Separating intermediate predicate logics of well-founded and dually well-founded structures by monadic sentences

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Abstract

We consider intermediate predicate logics defined by fixed well-ordered (or dually well-ordered) linear Kripke frames with constant domains where the order-type of the well-order is strictly smaller than ω^ω. We show that two such logics of different order-type are separated by a first-order sentence using only one monadic predicate symbol. Previous results by Minari, Takano and Ono, as well as the second author, obtained the same separation but relied on the use of predicate symbols of unbounded arity.

1 Introduction

There are at least three good reasons for studying predicate logics defined by linear Kripke frames with constant domains: These logics are typical examples of intermediate predicate logics, that is logics that lie between classical and intuitionistic logic (Horn, 1969; Ono, 1972/73), and bare relation to linear-time temporal logic (Nowak and Demri, 2007; Prior, 1967; Rohde, 1997). Furthermore, they have a strong link to one of the three main t-norm based logics called Gödel logics (Hájek, 1998): The logics defined by *countable* linear Kripke frames with constant domains coincide with the set of all Gödel logics (Beckmann and Preining, 2007). Finally, they have interesting connections to the theory of linear orders. For example, studying countable closed linear orderings with respect to continuous monotone embeddability has lead to the surprising result that there are only countably many Gödel logics (Beckmann et al., 2008; Laver, 1971).

The original motivation for this paper was to understand how much we can express in the world of linear Kripke frames with constant domain, if the language is restricted to one of the *simplest* reasonable first-order fragment which extends propositional logic, which is first-order formulas based on exactly one monadic predicate symbol. Very early guesses, that there are only four such logics ("What can we express more than infima and suprema and their order?"), were soon overthrown. In fact, our results in this paper show that there are countably infinite many such logics.

More specific, we will show (in Theorem 18) that for any ordinals $0 < \alpha < \beta < \omega^\omega$, the logics defined by α and β as well-founded linear Kripke frames

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with constant domains can be separated by a first-order sentence which uses only one monadic predicate symbol. The same holds if we take ordinals as *dually* well-founded Kripke frames (Theorem 23).

Related work The study of the relation between Kripke frames and ordinals carries a long tradition, and results related to ours have been obtained in Minari et al. (1990), which in turn is related to Minari (1985). Similar result have been obtained in Preining (2002). Minari (1985) showed that any ordinal ξ less then ω^ω is Kripke-definable, which in his interpretation means that there is a formula separating the logics of the Kripke frames based on ξ and $\xi + 1$.

We improve this result by firstly providing a formula in the monadic fragment with only one predicate symbol, and secondly, by separating any two logics of Kripke frames based on two different ordinals less than ω^ω. This also explains why the formulas we are providing are dependent on both ordinals.

Minari (1985) also discusses the definability of ordinals larger then ω^ω: He shows that no ordinal bigger then ω_1 is Kripke-definable (based on Löwenheim-Skolem), and conjectures that no ordinal between ω^ω and ω_1 is definable.

Relation to quantified propositional logic The monadic fragment under discussion can be seen as the linear fragment of Gabbay's $\mathsf{2h}$ logic (Gabbay, 1981), the second order propositional logic, which could also be called intuitionistic quantified propositional logic. The semantic of this logic is based on Kripke frames with the addition that the set of possible interpretations for atomic propositions is not necessarily the full set, but any arbitrary subset of the sets of all upsets of the Kripke frame (set D in (Gabbay, 1981)). Note that the restriction to evaluate atomic propositions into a restricted set does not apply to the extension to compound formulas. Each first order quantification, as its variable is only occurring within one monadic predicate symbol, can be replaced by a corresponding propositional quantification. In this way, each particular model of the second order propositional logic $\mathsf{2h}$ can be simulated by one particular model and evaluation of monadic first order linear Kripke logic. Thus, counter models can be translated from second order propositional logic $\mathsf{2h}$ to monadic first order logics of linear Kripke frames with constant domains, and vice versa.

A less direct relation exists to *quantified propositional Gödel logics* (Baaz et al., 2001, 2000), where the full set of truth values can act as possible interpretation for atomic propositions. In this case, counter models can be translated from quantified propositional Gödel logic to monadic first order Gödel logic, but not vice versa.

The present article also exploits and continues the connection between logics of linear Kripke frames and Gödel logics, obtained in Beckmann and Preining (2007). Due to the fact that evaluations in Kripke frames are governed by special rules with respect to the order — in other words, evaluations in Kripke frames are based on upsets — evaluations in Gödel logics have a much simpler structure. Furthermore we view our results as part of a wider research programme which connects the theory of linear orders to investigations of logics. In particular, we are interested in the question which order theoretic notion resembles the structure of logics best, see Section 6 for more details.

To keep the article self-contained we will introduce all necessary definitions, but refer the reader to the Handbook article on Gödel logics for more background (Baaz and Preining, 2011). Section 2 introduces logics of Kripke frames and ordinals as Kripke frames, as well as giving the necessary notations of Descriptive Set Theory. It furthermore contains the definitions of important formulas and the main technical Lemma 14 computing their evaluations. Although not strictly necessary, we treat the special case of ordinals of the form ω^n separately in Section 3 as it is simpler than the general case. The separation theorem for well-founded Kripke frames, Theorem 18, and it's proof is given in Section 4. Finally, the case of dually well-ordered Kripke frames with Theorem 23 is presented in Section 5. We conclude with plans for further extensions and possible future research in Section 6.

2 Preliminaries

2.1 Logics based on (linear) Kripke frames

In the following we are only concerned with linear Kripke frames with constant domains. Thus, the following definitions are targeted towards these cases. Note that the class of logics based on countable linear Kripke frames with constant domain is equivalent to the class of Gödel logics, as shown in Beckmann and Preining (2007).

Definition 1 (Kripke frame (linear, constant domain))**.** A *Kripke frame* is a triple $(K, R, \mathcal{U})$ where (K, R) is a (non-empty) quasi order (i.e., reflexive and transitive), and $\mathcal{U}$ is a countable infinite set of objects.

We often define Kripke frames by not mentioning its domain $\mathcal{U}$. In this case it is assumed that $\mathcal{U}$ is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.

In standard Kripke style semantics evaluations are considered via forcing relations in the worlds (i.e., elements of K), together with conditions that guarantee persistency, i.e., that if A holds in a world w, it also holds in all worlds w' such that $w\ R\ w'$. Instead of following this approach we will consider an equivalent approach in which valuations map to the set of upward closed subsets of (K, R), i.e., the latter will serve as the set of truth values for valuations.

Definition 2 (Upsets)**.** Let K be a Kripke frame. The *set of all upsets of* K, $\mathrm{Up}(K)$, consists of all upward closed subsets of K, where $X \subseteq K$ is upward closed iff $x \in X$ and $R(x, y)$ implies $y \in X$, for all $x, y \in K$.

In case of linear Kripke frames the set $\mathrm{Up}(K)$ is a complete total order with respect to $\subseteq$. It can also be viewed as a complete lattice under the usual set-theoretic lattice operations. We will often write $\leq$ instead of $\subseteq$, and we will denote with $\mathbf{0_K}$ the smallest element in $\mathrm{Up}(K)$, i.e., the empty set $\mathbf{0_K} = \emptyset$, and with $\mathbf{1_K}$ the largest element, i.e., the full frame $\mathbf{1_K} = K$.

We will furthermore freely use notations from linear orders, especially intervals like $[a, b]$ for a and b in $\mathrm{Up}(K)$, with the usual meaning $[a, b] = \{c \in \mathrm{Up}(K) : a \subseteq c \subseteq b\}$.

Let L be a language of first-order logic. Formulas and their free and bound variables are defined as usual. A formula is called *closed* if it does not contain free variables. Let $\mathcal{U}$ be a domain. With $L_{\mathcal{U}}$ we denote the language L extended by constant symbols for elements in $\mathcal{U}$. For $u \in \mathcal{U}$, we will identify the constant symbol and its corresponding element, and write u for both of them.

We will base our definition of semantics on $\mathrm{Up}(K)$ instead of K itself. This is straight-forward in the constant domain case.

Definition 3 (Valuation)**.** Let K be a Kripke frame. A function φ mapping closed atomic formulas in $L_{\mathcal{U}}$ into $\mathrm{Up}(K)$ is called a *valuation for K*

The extension of φ to all closed formulas in $L_{\mathcal{U}}$ is defined by structural induction as follows:

$$\varphi(A \wedge B) = \min\{\varphi(A), \varphi(B)\} \qquad \varphi(A \vee B) = \max\{\varphi(A), \varphi(B)\}$$

$$\varphi(\bot) = \mathbf{0_K} \qquad \varphi(A \to B) = \begin{cases} \mathbf{1_K} & \text{if } \varphi(A) \leq \varphi(B) \\ \varphi(B) & \text{otherwise} \end{cases}$$

$$\varphi(\forall x A(x)) = \inf\{\varphi(A(u)) : u \in \mathcal{U}\} \quad \varphi(\exists x A(x)) = \sup\{\varphi(A(u)) : u \in \mathcal{U}\}$$

where min, max, inf, sup are the usual order-theoretic operations on $\mathrm{Up}(K)$.

