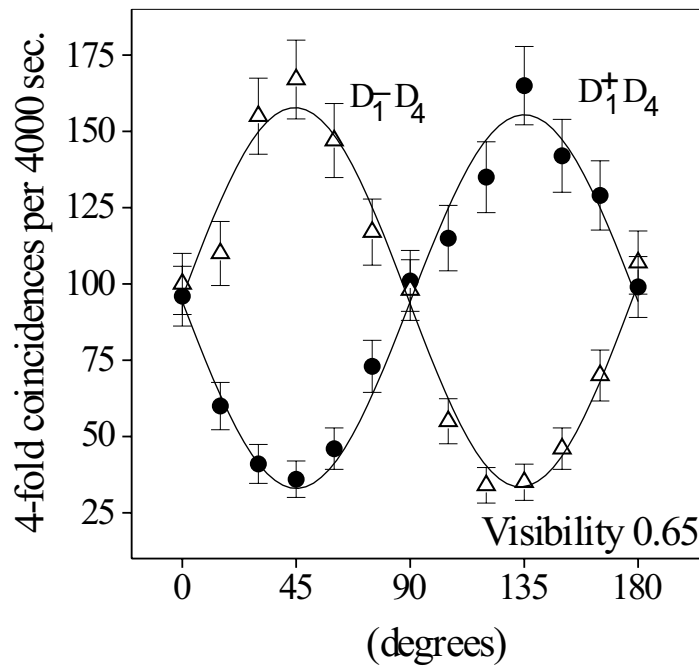


Fig. 1.17 Interference data from an entanglement swapping experiment.

The experimental data taken from Ref. [14] show an interference visibility of 65%, which is only slightly less than the value of 71% required to violate a Bell-type inequality. While the experimental data already clearly show that the correlations produced are stronger than any classical one could ever be, a violation of Bell's inequality would be a definitive indication of the success of the entanglement swapping method. The experimental limitation is mostly due to the difficulty of realizing perfect erasure of the information about the origin of the two independent particles subjected to the Bell-state analysis. Current experiments indicate that it will soon be possible to perform entanglement swapping with the desired visibility level.

#### 1.4.4 GHZ-preparation

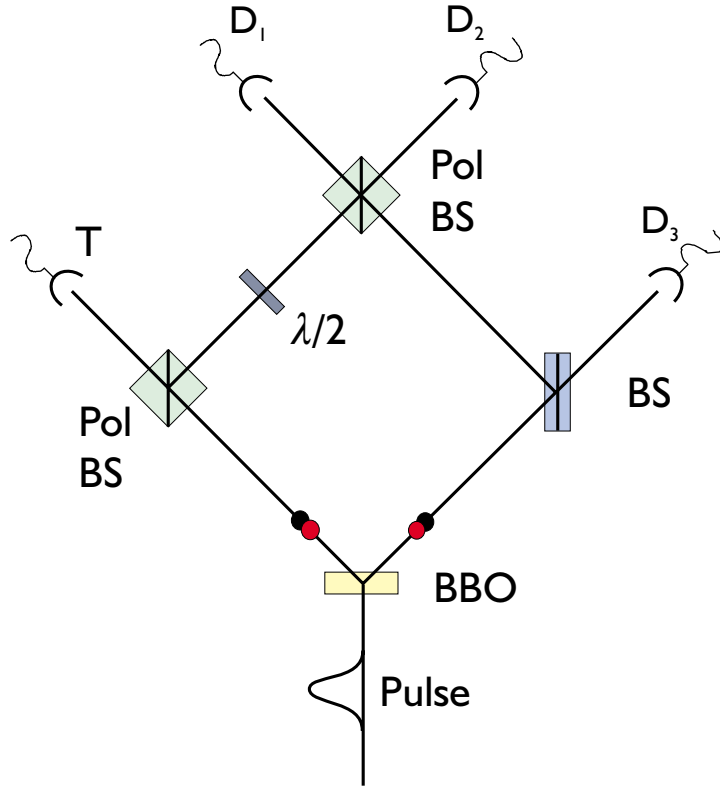
An intriguing application of both ordinary and polarising beam splitters is the creation of Greenberger-Horne-Zeilinger (GHZ) [7, 6] entangled states of three or more particles. GHZ states enable us to show a contradiction between the possible predictions of local realistic models and quantum physical predictions without having to resort to an assessment of imperfect statistical correlations as in Bell's inequality.

