Gödel logics and Cantor-Bendixon Analysis*

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Abstract. This paper presents an analysis of Gödel logics with countable truth value sets with respect to the topological and order theoretic structure of the underlying truth value set. Gödel logics have taken an important rôle in various areas of computer science, e.g. logic programming and foundations of parallel computing. As shown in a forthcoming paper all these logics are not recursively axiomatizable. We show that certain topological properties of the truth value set can distinguish between various logics. Complete separation of a class of countable valued logics will be proven and direction for further separation results given.

1 Gödel logics

Gödel logics, as introduced by Gödel in [Göd33] and later generalized by Dummett in [Dum59], are well known in computer science as they have been recognized as one of the most important formalizations of fuzzy logic [Háj98].

The unique property of Gödel logics in the group of many valued logics is the fact that the truth functions do not "compute" as in other many-valued logics, but are just projections onto one of the arguments. This implies that the truth value of a formula in a Gödel logic is solely defined by the topological and order theoretic structure of the underlying truth value set.

This interesting property naturally leads to a program of connecting logical properties with topological and order theoretic properties of the truth value set. The basic idea behind this paper is the description of Cantor-Bendixon ranks [Kec95] in logical terms. While this is of great interest by itself, the real aim of this program is the decision whether there are uncountable many first order Gödel logics or not. While this can be shown for logics defined by the entailment relation, it is not known for logics defined by the tautologies.

The separation within the class $G_{\downarrow *}$ of Gödel logics as given below is an orthogonal result to the one obtained by Baaz in [Baa96], where he separates logics with different numbers of limit points, while herein the Cantor-Bendixon rank of the truth value set is raised.

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The truth-values for Gödel logics can be taken from a set V such that $\{0,1\} \subseteq V \subseteq [0,1]$ with the designated truth value of 1. In the case of countable truth values some basic sets would be $V_\uparrow = \{1 - 1/i : i \geq 1\} \cup \{1\}$ and $V_\downarrow = \{1/n : n \geq 1\} \cup \{0\}$.

First-order Gödel logics are given by a first-order language, truth functions for the connectives and quantifiers, and a set of truth values. The language contains countably many free (a, b, c, ...) and bound (x, y, z, ...) variables, predicate symbols (P, Q, R, ...), connectives ($\neg$, $\wedge$, $\vee$, $\supset$) and quantifiers ($\exists$, $\forall$)

Interpretations are defined as usual:

Definition 1. *Let $V \subseteq [0,1]$ be some set of truth values which contains 0 and 1 and is closed under supremum and infimum. A* many-valued interpretation $\mathfrak{I} = \langle D, \mathbf{s} \rangle$ based on V *is given by the* domain D *and the* valuation function $\mathbf{s}$ *where* $\mathbf{s}$ *maps atomic formulas in* $\mathrm{Frm}(L^{\mathfrak{I}})$ *into V, n-ary function symbols to functions from D^n to D, and free variables to elements of D.*

$\mathbf{s}$ *can be extended in the obvious way to a function on all terms. The valuation for formulas is defined as follows:*

1. *$A \equiv P(t_1, \dots, t_n)$ is atomic: $\mathfrak{I}(A) = \mathbf{s}(P)(\mathbf{s}(t_1), \dots, \mathbf{s}(t_n))$.*
2. *$A \equiv \neg B$:*
$$\mathfrak{I}(\neg B) = \begin{cases} 0 & \text{if } \mathfrak{I}(B) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$
3. *$A \equiv B \wedge C$: $\mathfrak{I}(B \wedge C) = \min(\mathfrak{I}(B), \mathfrak{I}(C))$.*
4. *$A \equiv B \vee C$: $\mathfrak{I}(B \vee C) = \max(\mathfrak{I}(A), \mathfrak{I}(B))$.*
5. *$A \equiv B \supset C$:*
$$\mathfrak{I}(B \supset C) = \begin{cases} \mathfrak{I}(C) & \text{if } \mathfrak{I}(B) > \mathfrak{I}(C) \\ 1 & \text{if } \mathfrak{I}(B) \leq \mathfrak{I}(C). \end{cases}$$