We are interested in fragments of one monadic predicate symbol. Thus, we assume that our L contains a unary predicate symbol P, which we keep fixed throughout this article. To simplify notation, we will often use the following abbreviation for $c \in \mathcal{U}$:

$$\varphi(c) := \varphi(P(c))$$

Based on the above definition of a valuation we will define the validity of a formula and the notion of logic as the set of formulas valid under all valuations. Note that the definition of logic here does not involve entailment relations. In the case of Gödel logics (which is the same as countable linear Kripke frames with constant domains), entailment based logics and validity based logics do not coincide. For a more detailed discussion see Baaz and Preining (2011), Baaz et al. (2007), or Beckmann and Preining (2007).

Definition 4 (Validity)**.** Let K be a Kripke frame. The logic of a fixed Kripke frame K, $L(K)$, is the set of all closed formulas A in L, such that for all valuations φ for K, A evaluates to $\mathbf{1_K}$ under φ, i.e., $\varphi(A) = \mathbf{1_K}$.

Let α be an ordinal in the set-theoretic sense, that is, α is the set of all smaller ordinals. We will denote the less-than-or-equal relation on ordinals by $\preceq$ to distinguish it from the order relation on upsets. Ordinals as Kripke frames have already been studied in Minari (1985) and Minari et al. (1990). We will follow their approach and define what they called *normal ordinal logics* (for countable ordinals).

Definition 5 (Ordinal-based Kripke frames)**.** For an ordinal $\alpha \succ 0$, let K_α be the Kripke frame given by the set α and the relation $\preceq$. The upsets of K_α have the form $[\beta, \alpha) = \{\gamma : \beta \preceq \gamma \prec \alpha\}$, and $\beta \mapsto [\beta, \alpha)$ is an isomorphism between $(\alpha{+}1, \succeq)$ and $(\mathrm{Up}(K_\alpha), \subseteq)$. We will use $\beta^\uparrow$ to denote $[\beta, \alpha)$.

Consider K_α and the corresponding set of upsets $\mathrm{Up}(K_\alpha)$. We observe that $\alpha^\uparrow = \emptyset = \mathbf{0}_{\mathbf{K}_\alpha}$ and that $0^\uparrow = \mathbf{1}_{\mathbf{K}_\alpha}$. The order $\subseteq$ on upsets corresponds to $\succeq$ on ordinals.

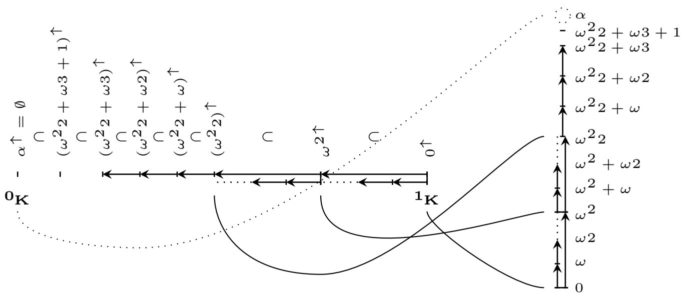
Definition 6 (Logics of ordinal-based Kripke frames)**.** We will denote the logic of K_α, $L(K_\alpha)$, by $L(\alpha)$.

For other studies related to logics of Kripke frames over ordinals we refer the reader to Minari et al. (1990).

Example. Consider the ordinal $\alpha = \omega^2 2 + \omega^1 3 + \omega^0 2$. As a set, α consists of all ordinals less than α:

$$\alpha = \{0, 1, \dots, \omega^2 2 + \omega^1 3 + \omega^0 1\}$$

Writing α as a Kripke frame we obtain a frame as displayed on the lower right. The origin of the frame is labelled with the ordinal 0, the last element of the frame is the ordinal $\omega^2 2 + \omega^1 3 + \omega^0 1$. Considering the upsets of the frame, drawn horizontally in the left part of the following figure, we get exactly the same sequence with one additional element at the left end, the empty set, which can be represented by the original ordinal α. Considering the ordering relations we see that in the order of the ordinals, we have $\omega^2 \prec \omega^2 + \omega$, and accordingly the subset relation of the upsets will give ${\omega^2}^\uparrow \supset (\omega^2 + \omega)^\uparrow$. We also indicate the imaginative point for α itself in the Kripke frame, that links to $\mathbf{0_K}$.



In the following we will keep the following conventions to ensure that the reader does not get lost:

- Kripke frames are drawn vertically
- elements of the Kripke frame are denoted with lowercase Greek letters
- the upsets of the Kripke frames are drawn horizontally
- upsets are denoted by $\alpha^\uparrow$
- comparing elements of the Kripke frame, i.e., ordinals, we use $\preceq$
- comparing elements of $\mathrm{Up}(K)$ we use either $\subseteq$ or $\leq$

2.2 Descriptive Set Theory

We recall a few necessary notions from the framework of Polish spaces, which are separable, completely metric topological spaces. For our discussion it is only necessary to know that the set of upsets are all Polish spaces. For a detailed exposition see Kechris (1995) or Moschovakis (1980).

Definition 7 ((iterated) Cantor-Bendixon derivative)**.** For any topological space X let

$$X' = \{x \in X : x \text{ is limit point of } X\}.$$

We call X' the *Cantor-Bendixon derivative* of X.

Using transfinite recursion we define the *iterated Cantor-Bendixon derivatives* X^α, α ordinal, as follows:

$$\begin{aligned} X^0 &= X \\ X^{\alpha+1} &= (X^\alpha)' \\ X^\lambda &= \bigcap_{\alpha<\lambda} X^\alpha, \text{if } \lambda \text{ is limit ordinal.} \end{aligned}$$

It is obvious that X' is closed, that X is perfect iff $X = X'$, and that X^α for ordinals $\alpha \succ 0$ is a decreasing transfinite sequence of closed subsets of X.

Theorem 8 (Cantor-Bendixon)**.** *Let X be a polish space. For some countable ordinal α_0, $X^\alpha = X^{\alpha_0}$ for all $\alpha \succeq \alpha_0$ (X^{α_0} is the perfect kernel).*

Definition 9 (Cantor-Bendixon rank, CB-rank)**.** The Cantor-Bendixon rank of an element $x \in X$, where X is countable closed polish space, is defined as

$$\mathrm{rk}_{\mathrm{CB}}(x) = \sup\{\alpha : x \in X^\alpha\}$$

The rank of X is defined as $\mathrm{rk}_{\mathrm{CB}}(X) = \sup\{\mathrm{rk}_{\mathrm{CB}}(x) : x \in X\}$.

Since $\mathrm{Up}(K)$ forms a Polish Space, we can compute the Cantor-Bendixon rank of elements of it.

Lemma 10. *Assume $\beta = \gamma + \omega^\xi \preceq \alpha$. Then $\mathrm{rk}_{CB}(\beta^\uparrow) = \xi$ in $\mathrm{Up}(L(\alpha))$.*

Proof. Under the assumption $\beta = \gamma + \omega^\xi \preceq \alpha$ we have that the Cantor normal form of β ends in ω^ξ. Let W_α be $\mathrm{Up}(L(\alpha)) \setminus \{0^\uparrow\}$, and W_α^ξ its Cantor-Bendixon derivations. As $0^\uparrow$ is isolated in $\mathrm{Up}(L(\alpha))$ we have that $W_\alpha^\xi = \big(\mathrm{Up}(L(\alpha))\big)^\xi$ for $\xi \succ 0$. Let

$$S^\xi \;=\; \left\{(\gamma+\omega^\nu)^\uparrow : \gamma + \omega^\nu \preceq \alpha \text{ and } \nu \succeq \xi\right\} \;.$$

We observe that

$$(\gamma+\omega^\nu)^\uparrow \in S^\xi \quad \text{iff} \quad \nu \succeq \xi \;. \tag{1}$$

Claim. *$W_\alpha^\xi = S^\xi$ for $\xi \succeq 0$.*

This claim immediately proves the lemma: The assumption $\beta = \gamma + \omega^\xi \preceq \alpha$ together with the claim and (1) shows that

$$\beta \in W_\alpha^\eta \quad \text{iff} \quad \beta \in S^\eta \quad \text{iff} \quad \xi \succeq \eta \;.$$

Thus $\mathrm{rk}_{\mathrm{CB}}(\beta^\uparrow) = \sup\{\eta : \beta \in W_\alpha^\eta\} = \xi$.

We are left to prove the claim, which we do by transfinite induction on ξ. For $\xi = 0$ we have

$$\begin{aligned} \beta^\uparrow \in W_\alpha^0 &\iff \beta^\uparrow \in \mathrm{Up}(L(\alpha)) \setminus \{0^\uparrow\} \\ &\iff \beta \preceq \alpha \quad \text{and} \quad \beta \succ 0 \\ &\iff \beta \preceq \alpha \quad \text{and} \quad \beta = \gamma + \omega^\nu \quad \text{and} \quad \nu \succeq 0 \\ &\iff \beta^\uparrow \in S^0 \end{aligned}$$

so the claim follows in this case.