Currently, parametric down-conversion is the standard source for the production of two-particle entangled states. However, for entangled states of more than two particles there is no convenient source available. As already foreseen in Ref. [18] and presented in Ref. [17] it is however possible to employ two or more photon pairs as a resource in the creation of higher-dimensional entangled states. The combination of the pairs is done with various beam splitter assemblies.

In the work of Ref. [3] the setup shown in Fig. 1.18 was used to create a three-particle entangled state conditioned on the detection of a fourth “trigger” photon. A type-II down-conversion crystal pumped by a femtosecond pulsed laser delivers polarisation entangled photon pairs in the state  $|\Psi^-\rangle = 1/\sqrt{2}(|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2)$ . The number of pairs emitted from a single pump pulse is not fixed. As we are considering a spontaneous process we expect thermal emission statistics and sometimes there will be two pairs, whereas the emission of more than two is much more unlikely. We now impose the restriction that we only look at events where all four detectors (T,  $D_1$ ,  $D_2$ , and  $D_3$ ) register a photon. Thus we need not go through the tedious calculation of the full state but it is sufficient to only account for the few cases that lead to such a four-fold coincidence event.

We immediately see that the trigger detector T will only register horizontally polarised photons<sup>4</sup> and four-fold events only happen if it registers exactly one such particle. Therefore there must be one vertical and one horizontal

<sup>4</sup>Remember that a totally polarising beam splitter will transmit horizontally polarised photons and reflect vertically polarised ones (polarised parallel to the beam splitting plane).



**Fig. 1.18** Schematic of the experimental setup to produce three-photon GHZ-states from two entangled pairs, consisting of a parametric down-conversion source ( $\beta$ -BariumBorate (BBO) pumped by a femtosecond pulsed UV laser), a beam splitter “BS”, two polarising beam splitters “Pol BS”, and a half-wave plate with its optic axis at  $22.5^\circ$  to vertical. The half-wave plate changes  $V$  photons into  $1/\sqrt{2}(|V\rangle + |H\rangle)$ .

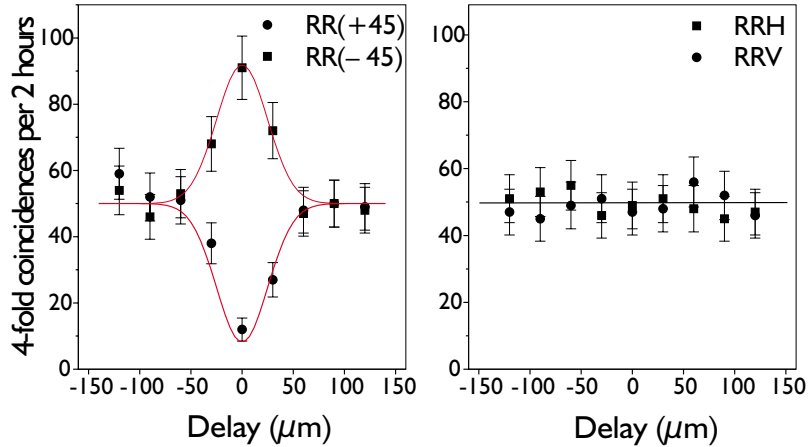
particle in the right arm. In that case these two photons must have split up at the beam splitter such that one went to  $D_3$  and one to the final polarising beam splitter. Remembering that in the left arm we have a  $V$  photon, which is transformed to  $1/\sqrt{2}(|V\rangle + |H\rangle)$  by the half-wave plate we find that there are now two ways in which a four-fold event can occur: either two vertically polarised photons are detected in  $D_1$  and  $D_2$  or two horizontally polarised ones. In either case we will register the opposite polarisation in  $D_3$ . If the two possibilities are indistinguishable by other means we will achieve a coherent

superposition yielding the state

$$\frac{1}{\sqrt{2}}|H\rangle_T(|H\rangle_1|H\rangle_2|V\rangle_3 + |V\rangle_1|V\rangle_2|H\rangle_3) \quad (1.26)$$

where the actual phase between the two components can be found in a more formal derivation.

In Fig. 1.19 (left pane) one can see the interferogram for two specific components of the above mentioned state (Eq. 1.26) when expressed in a basis that chooses circular polarisation measurement for the first two detectors and linear  $45^\circ$  for the third one. The observed contrast was higher than 84%. Fifteen more curves were measured to check a version of the GHZ-argument. For this purpose one needs to construct a simple inequality relating all sixteen observations which must be obeyed by any local realistic model. It could be shown [13] that, closing the detection efficiency loophole, indeed local realism is in conflict with the experiment.