The set $\{\mathfrak{I}(A(d)) : d \in D\}$ is called the distribution *of $A(x)$, we denote it by* $\mathrm{Distr}_{\mathfrak{I}}(A(x))$. *The quantifiers are, as usual, defined by infimum and supremum of their distributions.*

(6) $A \equiv \forall x B(x)$: $\mathfrak{I}(A) = \inf \mathrm{Distr}_{\mathfrak{I}}(B(x))$.
(7) $A \equiv \exists x B(x)$: $\mathfrak{I}(A) = \sup \mathrm{Distr}_{\mathfrak{I}}(B(x))$.

$\mathfrak{I}$ satisfies *a formula A, $\mathfrak{I} \models A$, if $\mathfrak{I}(A) = 1$.*

2 Topology of closed countable sets

All the following notations, lemmas, theorems are carried out within the framework of polish spaces, which are separable, completely metrizable topological spaces. For our discussion it is only necessary to know that $\mathbb{R}$ is such a polish space. In the presentation we follow [Kec95] where all the proofs are given, if not otherwise indicated.

Definition 2 (limit point, perfect space). *A* limit point *of a topological space is a point that is not isolated, i.e., for every open neighborhood U of x there is a point $y \subset U$ with $y \neq x$.*

A space is perfect *if all its points are limit points.*

Polish space can be partitioned into a perfect kernel and a countable rest. This is the well known Cantor-Bendixon Theorem:

Theorem 1 (Cantor-Bendixon). *Let X be a polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable open. The subset P is called* perfect kernel *of X.*

As a corollary we obtain that any uncountable polish space contains a perfect set, and therefor has cardinality $2^{\aleph_0}$.

2.1 Cantor-Bendixon Derivatives and Ranks

Definition 3 ((iterated) Cantor-Bendixon derivative). *For any topological space X let*

$$X' = \{x \in X : x \text{ is limit point of } X\}.$$

We call X' the Cantor-Bendixon derivative *of X.*

Using transfinite recursion we define the iterated Cantor-Bendixon derivatives *X^α, α ordinal, as follows:*

$$\begin{aligned} X^0 &= X \\ X^{\alpha+1} &= (X^\alpha)' \\ X^\lambda &= \bigcap_{\alpha<\lambda} X^\alpha, \text{ if } \lambda \text{ is limit ordinal.} \end{aligned}$$

It is obvious that X' is closed, that X is perfect iff $X = X'$ and that (X^α) for α ordinal is a decreasing transfinite sequence of closed subsets of X.

Theorem 2. *Let X be a polish space. For some countable ordinal α_0, $X^\alpha = X^{\alpha_0}$ for all $\alpha \geq \alpha_0$ and X^{α_0} is the perfect kernel of X.*

Thus is is possible to obtain the perfect kernel in a more constructive way. This leads to the definition of the Cantor-Bendixon rank:

Definition 4 (Cantor-Bendixon rank). *For any polish space X, the least ordinal α_0 as above is called the* Cantor-Bendixon rank *of X and is denoted by $|X|_{\mathrm{CB}}$. We will denote the perfect kernel of X with X^∞ or $X^{|X|_{\mathrm{CB}}}$.*

2.2 The structure of countable compact topological spaces

If the space X is countable then $X^\infty = \emptyset$, since every nonempty perfect set has at least cardinality of the continuum. Now it is possible to give a finer characterization of these countable sets by analyzing their structure under the CB-derivations. See [Win99] for a more detailed explanation.