For ξ a limit ordinal, we immediately obtain

$$\bigcap_{\eta \prec \xi} S^{\eta} \;=\; S^{\xi}$$

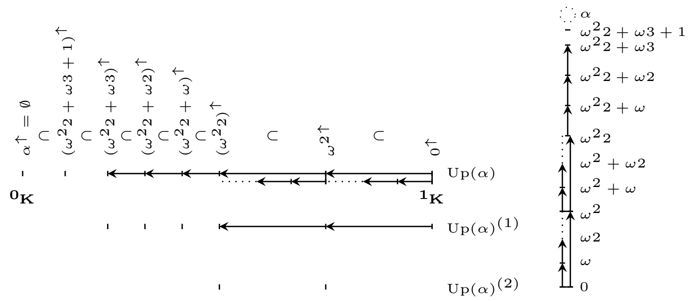
using (1). Hence the claim follows by induction.

Finally, assume $\xi = \eta + 1$. To prove the claim by induction, we have to show that $(S^{\eta})' = S^{\xi}$. It is easy to see that the isolated points in S^{η} are exactly of the form $(\gamma + \omega^{\eta})^{\uparrow}$. Therefore, using (1) we obtain

$$\begin{aligned}(S^{\eta})' &= \{(\gamma + \omega^{\nu})^{\uparrow} : \gamma + \omega^{\nu} \preceq \alpha \text{ and } \nu \succ \eta\} \\ &= \{(\gamma + \omega^{\nu})^{\uparrow} : \gamma + \omega^{\nu} \preceq \alpha \text{ and } \nu \succeq \xi\} = S^{\xi} \ . \end{aligned}$$

□

Example (cont). We are still considering the ordinal $\alpha = \omega^2 2 + \omega^1 3 + \omega^0 2$ from above. The first Cantor-Bendixon derivative of the Upsets of α, $\mathrm{Up}(\alpha)' = \mathrm{Up}(\alpha)^{(1)}$ as well as the second one, $\mathrm{Up}(\alpha)^{(2)}$, are shown on the left, while the original Kripke frame is still shown on the right. We see that at the second derivative only isolated points remain, thus at the next stage we will obtain the empty set.



In the following we will often have to consider the rank of an element in the context of a fixed valuation φ, e.g. when computing the evaluation of specific formulas under φ. In such situations, the ranks needs to be computed relative to the set of actually taken truth values under φ. We call this the *relativised* Cantor-Bendixon rank, since we relativise to the set of all actually taken truth values. Since this set is not necessarily a closed set, we will have to take the topological closure of it, and compute the rank within this closure.

Definition 11 (relativised CB-rank)**.** Given a specific valuation φ, we define the set of evaluations of atomic formulas and its closure as

$$\begin{aligned}\varphi(P(\mathcal{U})) &= \{\varphi(P(u)) : u \in \mathcal{U}\} \ , \\ V_{\mathcal{U}} &= \overline{\varphi(P(U))} \ , \end{aligned}$$

and the relativised CB-rank of c as

$$\mathrm{rk}_{\varphi\mathrm{CB}}(c) = \mathrm{rk}_{\mathrm{CB}}(c) \text{ in } V_{\mathcal{U}} \ .$$

If $\mathrm{rk}_{\varphi\mathrm{CB}}(c) = 0$ (i.e., $\varphi(c) = \varphi(P(c))$ is isolated in $V_{\mathcal{U}}$), we define the successor of this element, written as $\mathrm{succ}_{\varphi}(c)$, as its order-theoretic successor within $V_{\mathcal{U}}$.

In other words, the set $V_{\mathcal{U}}$ is the smallest subset of $\mathrm{Up}(K)$ that can serve as valuation base for the given φ. Note that in the above definition we will write only c in $\mathrm{rk}_{\varphi\mathrm{CB}}(c)$ instead of the more noisy $\mathrm{rk}_{\varphi\mathrm{CB}}(\varphi(P(c)))$.

We continue defining formulas that will be used to characterise infima of a certain degree. The first definition ($A \prec B$) is standard notation in Gödel logics, and expresses the strict less in the linear order of the truth values with the sole exception at $\mathbf{1_K}$ where we have $\mathbf{1_K} \prec \mathbf{1_K}$. The definition of infimum follows the intuitive, order theoretic, definition.

$$\begin{aligned} A \prec B &= ((B \to A) \to B) \\ Q(c) &= \forall x((Pc \prec Px) \to Px) \\ \mathrm{Inf}^0(x) &= \bot \to \bot \\ \mathrm{Inf}^{n+1}(x) &= \forall y((Px \prec Py) \to \exists z(\mathrm{Inf}^n(z) \wedge Px \prec Pz \prec Py)) \end{aligned}$$

Later in the proofs of the central theorems we will often refer to the following lemmas computing the evaluations of the above defined formulas, depending on topological properties of the evaluation of $P(c)$. As mentioned already above, the formula $A \prec B$ defines the strictly-less relation on the truth values, with the exception at $\mathbf{1_K}$:

Lemma 12. *Let φ be a valuation for a Kripke frame K. Then*

$$\varphi(A \prec B) = \begin{cases} \mathbf{1_K} & \varphi(A) < \varphi(B) \\ \varphi(B) & \textit{otherwise} \end{cases}$$

Proof. If $\varphi(A) < \varphi(B)$, then by definition $\varphi(B \to A) = \varphi(A)$, and thus again by definition $\varphi((B \to A) \to B) = \varphi(A \to B) = \mathbf{1_K}$. Otherwise we have $\varphi(B \to A) = \mathbf{1_K}$ and thus (adding $\top$ temporarily) $\varphi((B \to A) \to B) = \varphi(\top \to B) = \varphi(B)$. $\square$

The next lemma exhibits an important property, namely that we can distinguish between isolated points (that is $\mathrm{rk}_{\varphi\mathrm{CB}}(c) = 0$) and not-isolated points.

Lemma 13. *Let φ be a valuation for K_α. Then*

$$\varphi(Q(c)) = \begin{cases} \varphi(c) & \textit{if } \varphi(c) = \mathbf{1_K} \textit{ or } \mathrm{rk}_{\varphi CB}(c) \geq 1 \\ \mathrm{succ}_\varphi(c) & \textit{otherwise} \end{cases}$$

Proof. First, assume that $\varphi(c) = \varphi(P(c)) = \mathbf{1_K}$. Then for all $u \in U$ we have $\varphi(Pc \prec Pu) = \varphi(P(u))$, and thus $(Pc \prec Pu) \to Pu$ as well as $Q(c)$ evaluates to $\mathbf{1_K}$, which indeed is $\varphi(c)$.

Consider now an arbitrary $u \in U$: If $\varphi(c) \geq \varphi(u)$, then $\varphi(Pc \prec Pu) = \varphi(Pu)$, and the inner part of $Q(c)$ evaluates to $\varphi(Pu \to Pu) = \mathbf{1_K}$. If $\varphi(c) < \varphi(u)$, then the inner part evaluates to $\varphi(u)$. As a consequence we have that

$$\varphi(Q(c)) = \inf\{\varphi(u) : \varphi(u) > \varphi(c)\}.$$

Considering now the rank of $\varphi(c) = \varphi(P(c))$ in $V_{\mathcal{U}}$: Let us assume first that the rank of $\varphi(c)$ in $V_{\mathcal{U}}$ is bigger than 1, i.e., $\mathrm{rk}_{\varphi\mathrm{CB}}(c) \geq 1$. Since we are discussing only ordinals, we know that there has to be a strictly decreasing sequence to $\varphi(P(c))$ in $V_{\mathcal{U}}$. That is, there are $u_n \in U$ such that $\varphi(P(u_n))$ is a strictly decreasing sequence with limit $\varphi(P(c))$. This shows that

$$\varphi(Q(c)) = \inf\{\varphi(u) : \varphi(u) > \varphi(c)\} = \varphi(c).$$

In the case that the rank of $\varphi(c)$ in $V_{\mathcal{U}}$ is 0, i.e., $\varphi(c)$ is isolated in $V_{\mathcal{U}}$, we know there is a successor, and the above infimum evaluates exactly to the successor of $\varphi(c)$ in $V_{\mathcal{U}}$, that is

$$\varphi(Q(c)) = \inf\{\varphi(u) : \varphi(u) > \varphi(c)\} = \mathrm{succ}_{\varphi}(c). \qquad \square$$

Lemma 14. *Let φ be a valuation for K_α, then*

$$\varphi(\mathrm{Inf}^n(c)) = \begin{cases} \mathbf{1_K} & \text{if } \varphi(c) = \mathbf{1_K} \text{ or } \mathrm{rk}_{\varphi CB}(c) \geq n \\ \varphi(c) & \text{if } \varphi(c) \neq \mathbf{1_K} \text{ and } 0 < \mathrm{rk}_{\varphi CB}(c) < n \\ \mathrm{succ}_{\varphi}(c) & \text{otherwise, i.e., } \varphi(c) \neq \mathbf{1_K},\ \mathrm{rk}_{\varphi CB}(c) = 0 \text{ and } n > 0. \end{cases}$$

Proof. The proof is by induction on n. Let us recall the definition of $\mathrm{Inf}^n(c)$:

$$\begin{aligned} \mathrm{Inf}^0(c) &= \bot \to \bot \\ \mathrm{Inf}^{n+1}(c) &= \forall y((Pc \prec Py) \to \exists z(\mathrm{Inf}^n(z) \wedge Pc \prec Pz \prec Py)) \end{aligned}$$

Obviously, $\varphi(\mathrm{Inf}^0(c)) = \mathbf{1_K}$ for any $c \in \mathcal{U}$, which proves the lemma for the case $n = 0$, since $\mathrm{rk}_{\varphi \mathrm{CB}}(c) \geq 0$ for all c.