**Fig. 1.19** Measurement data from the GHZ-experiment. As the length difference of the left and right paths in Fig. 1.18 is varied around zero one observes the interferogram for a specific basis of polarisation analysis. In the circular, circular, linear  $45^\circ$  basis we observe interference (**left**), while there is no signal in the circular, circular, linear  $0^\circ$  (H-V) basis (**right**).

## 1.5 SUMMARY

We saw that beam splitters operating on various single photon states can yield a great variety of behaviour. Most of the basic and even some more advanced



schemes of quantum communication can thus be implemented without any (nonlinear) photon-photon interaction. Dense coding, teleportation, entanglement swapping, and even the production of GHZ states could successfully be demonstrated using only linear beam splitters and projective measurement. There are some other proposals that still miss their realisation, such as purification of entanglement, quantum error correction, and many more elaborate ones. While probably in more complex protocols the decreasing efficiency of the above techniques may eventually limit their technological use, there is still much room for experiments that not only lead us the way to a deeper understanding of the theory but also give us a better intuition to find intriguing new applications.

### **Acknowledgments**

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## References

1. Ch. H. Bennett. Quantum cryptography using any two nonorthogonal states. *Phys. Rev. Lett.*, 68(21):3121–3124, 1992.
2. Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.*, 70(13):1895–1899, 1993.
3. D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurter, and A. Zeilinger. Observation of three-photon Greenberger-Horne-Zeilinger entanglement. *Phys. Rev. Lett.*, 82:1345–1349, 1999.
4. D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger. Experimental quantum teleportation. *Nature*, 390:575–579, 1997.
5. N. J. Cerf, C. Adami, and P. G. Kwiat. Optical simulation of quantum logic. *Phys. Rev. A*, 57:R1477–R1480, 1998.
6. D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger. Bell’s theorem without inequalities. *Am. J. Phys.*, 58(12):1131–1143, 1990.
7. Daniel M. Greenberger, Michael Horne, and Anton Zeilinger. Going beyond Bell’s theorem. In M. Kafatos, editor, *Bell’s Theorem, Quantum Theory, and Conceptions of the Universe*, pages 69–72. Kluwer Academic, Dordrecht, The Netherlands, 1989.
8. C. K. Hong, Z. Y. Ou, and L. Mandel. Measurement of subpicosecond time intervals between two photons by interference. *Phys. Rev. Lett.*, 59(18):2044, 1987.
9. K. Mattle, M. Michler, H. Weinfurter, A. Zeilinger, and M. Żukowski. Non-classical statistics at multiphot beamsplitters. *Appl. Phys. B*, 60:S111–S117, 1995.
10. Klaus Mattle, Harald Weinfurter, Paul G. Kwiat, and Anton Zeilinger. Dense coding in experimental quantum communication. *Phys. Rev. Lett.*, 76(25):4656–4659, 1996.

11. Markus Michler, Klaus Mattle, Harald Weinfurter, and Anton Zeilinger. Interferometric Bell-state analysis. *Phys. Rev. A*, 53:R1209–12, 1996.
12. Markus Oberparleiter. Bosonische und fermionische zweiphotonenstatistik am strahlteiler. Master’s thesis, Universität Innsbruck, 1997.
13. J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger. Experimental test of quantum nonlocality in three-photon Greenberger-Horne-Zeilinger entanglement. *Nature*, 403:515–519, 2000.
14. Jian-Wei Pan, Dik Bouwmeester, Harald Weinfurter, and Anton Zeilinger. Experimental entanglement swapping: Entangling photons that never interacted. *Physical Review Letters*, 80:3891–3894, 1998.
15. Michael Reck. *Quantum Interferometry with Multiports: Entangled Photons in Optical Fibers*. PhD thesis, Universität Innsbruck, 1996.
16. Michael Reck, Anton Zeilinger, Herbert J. Bernstein, and Philip Bertani. Experimental realization of any discrete unitary operator. *Phys. Rev. Lett.*, 73(1):58–61, 1994.
17. A. Zeilinger, M. A. Horne, H. Weinfurter, and M. Żukowski. Three-particle entanglements from two entangled pairs. *Phys. Rev. Lett.*, 78:3031–3034, 1997.
18. M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert. “Event-ready-detectors” Bell experiment via entanglement swapping. *Phys. Rev. Lett.*, 71(26):4287–4290, 1993.
19. M. Żukowski, A. Zeilinger, and H. Weinfurter. Entangling photons radiated by independent pulsed sources. *Annals of the N. Y. Acad. of Sciences*, 755:91–102, 1995.