Definition 5 (rank of an element, topological type of X). *For any $x \in X$, we can define its (Cantor-Bendixon-)rank*

$$\mathrm{rg}(x) = \sup\{\alpha : x \in X^{\alpha}\}.$$

Thus we also can define the rank of X equivalently with

$$|X|_{\mathrm{CB}} = \sup\{\mathrm{rg}(x) : x \in X\}.$$

If X is countable we call

$$\tau(X) = (\alpha, n), \quad \text{with } \alpha = \alpha(X) = |X|_{\mathrm{CB}}, \quad n = n(X) = |X^{|X|_{\mathrm{CB}}}|$$

the topological type *of X.*

3 Separating Gödel logics with truth values sets of different type

In the previous part we described a topological structuring of the countable closed subsets of the $[0,1]$ interval. Gödel logics are in a sense "topological" that the absolute truth values are not of primary interest, but the order of the truth values. This is due to the fact that the truth functions do not "compute" as in other many-valued logics, but are just projections onto one of the arguments. Therefor it is of interest to analyze how many properties of the underlying truth value set can be represented in the logical framework. Similar ideas have been used in [BV98] to prove that there are uncountable many quantified propositional Gödel logics.

To come back to [Baa96] where Baaz proves separation of certain logics, we see now that the distinct logics in the paper all have Cantor-Bendixon rank 1, or more specific, a topological type of $\tau = (1, n)$. In this paper we will restrict ourself to a second component of 1, while the separation is in the first component, i.e. the truth value set of the logics separated here have topological type $\tau = (n, 1)$, which justifies the notion of orthogonality to [Baa96] given in the introduction.

3.1 The class $G_{\downarrow *}$ of *descending* logics

As already mentioned in the first section the logic $G_{\downarrow}$ is defined as the Gödel logic over the truth value set $V_{\downarrow}$ with

$$V_{\downarrow} = \{1/n : n \geq 1\} \cup \{0\}.$$

This truth value set has topological type $(1,1)$. We will extend this truth value set by approximating every truth value in $V_{\downarrow}$ from above and iterating this procedure. Thus we obtain a truth value set with topological type $(n,1)$. Unfortunately the characterization by its topological type $(n,1)$ is still too coarse, because it does not distinguish between approximations from above, below or both sides.

Definition 6. *Let $\mathbb{N}'$ be the set of natural numbers without 0 and 1, i.e.*

$$\mathbb{N}' = \mathbb{N} \setminus \{0, 1\},$$

then we define the sets S_n, S and $S{\downarrow}_n$ as follows:

$$S_n = \mathbb{N}'^n \qquad S = \bigcup_{n \geq 0} S_n \qquad S{\downarrow}_n = \bigcup_{k=0}^{n} S_k$$

On the set S we can define the following function f mapping into the unit interval:

$$f(()) = 0 \qquad f(\overrightarrow{s_n}, s_{n+1}) = f(\overrightarrow{s_n}) + \frac{1}{\prod_{i=1}^{n} s_i^2} \cdot \frac{1}{s_{n+1}}$$

(with the definition of $\prod_{i=1}^{0} = 1$)

The definition of the function f has been chosen such that each descending sequence $f(\overrightarrow{s_n}, s_{n+1})$ is in the interval right to $f(\overrightarrow{s_n})$. A sketch of a small part of $f(S)$ is shown in fig. 1 below.

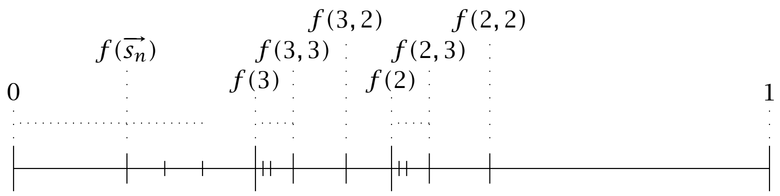

Fig. 1. The set $\{f(s) : s \in S\}$

We use the abbreviation $f(\overrightarrow{s_n})$ for $f(s_1, \ldots, s_n)$. As a convenient notational shortcut we will sometimes write $f(\overrightarrow{s_{n-1}}, s_n - 1)$. Since $s_n \in \mathbb{N}'$ this is for $s_n = 2$ defined as $f(\overrightarrow{s_{n-2}}, s_{n-1} - 1)$, which in turn may be defined by such a term. The base case for this definition is $f(s_1 - 1)$ with $s_1 = 2$ is defined as $f(s_1 - 1) = 1$. This definition just gives the next greater truth value from the set of the same or greater rank.