Now assume $n > 0$. Let

$$C(x, y, z) \ = \ \mathrm{Inf}^{n-1}(z) \wedge Px \prec Pz \prec Py$$

and

$$B(x, y) \ = \ \exists z C(x, y, z)$$

then $\mathrm{Inf}^n(x)$ can be written as $\forall y((Px \prec Py) \to B(x, y))$.

Claim.
$$\varphi(B(c, d)) \ \geq \ \min(\varphi(c), \varphi(d)) \tag{2}$$
and
$$\varphi\big((Pc \prec Pd) \to B(c, d)\big) \ \geq \ \varphi(c) \tag{3}$$
for any $c, d \in \mathcal{U}$. Thus,
$$\varphi(\mathrm{Inf}^n(c)) \geq \varphi(c) \tag{4}$$
for any $c \in \mathcal{U}$.

Proof of Claim. To prove the first part, it is enough to show

$$\varphi(C(c, d, c)) \geq \min(\varphi(c), \varphi(d)) \ .$$

By induction hypothesis we have $\varphi(\mathrm{Inf}^{n-1}(c)) \geq \varphi(c)$. Lemma 12 shows that $\varphi(Pc \prec Pc) = \varphi(c)$ and $\varphi(Pc \prec Pd) \geq \varphi(c)$. Putting things together we obtain $\varphi(C(c, d, c) \geq \min(\varphi(c), \varphi(d))$, hence (2) follows.

The second part follows using (2) and distinguishing cases depending on whether $\varphi(c) < \varphi(d)$ or not. If $\varphi(c) < \varphi(d)$, then

$$\varphi((Pc \prec Pd) \to B(c,d)) = \varphi(B(c,d)) \geq \min(\varphi(c), \varphi(d)) = \varphi(c) \ .$$

Otherwise, $\varphi(c) \geq \varphi(d)$, and

$$\varphi(B(c,d)) \geq \min(\varphi(c), \varphi(d)) = \varphi(d) = \varphi(Pc \prec Pd) \ .$$

Hence,

$$\varphi((Pc \prec Pd) \to B(c,d)) = \mathbf{1_K} \geq \varphi(c) \ .$$

The last part follows directly from (3). □

Let $c \in \mathcal{U}$. We consider cases according to the properties satisfied by c. If $\varphi(c) = \mathbf{1_K}$ then $\varphi(\mathrm{Inf}^n(c)) = \mathbf{1_K}$ by (4).

Assume $\varphi(c) < \mathbf{1_K}$. Let m be $\mathrm{rk}_{\varphi\mathrm{CB}}(c)$.

If $m \geq n$, we have to show that $\varphi((Pc \prec Pd) \to B(c,d)) = \mathbf{1_K}$ for any $d \in \mathcal{U}$. Let $d \in \mathcal{U}$ be given. If $\varphi(d) \leq \varphi(c)$, then, by Lemma 12 and (2),

$$\varphi(Pc \prec Pd) = \varphi(d) = \min(\varphi(c), \varphi(d)) \leq \varphi(B(c,d))$$

hence $\varphi((Pc \prec Pd) \to B(c,d)) = \mathbf{1_K}$. In the other case $\varphi(d) > \varphi(c)$. By definition of Cantor-Bendixon rank, as $\mathrm{rk}_{\varphi\mathrm{CB}}(c) = m$, there exists $b \in \mathcal{U}$ such that $\mathrm{rk}_{\varphi\mathrm{CB}}(b) = m{-}1$ and $\varphi(c) < \varphi(b) < \varphi(d)$. Induction hypothesis yields $\varphi(\mathrm{Inf}^{n-1}(b)) = \mathbf{1_K}$ as $\mathrm{rk}_{\varphi\mathrm{CB}}(b) = m{-}1 \geq n{-}1$. Hence, by Lemma 12,

$$\varphi(C(c,d,b)) \ = \ \varphi\big(\mathrm{Inf}^{n-1}(b) \wedge (Pc \prec Pb) \wedge (Pb \prec Pd)\big) \ = \ \mathbf{1_K} \ .$$

Thus, $\varphi(B(c,d)) = \mathbf{1_K}$ and the assertion follows for $m \geq n$.

Now assume $0 < m < n$. We have to show $\varphi(\mathrm{Inf}^n(c)) = \varphi(c)$.

By definition of Cantor-Bendixon rank, there exist some $d_i \in \mathcal{U}$ such that $\varphi(c) < \varphi(d_i)$,

$$\forall b{\in}\mathcal{U}\big(\varphi(c) < \varphi(b) < \varphi(d_i) \Rightarrow \mathrm{rk}_{\varphi\mathrm{CB}}(b) < m\big)$$

and $\varphi(c) = \inf_i(\varphi(d_i))$. We will show that $\varphi(B(c,d_i)) \leq \varphi(d_i)$ for all i, which will prove this case because

$$\begin{aligned}\varphi(\mathrm{Inf}^n(c)) &\leq \inf_i \big(\varphi((Pc \prec Pd_i) \to B(c,d_i))\big) \\ &= \inf_i \big(\varphi(B(c,d_i))\big) \leq \inf_i \big(\varphi(d_i)\big) = \varphi(c)\end{aligned}$$

and $\varphi(\mathrm{Inf}^n(c)) \geq \varphi(c)$ using (4).

To prove $\varphi(B(c,d_i)) \leq \varphi(d_i)$ we will show $\varphi(C(c,d_i,b)) \leq \varphi(d_i)$ for any $b \in \mathcal{U}$, by distinguishing cases according to the comparison of $\varphi(b)$ with $\varphi(c)$ and $\varphi(d_i)$. Let $b \in \mathcal{U}$ be given. $C(c,d_i,b)$ is of the form

$$\mathrm{Inf}^{n-1}(b) \wedge (Pc \prec Pb) \wedge (Pb \prec Pd_i) \ .$$

If $\varphi(b) \leq \varphi(c)$ then

$$\varphi(C(c,d_i,b)) \leq \varphi(Pc \prec Pb) = \varphi(b) \leq \varphi(c) < \varphi(d_i) \ .$$

If $\varphi(b) \geq \varphi(d_i)$ then

$$\varphi(C(c, d_i, b)) \leq \varphi(Pb \prec Pd_i) = \varphi(d_i) \ .$$

If $\varphi(c) < \varphi(b) < \varphi(d_i)$ then $\mathrm{rk}_{\varphi\mathrm{CB}}(b) < m \leq n-1$ by assumption, thus, using the induction hypothesis,

$$\varphi(C(c, d_i, b)) \leq \varphi(\mathrm{Inf}^{n-1}(b)) \leq \mathrm{succ}_\varphi(b) \leq \varphi(d_i) \ .$$

For the final case, assume $m = 0 < n$. We have to show $\varphi(\mathrm{Inf}^n(c)) = \mathrm{succ}_\varphi(c)$.

By definition of Cantor-Bendixon rank, there exists some $d \in \mathcal{U}$ such that $\varphi(c) < \varphi(d)$ and, for all $b \in \mathcal{U}$, either $\varphi(b) \leq \varphi(c)$ or $\varphi(d) \leq \varphi(b)$. We first prove

$$\varphi(B(c, d)) = \varphi(d) \tag{5}$$

by showing that $\varphi(C(c, d, b)) \leq \varphi(d)$ for any $b \in \mathcal{U}$, and that $\varphi(C(c, d, d)) = \varphi(d)$. Let $b \in \mathcal{U}$ be arbitrary. $C(c, d, b)$ is of the form

$$\mathrm{Inf}^{n-1}(b) \wedge (Pc \prec Pb) \wedge (Pb \prec Pd) \ .$$

If $\varphi(b) \leq \varphi(c)$ then

$$\varphi(C(c, d, b)) \leq \varphi(Pc \prec Pb) = \varphi(b) \leq \varphi(c) < \varphi(d) \ .$$

If $\varphi(b) > \varphi(c)$, then $\varphi(b) \geq \varphi(d)$ by choice of d, hence

$$\varphi(C(c, d, b)) \leq \varphi(Pb \prec Pd) = \varphi(d) \ .$$

To evaluate $\varphi(C(c, d, d))$, we compute $\varphi(\mathrm{Inf}^{n-1}(d)) \geq \varphi(d)$ by induction hypothesis, $\varphi(Pc \prec Pd) = \mathbf{1_K}$, and $\varphi(Pd \prec Pd) = \varphi(d)$. This finishes the proof of (5).