The image of $S{\downarrow}_n$ under f is a set as described above, i.e. it is a set of limit points of limit points etc.

Lemma 1. *The following properties hold*

1. $f(\overrightarrow{s_n}) < f(\overrightarrow{s_n}, s_{n+1}) < f(\overrightarrow{s_{n-1}}, s_n - 1)$
2. $\inf_{s_{n+1} \in \mathbb{N}'} f(\overrightarrow{s_{n+1}}) = f(\overrightarrow{s_n})$

Proof: ad 1. The one inclusion is given by

$$f(\overrightarrow{s_n}, s_{n+1}) = f(\overrightarrow{s_n}) + \left(\prod_{i_1}^{n} \frac{1}{s_i^2} \right) \frac{1}{s_{n+1}} > f(\overrightarrow{s_n})$$

and the second by

$$
\begin{aligned}
f(\overrightarrow{s_{n-1}}, s_n - 1) - f(\overrightarrow{s_n}) = \\
= f(\overrightarrow{s_{n-1}}) + \left(\prod_{i=1}^{n-1} \frac{1}{s_i^2}\right) \frac{1}{s_n - 1} - f(\overrightarrow{s_{n-1}}) - \left(\prod_{i=1}^{n-1} \frac{1}{s_i^2}\right) \frac{1}{s_n} = \\
= \left(\prod_{i=1}^{n-1} \frac{1}{s_i^2}\right) (\frac{1}{s_n - 1} - \frac{1}{s_n}) = \left(\prod_{i=1}^{n-1} \frac{1}{s_i^2}\right) \frac{1}{(s_n - 1)s_n} > \left(\prod_{i=1}^{n-1} \frac{1}{s_i^2}\right) \frac{1}{s_n^2} = \\
= \left(\prod_{i=1}^{n} \frac{1}{s_i^2}\right) > \left(\prod_{i=1}^{n} \frac{1}{s_i^2}\right) \frac{1}{s_{n+1}} = f(\overrightarrow{s_n}, s_{n+1}) - f(\overrightarrow{s_n})
\end{aligned}
$$

which yields

$$f(\overrightarrow{s_{n-1}}, s_n - 1) > f(\overrightarrow{s_n}, s_{n+1}).$$

What is left to be shown is that $f(2, 2, 2, \ldots)$ is always less then 1, which is obvious from the definition and the sum formula of a power series.

ad 2.

$$
\begin{aligned}
\inf_{s_{n+1} \in \mathbb{N}'} f(\overrightarrow{s_{n+1}}) = \inf_{s_{n+1} \in \mathbb{N}'} \left(f(\overrightarrow{s_n}) + \left(\prod_{i=1}^{n} \frac{1}{s_i^2}\right) \frac{1}{s_{n+1}} \right) = \\
f(\overrightarrow{s_n}) + \inf_{s_{n+1} \in \mathbb{N}'} (\frac{1}{s_{n+1}}) \cdot \prod_{i=1}^{n} \frac{1}{s_i^2} = f(\overrightarrow{s_n})
\end{aligned}
$$

□

After this preliminaries we can proceed to the definition of the truth value sets:

Definition 7. *Let I_n be defined as*

$$I_n = f(S{\downarrow}_n) \cup \{0, 1\}$$

The following lemma is obvious which gives us the possibility to use I_n as a truth value set for Gödel logic since it is closed.