At last, to show $\varphi(\mathrm{Inf}^n(c)) = \varphi(d)$ it suffices to show

$$\varphi\big((Pc \prec Pb) \to B(c, b)\big) \ \geq \ \varphi(d)$$

for all $b \in \mathcal{U}$, because $\varphi((Pc \prec Pd) \to B(c, d)) = \varphi(d)$ follows using (5). Let $b \in \mathcal{U}$ be arbitrary. If $\varphi(b) \leq \varphi(c)$ then $\varphi(Pc \prec Pb) = \varphi(b)$ and $\varphi(B(c, b)) \geq \min(\varphi(c), \varphi(b)) = \varphi(b)$ using (2). Thus,

$$\varphi((Pc \prec Pb) \to B(c, b)) = \mathbf{1_K} \geq \varphi(d) \ .$$

Otherwise, $\varphi(b) > \varphi(c)$, hence $\varphi(b) \geq \varphi(d)$ by choice of d. Thus

$$\varphi((Pc \prec Pb) \to B(c, b)) = \varphi(B(c, b)) \geq \varphi\big(\mathrm{Inf}^{n-1}(d) \wedge (Pc \prec Pd) \wedge (Pd \prec Pb)\big) \ .$$

We have $\varphi(\mathrm{Inf}^{n-1}(d)) \geq \varphi(d)$ by induction hypothesis, $\varphi(Pc \prec Pd) = \mathbf{1_K}$, and $\varphi(Pd \prec Pb) \geq \varphi(b) \geq \varphi(d)$. Hence

$$\varphi((Pc \prec Pb) \to B(c, b)) \geq \varphi(d)$$

which finishes the proof. □

In the following, our aim is to distinguish the logics $L(\alpha)$ and $L(\beta)$ based on the Kripke frames K_α and K_β according to Definition 5 on page 4, for different ordinals α and β.

3 The simple case

In a first step we separate the logics determined by ordinals of the form $\alpha_n = \omega^n$, for $n \geq 0$. In the following we will refer to the linear Kripke frame where the carrier set is equivalent to the ordinal ω^n as K^n.

Definition 15. Let

$$A^n = \forall x \forall y (\mathrm{Inf}^n(x) \wedge \mathrm{Inf}^n(y) \wedge Q(x) \to Q(y))$$

Theorem 16. *With the definitions above we have $A^n \notin L(K^n)$, but $A^n \in L(K^m)$ for all $m < n$.*

Proof. We first show that $A^n \notin L(K^n)$ by providing a model in which $\varphi(A^n) \neq \mathbf{1_K}$, i.e., does not evaluate to the full Kripke frame.

Since the outer quantifiers are universal, we have to give instances for x and y rendering the inner part false. Let U, the universe of objects, be equal to the upset of K, $U = \mathrm{Up}(K)$, and fix the valuations in the canonical way, i.e.,

$$\varphi(u) = u$$

Now let $x = \mathbf{1_K}$ and $y = \mathbf{0_K}$. Thus, according to Lemma 14, we obtain that

$$\begin{aligned} \varphi(\mathrm{Inf}^n(x)) &= \varphi(\mathrm{Inf}^n(\mathbf{1_K})) = \mathbf{1_K} && \text{due to } \varphi(\mathbf{1_K}) = \mathbf{1_K} \\ \varphi(\mathrm{Inf}^n(y)) &= \varphi(\mathrm{Inf}^n(\mathbf{0_K})) = \mathbf{1_K} && \text{due to } \mathrm{rk}_{\varphi\mathrm{CB}}(\mathbf{0_K}) = n > 0. \end{aligned}$$

According to Lemma 13, we obtain that

$$\begin{aligned} \varphi(Q(x)) &= \varphi(Q(\mathbf{1_K})) = \mathbf{1_K} && \text{due to } \varphi(\mathbf{1_K}) = \mathbf{1_K} \\ \varphi(Q(y)) &= \varphi(Q(\mathbf{0_K})) = \mathbf{0_K} && \text{due to } \mathrm{rk}_{\varphi\mathrm{CB}}(\mathbf{0_K}) \geq 1. \end{aligned}$$

Combining these, we obtain that

$$\varphi(A^n) = \mathbf{1_K} \wedge \mathbf{1_K} \wedge \mathbf{1_K} \to \mathbf{0_K} = \mathbf{0_K},$$

which shows that $A^n \notin L(K^n)$.

Next we prove that $A^n \in L(K^m)$ for all $m < n$. We have to show that for each evaluation and each selection of x and y, the inner part evaluates to true, i.e., $\mathbf{1_K}$. This is done by case distinction over value and rank of x and y, see Table 1 on the next page. Thus, $A^n \in L(K^m)$ for $m < n$. ☐

4 The general case

The general case builds upon the ideas presented in the previous section, and extends it to any logic based on a Kripke frame with carrier set determined by an ordinal less than ω^ω.

At the core of the proof is the idea that also axiomatises finitely valued logics, namely that if we have a disjunction of n implications, but less than n many truth values, then the formula evaluates to true. More formally, define $\mathrm{chain}(x_1, \ldots, x_n)$ as follows:

$$\mathrm{chain}(x_1, \ldots, x_n) = (P(x_1) \to Q(x_2)) \vee \bigvee_{i=2}^{n-1} (P(x_i) \to P(x_{i+1})) \ .$$

Table 1: Evaluations of A^n and its sub-formulas

IF				THEN				
x	y	$\mathrm{rk}_{\varphi\mathrm{CB}}(x)$	$\mathrm{rk}_{\varphi\mathrm{CB}}(y)$	$\mathrm{Inf}^n(x)$	$\mathrm{Inf}^n(y)$	$Q(x)$	$Q(y)$	A^n
$\mathbf{1_K}$	$\mathbf{1_K}$	/	/	$\mathbf{1_K}$	$\mathbf{1_K}$	$\mathbf{1_K}$	$\mathbf{1_K}$	$\mathbf{1_K}$
$< \mathbf{1_K}$	$\mathbf{1_K}$	$0 < . < n$	/	x	$\mathbf{1_K}$	x	$\mathbf{1_K}$	$\mathbf{1_K}$
		0	/	$S(x)$	$\mathbf{1_K}$	$S(x)$	$\mathbf{1_K}$	$\mathbf{1_K}$
$\mathbf{1_K}$	$< \mathbf{1_K}$	/	$0 < . < n$	$\mathbf{1_K}$	y	$\mathbf{1_K}$	y	$\mathbf{1_K}$
		/	0	$\mathbf{1_K}$	$S(y)$	$\mathbf{1_K}$	$S(y)$	$\mathbf{1_K}$
$< \mathbf{1_K}$	$< \mathbf{1_K}$	$0 < . < n$	$0 < . < n$	x	y	x	y	$\mathbf{1_K}$
		$0 < . < n$	0	x	$S(y)$	x	$S(y)$	$\mathbf{1_K}$
		0	$0 < . < n$	$S(x)$	y	$S(x)$	y	$\mathbf{1_K}$
		0	0	$S(x)$	$S(y)$	$S(x)$	$S(y)$	$\mathbf{1_K}$

$S(x)$ is short for $\mathrm{succ}_\varphi(x)$; if x or $y = \mathbf{1_K}$, then the respective $\mathrm{rk}_{\varphi\mathrm{CB}}$ is irrelevant (/)

In the following we will mainly consider infinite ordinals. For the finite case we only mention that by replacing in the above definition Q with P one obtains the standard chain definition that also provides an axiomatisation of the logics of finite linear Kripke frames with constant domains (Baaz et al., 2007).

An obvious consequence is

Proposition 17. *Let φ be a valuation, and $a_1, \ldots, a_n$ elements in U. Then we have the following:*

1. *If $\varphi(a_1) > \mathrm{succ}_\varphi(a_2)$ and $\varphi(a_2) > \varphi(a_3) > \cdots > \varphi(a_n)$, then*
$$\varphi(\mathrm{chain}(a_1, \ldots, a_n)) = \mathrm{succ}_\varphi(a_2) < \mathbf{1_K} \ .$$
2. *If one comparison in the above sequence is not strictly decreasing, then*
$$\varphi(\mathrm{chain}(a_1, \ldots, a_n)) = \mathbf{1_K} \ .$$

Proof. In the first case, we have $\varphi(P(a_1) \to Q(a_2)) = \varphi(Q(a_2))$ and $\varphi(P(a_i) \to P(a_{i+1})) = \varphi(P(a_{i+1}))$ for $i \geq 2$, thus

$$\begin{aligned}\varphi(\mathrm{chain}(a_1, \ldots, a_n)) &= \max\{\varphi(Q(a_2)), \varphi(a_3), \ldots, \varphi(a_n)\} \\ &= \max\{\mathrm{succ}_\varphi(a_2), \varphi(a_3), \ldots, \varphi(a_n)\} = \mathrm{succ}_\varphi(a_2) < \varphi(a_1) \leq \mathbf{1_K} \ .\end{aligned}$$

In the second case, either $\varphi(a_1) \leq \mathrm{succ}_\varphi(a_2)$ or $\varphi(a_i) \leq \varphi(a_{i+1})$ for some $i \geq 2$. If $\varphi(a_1) \leq \mathrm{succ}_\varphi(a_2)$ we have

$$\varphi(\mathrm{chain}(a_1, \ldots, a_n)) \geq \varphi(P(a_1) \to Q(a_2)) = \mathbf{1_K} \ ,$$

and otherwise

$$\varphi(\mathrm{chain}(a_1, \ldots, a_n)) \geq \varphi(P(a_i) \to P(a_{i+1})) = \mathbf{1_K} \ . \qquad \square$$