Lemma 2. *1. I_n is a closed set*
2. the Cantor-Bendixon derivation of I_n is I_{n-1}, i.e. $I'_n = I_{n-1}$

Proof: Both are obvious from the construction and lemma 1. □

3.2 Separation within the class $G_{\downarrow *}$

Definition 8. *A Gödel logic over a truth value set V is contained in $G_{\downarrow n}$ iff there is an isomorphism from*

$$(V, \leq, \inf, \sup) \mapsto (I_n, \leq, \inf, \sup)$$

$G_{\downarrow *}$ is the union of all $G_{\downarrow n}$, i.e.

$$G_{\downarrow n} = \{G_V : \textit{there is an isomorphism } f : (V, \leq, \inf, \sup) \mapsto (I_n, \leq, \inf, \sup)\}$$

and

$$G_{\downarrow *} = \bigcup_{n>0} G_{\downarrow n}$$

Our aim is to give formulas which separate these classes, i.e. we are looking for formulas A_n such that all logics from $G_{\downarrow n}$ and upward disprove A_n and that A_n is valid in all logics below $G_{\downarrow n}$.

For this aim it is useful to know that although the principle of the excluded middle is not generally valid in Gödel logics, but something very closely related is valid for certain formulas expressing properties of the infimum:

$$\exists x(A(x) \supset \forall y A(y)) \vee (\exists x(A(x) \supset \forall y A(y)) \supset \forall y A(y))$$

This formula expresses that either the infimum is a minimum or that all infima are strict one, i.e. not minima (or all values are 1).

We will use a variant of this formula to define the separation sequence as follows:

Definition 9. *Let*

$$L_0(X, x_i) = \exists x_i(X(x_i) \supset \forall x_i' X(x_i')) \supset \forall x_i X(x_i)$$

where X can be any formula with a designated variable occurrence,

$$L_{i,n}(P) = \forall x_1 \ldots \forall x_{i-1} L_0(\forall x_{i+1} \ldots \forall x_n P(x_1, \ldots, x_{i_i}, \cdot, x_{i+1}, \ldots, x_n), x_i)$$

with P a predicate symbol and

$$A_n = \bigwedge_{i=1}^{n} L_{i,n}(P, x_i) \supset \forall x_1 \ldots \forall x_n P(x_1, \ldots, x_n)$$

Theorem 3. *The following two results separate the classes $G_{\downarrow n}$:*

1. $\forall k \geq n \, \forall G_V \in G_{\downarrow k} : A_n \notin G_V$
2. $\forall k < n \, \forall G_V \in G_{\downarrow k} : A_n \in G_V$

Proof: We will use the following model in our computations: The domain D will be $\mathbb{N}'$ and

$$\mathfrak{I}(P(s_1, \ldots, s_n)) = f(s_1, \ldots, s_n)$$

We will use the following abbreviations:

$$\begin{aligned}
\forall x_{i,j} &= \forall x_i \forall x_{i+1} \ldots \forall x_j \\
Q_i(x_1, \ldots, x_i) &= \forall x_{i+1,n} P(x_1, \ldots, x_n) \\
R_i(x_i) &= Q_i(s_1, \ldots, s_{i-1}, x_i) \\
&= \forall x_{i+1,n} P(s_1, \ldots, s_{i-1}, x_i, \ldots, x_n)
\end{aligned}$$

All infima and suprema are with respect to $\mathbb{N}'$. First note that

$$\begin{aligned}\mathfrak{I}[R_i(s_i)] &= \mathfrak{I}[\forall x_{i+1,n} P(s_1,\dots,s_i,x_{i+1},\dots,x_n)] = \\ &= \inf_{s_{i+1}} \dots \inf_{s_n} \mathfrak{I}[P(s_1,\dots,s_n)] = \\ &= \inf_{s_{i+1}} \dots \inf_{s_n} f(s_1,\dots,s_n) = \\ &= f(s_1,\dots,s_i)\end{aligned}$$

Note that for a domain which is not deep enough, i.e. for all domains I_k with $k < (n-i-1)$ we will obtain $f()$ which is 0.