Let $0 \prec \alpha \prec \beta \prec \omega^\omega$, and assume that β is infinite. The Cantor normal form of α and β can be written as

$$\begin{aligned} \alpha &= \omega^n k_n + \cdots + \omega^0 k_0 \\ \beta &= \omega^n l_n + \cdots + \omega^0 l_0 \end{aligned} \tag{6}$$

for some finite $n, l_0, \ldots, l_n, k_0, \ldots, k_n$ with $n > 0$ (as $\beta \succeq \omega$) and $l_n > 0$ (some or all of the other k_i and l_i may be 0). As $\alpha \prec \beta$, there is some $d \leq n$ such that $k_d < l_d$. Choose d maximal, i.e., $k_i = l_i$ for $i = d+1, \ldots, n$. As $0^\uparrow = \mathbf{1_K}$ and $\mathbf{1_K}$ satisfies all of the Inf^n-formulas, we need an additional variable x_1^{n+1} reserved to deal with this situation. Then, let

$$\vec{x} = (x_1^{n+1}, x_1^n, \ldots, x_{l_n}^n, \ldots, x_1^d, \ldots, x_{l_d}^d),$$

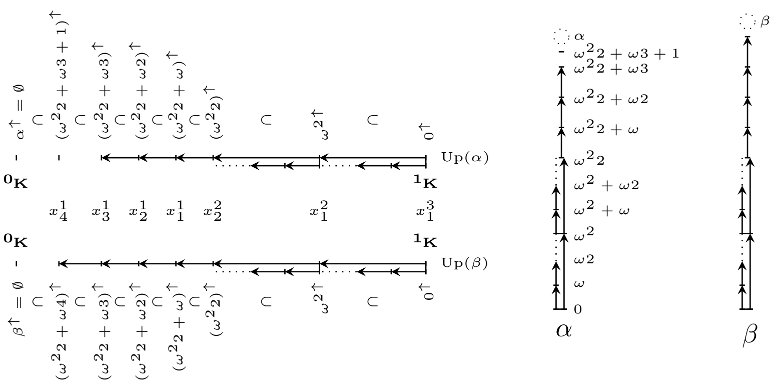
and define $A_{\alpha,\beta}(\vec{x})$ and $A_{\alpha,\beta}$ as follows:

$$A_{\alpha,\beta}(\vec{x}) = \Big(\bigwedge_{u=d}^{n} \bigwedge_{i=1}^{l_u} \mathrm{Inf}^u(x_i^u) \Big) \rightarrow \mathrm{chain}(\vec{x})$$

and

$$A_{\alpha,\beta} = \forall \vec{x} A_{\alpha,\beta}(\vec{x}).$$

Example (cont). Continuing our example, let $\alpha = \omega^2 2 + \omega^1 3 + \omega^0 2$ and $\beta = \omega^2 2 + \omega^1 4 + \omega^0 1$. That is, $k_0 = 2$, $k_1 = 3$, $k_2 = 2$, $l_0 = 1$, $l_1 = 4$, $l_2 = 2$. We obtain $d = 1$, since $k_1 = 3 < 4 = l_1$, and it is the largest one.



Since we add an additional term for $\mathbf{1_K}$, we also have $k_3 = l_3 = 1$. The variable vector becomes

$$\vec{x} = (x_1^3, x_1^2, x_2^2, x_1^1, x_2^1, x_3^1, x_4^1)$$

and $A_{\alpha,\beta}$ is

$$\forall \vec{x} (\mathrm{Inf}^2(x_1^2) \wedge \ldots \wedge \mathrm{Inf}^1(x_3^1) \wedge \mathrm{Inf}^1(x_4^1) \rightarrow \mathrm{chain}(\vec{x}))$$

The diagram on the side shows the two ordinals, and the intended interpretation of the variables as the respective infima. If the x_j^i are chosen as shown, the formula $A_{\alpha,\beta}$ will evaluate to $\mathbf{1_K}$ in $L(\alpha)$ but something smaller in $L(\beta)$.

Theorem 18. *If $0 \prec \alpha \prec \beta \prec \omega^\omega$ with $\beta \succeq \omega$, then $A_{\alpha,\beta} \in L(\alpha)$, but $A_{\alpha,\beta} \notin L(\beta)$.*

Proof. Let K be K_β. First we show that $A_{\alpha,\beta} \notin L(\beta)$ in the very same way as in the in the proof of the simple case, Theorem 16, namely by defining the universe of our valuation to be $\mathrm{Up}(K)$, and the valuation to be $\varphi(P(c)) = c$ for $c \in \mathrm{Up}(K)$. We have to provide an instance $\vec{a}$ such that the formula $A_{\alpha,\beta}(\vec{a})$ evaluates to something different from $\mathbf{1_K}$. We choose $\vec{a}$ as the canonical points specified by the Cantor normal form of ordinal β; that is, let

$$a^i_j = (\omega^n l_n + \cdots + \omega^{i+1} l_{i+1} + \omega^i j)^{\uparrow}$$

for $d \le i \le n{+}1$ and $0 < j \le l_i$. In particular, $\varphi(P(a^{n+1}_1)) = 0^{\uparrow} = \mathbf{1_K}$. Using Lemma 10, we have $\mathrm{rk}_{\varphi\mathrm{CB}}(a^i_j) = i$ for $d \le i \le n$ and $0 < j \le l_i$. Hence, Lemma 14 shows $\varphi\big(\mathrm{Inf}^i(a^i_j)\big) = \mathbf{1_K}$ for $d \le i \le n{+}1$ and $0 < j \le l_i$.

Let

$$\vec{a} = (a^{n+1}_1, a^n_1, \ldots, a^n_{l_n}, \ldots, a^d_1, \ldots, a^d_{l_d}) \ .$$

Then $\varphi(a^{n+1}_1) > \varphi(a^n_1) > \ldots > \varphi(a^d_{l_d})$ and $\varphi(a^n_1) = \mathrm{succ}_\varphi(a^n_1)$ as β is infinite. Thus, Proposition 17 shows

$$\varphi\big(\,\mathrm{chain}(\vec{a})\big) \ = \ \mathrm{succ}_\varphi(a^n_1) \ = \ \varphi(a^n_1) \ < \ \varphi(a^{n+1}_1) \ = \ \mathbf{1_K} \ .$$

This proves the first direction.

In order to show $A_{\alpha,\beta} \in L(\alpha)$, let φ be a valuation, and let a^i_j be any choice of elements in $\mathcal{U}$, for $d \le i \le n{+}1$ and $0 < j \le l_i$. We have to show

$$\varphi(\bigwedge_{i=d}^{n} \bigwedge_{j=1}^{l_i} \mathrm{Inf}^i(a^i_j) \to \mathrm{chain}(\vec{a})) = \mathbf{1_K} \ .$$

To this end, assume $\varphi(\mathrm{chain}(\vec{a})) < \mathbf{1_K}$. By Proposition 17 we must have $\varphi(a^{n+1}_1) > \mathrm{succ}_\varphi(a^n_1)$,

$$\varphi(a^n_1) > \cdots > \varphi(a^n_{l_n}) > \varphi(a^{n-1}_1) > \cdots > \varphi(a^d_{l_d}) \tag{7}$$

and $\varphi(\mathrm{chain}(\vec{a})) = \mathrm{succ}_\varphi(a^n_1)$.

Assume for the sake of contradiction that we have $\mathrm{rk}_{\varphi\mathrm{CB}}(a^i_j) \ge i$ for all $d \le i \le n$ and $0 < j \le l_i$. By induction on the position of $\varphi(a^i_j)$ in (7), and using Lemma 10, we can show

$$\varphi(a^i_j) \ \le \ (\omega^n l_n + \cdots + \omega^{i+1} l_{i+1} + \omega^i j)^{\uparrow}$$

for $d \le i \le n$ and $0 < j \le l_i$. E.g., consider $\varphi(a^n_1)$. By assumption, $\mathrm{rk}_{\varphi\mathrm{CB}}(a^n_1) \ge n$, but the biggest element in $\mathrm{Up}(K_\alpha)$ of CB-rank $\ge n$ is $(\omega^n)^{\uparrow}$. Hence $\varphi(a^n_1) \le (\omega^n)^{\uparrow}$. Assume, the next element in (7) is $\varphi(a^n_2)$ (it could also be $\varphi(a^{n-1}_1)$ which would require a similar argumentation). Again, $\mathrm{rk}_{\varphi\mathrm{CB}}(a^n_2) \ge n$ by assumption, and the biggest element in $\mathrm{Up}(K_\alpha)$ of CB-rank $\ge n$, which is smaller than $\varphi(a^n_1)$, must also be smaller than $(\omega^n)^{\uparrow}$ as we have just shown, and thus is $\le (\omega^n 2)^{\uparrow}$. Hence $\varphi(a^n_2) \le (\omega^n 2)^{\uparrow}$. And so on. Thus, we obtain for $a^d_{k_d}$

$$\varphi(a^d_{k_d}) \ \le \ (\omega^n k_n + \cdots + \omega^d k_d)^{\uparrow}$$

(remember $k_n = l_n, \ldots, k_{d+1} = l_{d+1}$). As $l_d > k_d$, we obtain $\mathrm{rk}_{\varphi\mathrm{CB}}(a^d_{k_d+1}) \geq d$ by assumption, and also

$$\varphi(a^d_{k_d+1}) < \varphi(a^d_{k_d}) \leq (\omega^n k_n + \cdots + \omega^d k_d)^{\uparrow} \ .$$

By examining the Cantor normal form of α, we see that such an element does not exist in $\mathrm{Up}(K_\alpha)$, so we have obtained a contradiction.

Hence, we must have $\mathrm{rk}_{\varphi\mathrm{CB}}(a^i_j) < i$ for some $i \leq n$ and $0 < j \leq k_i$. Using Lemma 14, we obtain

$$\varphi(\bigwedge_{i=d}^{n} \bigwedge_{j=1}^{l_i} \mathrm{Inf}^i(a^i_j)) \leq \varphi(\mathrm{Inf}^i(a^i_j)) \leq \mathrm{succ}_\varphi(a^i_j) \leq \mathrm{succ}_\varphi(a^n_1) = \varphi(\mathrm{chain}(\vec{a}))$$

which proofs the claim. □

5 Dually well-founded Kripke frames based on ordinals

For well-founded Kripke frames, the ordering on the upsets contains proper infima, but no proper suprema — we have studied well founded Kripke frames based on ordinals $< \omega^\omega$ in detail in the previous section. For *dually* well-founded Kripke frames, the ordering on the upsets does not contain proper infima, but may contain proper suprema (of different Cantor-Bendixon rank). In this section we will consider this kind of dual situation to the previous section, and prove similar results. In particular we will define a kind of dual to the Inf formulas based on suprema, and will separate logics based on dually well-founded Kripke frames based on ordinals $< \omega^\omega$.

Definition 19 (Dually well-founded Kripke frames based on ordinals)**.** Given an ordinal α, let K^*_α be the Kripke frame given by the set α and the relation $\succeq$.

Using the set-theoretic view of ordinals, the upsets of K^*_α are literally the same as the ordinals $\beta \preceq \alpha$. Nevertheless, we will use $\beta^\uparrow$ to conceptually distinguish upsets from worlds in the Kripke frame — mathematically, $\beta^\uparrow$ is just the identity, $\beta^\uparrow = \beta$. Hence, we have $\mathrm{Up}(K^*_\alpha) = \{\beta^\uparrow : \beta \preceq \alpha\}$. We observe that $0^\uparrow = \emptyset = \mathbf{0}_{\mathbf{K}^*_\alpha}$ and that $\alpha^\uparrow = \mathbf{1}_{\mathbf{K}^*_\alpha}$. The order $\subseteq$ on upsets this time corresponds to $\preceq$ on ordinals, i.e., $\beta^\uparrow \subseteq \gamma^\uparrow$ iff $\beta \preceq \gamma$.

Lemma 10 on computing Cantor-Bendixon ranks carries over to the dually well-founded case, with the same proof.

Lemma 20. *Let $\beta = \gamma + \omega^\xi \preceq \alpha$. Then $\mathrm{rk}_{CB}(\beta^\uparrow) = \xi$ in $\mathrm{Up}(L(K^*_\alpha))$.* □

The following Sup^n formulas express a kind of dual to the Inf^n formulas from the previous section.

$$\begin{aligned} \mathrm{Sup}^0(x) &= \bot \to \bot \\ \mathrm{Sup}^{n+1}(x) &= [(\forall y Py) \prec Px] \\ &\quad \wedge \forall y[(Py \prec Px) \to \exists z(\mathrm{Sup}^n(z) \wedge Py \prec Pz \prec Px)] \end{aligned}$$

Lemma 21. *Let φ be a valuation for K^*_α, then*

$$\varphi(\mathrm{Sup}^n(c)) = \begin{cases} \mathbf{1_K} & \textit{if } \varphi(c) = \mathbf{1_K} \textit{ or } \mathrm{rk}_{\varphi CB}(c) \geq n \\ \varphi(c) & \textit{otherwise.} \end{cases}$$

Proof. The proof is by induction on n. Obviously, $\varphi(\mathrm{Sup}^0(c)) = \mathbf{1_K}$ for any $c \in \mathcal{U}$, which proves the lemma for the case $n = 0$.

Now assume $n > 0$. Let

$$B(x, y) = \exists z(\mathrm{Sup}^{n-1}(z) \wedge Py \prec Pz \prec Px)$$

then $\mathrm{Sup}^n(x)$ can be written as

$$((\forall y Py) \prec Px) \wedge \forall y((Py \prec Px) \rightarrow B(x, y))$$

Claim.

$$\varphi(B(c, d)) \geq \varphi(c) \tag{8}$$

for any $c, d \in \mathcal{U}$. Thus,

$$\varphi(\mathrm{Sup}^n(c)) \geq \varphi(c) \tag{9}$$

for any $c \in \mathcal{U}$.

Proof of Claim. For the first part, we obtain by Lemma 12 that $\varphi(Pd \prec Pc) \geq \varphi(c)$ and $\varphi(Pc \prec Pc) = \varphi(c)$, and by induction hypothesis that $\varphi(\mathrm{Sup}^{n-1}(c)) \geq \varphi(c)$. Thus $\varphi(\mathrm{Sup}^{n-1}(c) \wedge Pd \prec Pc \prec Pc) = \varphi(c)$, hence (8) follows.

The second part follows because, again by Lemma 12, $\varphi((\forall y Py) \prec Pc) \geq \varphi(c)$, and $\varphi((Pd \prec Pc) \rightarrow B(c, d)) \geq \varphi(B(c, d)) \geq \varphi(c)$ for any $d \in \mathcal{U}$. □

Let $c \in \mathcal{U}$. We consider cases according to the properties satisfied by c. If $\varphi(c) = \mathbf{1_K}$ then $\varphi(\mathrm{Sup}^n(c)) = \mathbf{1_K}$ by (9).

Assume $\varphi(c) < \mathbf{1_K}$. Let m be $\mathrm{rk}_{\varphi\mathrm{CB}}(c)$.

If $m \geq n$, there exists, by definition of Cantor-Bendixon rank, $c_i \in \mathcal{U}$ such that $\mathrm{rk}_{\varphi\mathrm{CB}}(c_i) = m-1$, $\varphi(c_i) < \varphi(c)$ and $\sup_i \varphi(c_i) = \varphi(c)$. In particular, $\varphi(\forall y Py) \leq \varphi(c_0) < \varphi(c)$, hence $\varphi((\forall y Py) \prec Pc) = \mathbf{1_K}$ by Lemma 12. Thus, we have to show that $\varphi((Pd \prec Pc) \rightarrow B(c, d)) = \mathbf{1_K}$ for any $d \in \mathcal{U}$.

Let $d \in \mathcal{U}$ be given. If $\varphi(d) \geq \varphi(c)$, then, by Lemma 12 and (8),

$$\varphi(Pd \prec Pc) \leq \varphi(c) \leq \varphi(B(c, d))$$

hence $\varphi((Pd \prec Pc) \rightarrow B(c, d)) = \mathbf{1_K}$. In the other case $\varphi(d) < \varphi(c)$, hence there is some i such that $\varphi(d) < \varphi(c_i) < \varphi(c)$. By induction hypothesis, $\varphi(\mathrm{Sup}^{n-1}(c_i)) = \mathbf{1_K}$ as $\mathrm{rk}_{\varphi\mathrm{CB}}(c_i) = m-1 \geq n-1$. Hence, by Lemma 12,

$$\varphi\big(\mathrm{Sup}^{n-1}(c_i) \wedge Pd \prec Pc_i \prec Pc\big) = \mathbf{1_K} \ .$$

Thus, $\varphi(B(c, d)) = \mathbf{1_K}$ and the assertion follows for $m \geq n$.

Now assume $m < n$. We have to show $\varphi(\mathrm{Sup}^n(c)) = \varphi(c)$.

By Lemma 12, $\varphi((\forall y Py) \prec Pc) \geq \varphi(c)$. Thus, by (8), it is enough to show that $\varphi((Pd \prec Pc) \rightarrow B(c, d)) \leq \varphi(c)$ for some $d \in \mathcal{U}$.

By definition of Cantor-Bendixon rank, there exists some $d \in \mathcal{U}$ such that $\varphi(d) < \varphi(c)$ and

$$\forall b {\in} \mathcal{U}(\varphi(d) < \varphi(b) < \varphi(c) \Rightarrow \mathrm{rk}_{\varphi\mathrm{CB}}(b) < m \ .$$

We claim that $\varphi(B(c,d)) \leq \varphi(c)$ which proves the assertion as $\varphi(Pd \prec Pc) = \mathbf{1_K}$. That is, we have to show for

$$C(x,y,z) \;=\; \mathrm{Sup}^{n-1}(z) \wedge (Py \prec Pz) \wedge (Pz \prec Px)$$

that

$$\varphi(C(c,d,b)) \leq \varphi(c)$$

for any $b \in \mathcal{U}$.

Let $b \in \mathcal{U}$ be given. We distinguish cases according to the comparison of $\varphi(b)$ with $\varphi(c)$ and $\varphi(d)$. If $\varphi(b) \leq \varphi(d)$ then $\varphi(C(c,d,b)) \leq \varphi(Pd \prec Pb) = \varphi(b) \leq \varphi(d) < \varphi(c)$. If $\varphi(b) \geq \varphi(c)$ then $\varphi(C(c,d,b)) \leq \varphi(Pb \prec Pc) = \varphi(c)$. If $\varphi(d) < \varphi(b) < \varphi(c)$ then $\mathrm{rk}_{\varphi\mathrm{CB}}(b) < m \leq n-1$ by assumption, thus, using the induction hypothesis, $\varphi(C(c,d,b)) \leq \varphi(\mathrm{Sup}^{n-1}(b)) = \varphi(b) < \varphi(c)$. □

The following formula is similar to the corresponding one in the previous section, It is slightly simpler as it does not involve the predicate Q.

$$\mathrm{chain}^*(x_1,\ldots,x_n) = \bigvee_{i=1}^{n-1} (P(x_i) \to P(x_{i+1})) \;\; .$$

The following proposition is proven in the same way as Proposition 17.

Proposition 22. *Let φ be a valuation for K^*_α, and $a_1,\ldots,a_n$ elements in $\mathcal{U}$. Then we have the following:*

1. *If $\varphi(a_1) > \varphi(a_2) > \varphi(a_3) > \cdots > \varphi(a_n)$, then*

$$\varphi(\mathrm{chain}(a_1,\ldots,a_n)) = \varphi(a_2) < \mathbf{1_K} \;\; .$$

2. *If the above sequence is not strictly decreasing, then*

$$\varphi(\mathrm{chain}(a_1,\ldots,a_n)) = \mathbf{1_K} \;\; .$$

The separating formulas for logics based on dually well-founded Kripke frames based on ordinals $\prec \omega^\omega$ are defined similarly to the case of well-founded Kripke frames. Let $\alpha \prec \omega^\omega$, and write $\alpha{+}1$ in Cantor normal form as

$$\alpha{+}1 \;=\; \omega^n k_n + \cdots + \omega^0 k_0 \tag{10}$$

for some finite $n, k_0, \ldots, k_n$ with $k_n > 0$. We also have $k_0 > 0$, but some other k_i may be 0. Let

$$\vec{x} = (x^0_{k_0}, \ldots, x^0_1, \ldots, x^n_{k_n}, \ldots, x^n_1) \;\; ,$$

and define $A^*_\alpha(\vec{x})$ and A^*_α as follows:

$$A^*_\alpha(\vec{x}) = \Big(\bigwedge_{i=0}^{n} \bigwedge_{j=1}^{k_i} \mathrm{Sup}^i(x^i_j) \Big) \to \mathrm{chain}^*(\vec{x})$$

and

$$A^*_\alpha = \forall \vec{x} A^*_\alpha(\vec{x}).$$

Theorem 23. *If $0 \prec \alpha \prec \beta \prec \omega^\omega$, then $A^*_\alpha \in L(K^*_\alpha)$, but $A^*_\alpha \notin L(K^*_\beta)$.*

Proof. Let K be K^*_β. First, we show that $A^*_\alpha \notin L(K^*_\beta)$ in a similar way as in the in the proof of Theorem 18, by defining the universe of our valuation to be $\mathrm{Up}(K^*_\beta)$, and the valuation to be $\varphi(P(c)) = c$ for $c \in \mathrm{Up}(K^*_\beta)$. We have to provide an instance $\vec{a}$ such that the formula $A^*_\alpha(\vec{a})$ evaluates to a value less than $\mathbf{1_K}$. To this end, choose $\vec{a}$ as the canonical points specified by the Cantor normal form of ordinal β, that is, let

$$a^i_j = (\omega^n k_n + \cdots + \omega^{i+1} k_{i+1} + \omega^i j)^\uparrow$$

for $i = 0, \ldots, n$ and $j \leq k_i$. Observe $a^i_0 = a^{i+1}_{k_{i+1}}$ for $i < n$. Using Lemma 20, we have $\mathrm{rk}_{\varphi\mathrm{CB}}(a^i_j) = i$ for $i \leq n$ and $0 < j \leq k_i$. Hence, Lemma 21 shows $\varphi\big(\mathrm{Sup}^i(a^i_j)\big) = \mathbf{1_K}$ for $i \leq n$ and $0 < j \leq k_i$.

Let
$$\vec{a} = (a^0_{l_0}, \ldots, a^0_1, \ldots, a^n_{l_n}, \ldots, a^n_1) \ .$$
As $\varphi(a^0_{l_0}) > \varphi(a^0_{l_0-1}) > \ldots > \varphi(a^n_1)$, Proposition 22 shows

$$\varphi\big(\,\mathrm{chain}^*(\vec{a})\big) \ = \ \varphi(a^0_{l_0-1}) \ < \ \varphi(a^0_{l_0}) \ \leq \ \mathbf{1_K} \ .$$

This proves the first direction.

In order to show $A^*_\alpha \in L(K^*_\alpha)$, let φ be any valuation for K^*_α, and let a^i_j be any choice of elements in $\mathcal{U}$ for $i \leq n$ and $0 < j \leq k_i$. We have to show

$$\varphi\Big(\bigwedge_{i=0}^{n} \bigwedge_{j=1}^{k_i} \mathrm{Sup}^i(a^i_j)\Big) \leq \varphi\big(\,\mathrm{chain}^*(\vec{a})\big) \ .$$

To this end, assume $\varphi(\mathrm{chain}^*(\vec{a})) < \mathbf{1_K}$. By Proposition 22, in this case we must have
$$\varphi(a^0_{k_0}) > \varphi(b) > \cdots > \varphi(a^n_1) \tag{11}$$
and $\varphi(\mathrm{chain}(\vec{a})) = \varphi(b)$ for $b = a^0_{k_0-1}$, where we let $a^0_0 = a^1_{k_1}$ in case $k_0 = 1$.

Assume for the sake of contradiction that we have $\mathrm{rk}_{\varphi\mathrm{CB}}(a^i_j) \geq i$ for all $i \leq n$ and $0 < j \leq k_i$, except maybe for $a^0_{l_0}$. Using Lemma 20, we then have

$$\varphi(a^i_j) \ \geq \ (\omega^n k_n + \cdots + \omega^{i+1} k_{i+1} + \omega^i j)^\uparrow$$

by induction of the position of $\varphi(a^i_j)$ in (11). E.g., consider the smallest element, $\varphi(a^n_1)$. By assumption, $\mathrm{rk}_{\varphi\mathrm{CB}}(a^n_1) \geq n$, but the smallest element in $\mathrm{Up}(K^*_\alpha)$ of CB-rank $\geq n$ is $(\omega^n)^\uparrow$. Hence $\varphi(a^n_1) \geq (\omega^n)^\uparrow$. The next element in (11) may be of the form $\varphi(a^n_2)$. The smallest element in $\mathrm{Up}(K^*_\alpha)$ of CB-rank $\geq n$, which is bigger than $\varphi(a^n_1)$, must be bigger than $(\omega^n)^\uparrow$, and thus is at least as big as $(\omega^n 2)^\uparrow$. Hence $\varphi(a^n_2) \geq (\omega^n 2)^\uparrow$. And so on. Thus, we obtain

$$\varphi(b) \ = \ \varphi(a^0_{l_0-1}) \ \geq \ (\omega^n k_n + \cdots + \omega^0 (l_0 - 1))^\uparrow \ = \ \alpha^\uparrow \ = \ \mathbf{1_K} \ .$$

But then we have $\varphi(b) = \varphi(a^0_{l_0}) = \mathbf{1_K}$, contradicting (11).

Hence, we must have $\mathrm{rk}_{\varphi\mathrm{CB}}(a_j^i) < i$ for some $i \le n$ and $0 < j \le k_i$ with $(i,j) \ne (0, l_0)$. Using Lemma 21, we obtain

$$\varphi\Big(\bigwedge_{i=0}^{n}\bigwedge_{j=1}^{k_i} \mathrm{Sup}^i(a_j^i)\Big) \;\le\; \varphi(\mathrm{Sup}^i(a_j^i)) \;=\; \varphi(a_j^i) \;\le\; \varphi(b) \;=\; \varphi(\mathrm{chain}^*(\vec{a}))$$

which proves the claim. □

6 Conclusion

Theorems 18 and 23 provide a clear separation of the logics of Kripke frames based on well-ordered and dually well-ordered ordinals up to ω^ω, already at level of only one monadic predicate symbol. While some of the results are known from the literature, the extension to dually well-ordered Kripke frames, and more importantly, the separation already with one monadic predicate symbol, is new. It can be considered as one more step in the examination of expressive power of standard first order language over order.

The above two theorems also set the stage for further investigations into separation within the class of logics of linear Kripke frames. As mentioned in Preining (2002), the Cantor-Bendixon analysis is not fine-grained enough for our purposes, since it doesn't make a distinction between suprema and infima. By using ordinal notations and comparison we hope to extended the current results to a broader class of Kripke frames, where at each stage of the Cantor-Bendixon derivation either all limit points are infima, or all are suprema, or all are both infima and suprema (but not mixture). In this case a ordinal description language can be used and ideas of the above proofs could lead to separation formulas between the respective logics.

Other possible directions of research are related to satisfiability. It has to be noted that, contrary to classical logic, in the current setting validity and satisfiability are not dual. As a consequence we cannot obtain any insights into satisfiability, in particular 1-satisfiability, with the above theorems.

Finally, an even more challenging open question: It seems that all these fragments under discussion in this article are decidable by a suitable embedding into Rabin's S2S. Although we have some initial ideas how to tackle this questions, the more general one, whether the same holds for the 1 predicate fragments of arbitrary linear countable Kripke, remains open.

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