ad 1. Let us now compute $\mathfrak{I}(L_{i,n}(P,x_i))$:

$$\begin{aligned}\mathfrak{I}[L_{i,n}(P,x_i)] &= \mathfrak{I}[\forall x_{1,i-1} L_0(Q_i(x_1,\dots,x_{i-1},\cdot),x_i)] \\ &= \inf_{s_1} \dots \inf_{s_{i-1}} \mathfrak{I}[L_0(Q_i(s_1,\dots,s_{i-1},\cdot),x_i)] \\ &= \inf_{s_1} \dots \inf_{s_{i-1}} \mathfrak{I}[L_0(R_i(\cdot),x_i)] \\ &= \inf_{s_1} \dots \inf_{s_{i-1}} \mathfrak{I}[\exists x_i(R_i(x_i) \supset \forall x_i' R_i(x_i')) \supset \forall x_i R_i(x_i)]\end{aligned}$$

The truth value of the left part of the implication is computed as follows:

$$\begin{aligned}\mathfrak{I}[\exists x_i(R_i(x_i) \supset \forall x_i' R_i(x_i'))] &= \sup_{s_i} \mathfrak{I}[R_i(s_i) \supset \forall x_i' R_i(x_i')] \\ &= \sup_{s_i} \mathfrak{I}(\supset)(\mathfrak{I}[R_i(s_i)], \mathfrak{I}[\forall x_i' R_i(x_i')]) \\ &= \sup_{s_i} \mathfrak{I}(\supset)(f(s_1,\dots,s_i), \inf_{s_i'} \mathfrak{I}[R_i(s_i')]) \\ &= \sup_{s_i} \mathfrak{I}(\supset)(f(s_1,\dots,s_i), \inf_{s_i'} f(s_1,\dots,s_i')) \\ &= \sup_{s_i} \mathfrak{I}(\supset)(f(s_1,\dots,s_i), f(s_1,\dots,s_{i-1})) \\ &= \sup_{s_i} f(s_1,\dots,s_{i-1}) \\ &= f(s_1,\dots,s_{i-1})\end{aligned}$$

The truth value of the right part of the above implication is

$$\begin{aligned}\mathfrak{I}[\forall x_i R_i(x_i)] &= \inf_{s_i} \mathfrak{I}[R_i(x_i)] \\ &= \inf_{s_i} f(s_1,\dots,s_i) \\ &= f(s_1,\dots,s_{i-1})\end{aligned}$$

Therefor the truth value of $L_{i,n}(P,x_i)$ is 1 for all i, and by this the truth value of the conjunction $\bigwedge_{i=1}^{n} L_{i,n}(P,x_i)$ is also 1, while the truth value of $\forall x_{1,n} P(x_1,\dots,x_n)$ is equal to 0 and so we obtain that

$$\mathfrak{I}[A_n] = 0$$

which proves the first part of the theorem.

ad 2. Proceeding to the second part, we will compute the truth value of $L_{1,n}(P)$:

$$\begin{aligned}\mathfrak{I}[L_{1,n}(P)] &= \mathfrak{I}[L_0(R_1(\cdot),x_1)]\\ &= \mathfrak{I}[\exists x_1(R_1(x_1)\supset \forall x_1' R_1(x_1'))\supset \forall x_1 R_1(x_1)]\end{aligned}$$

The truth value of the left side of the implication is computed as follows:

$$\begin{aligned}\mathfrak{I}[\exists x_1(R_1(x_1)\supset \forall x_1' R_1(x_1'))] &= \sup_{s_1}\mathfrak{I}[R_1(s_1)\supset \forall x_1' R_1(x_1')]\\ &= \sup_{s_1}\mathfrak{I}(\supset)(\mathfrak{I}[R_1(s_1)],\mathfrak{I}[\forall x_1' R_1(x_1')])\\ &= \sup_{s_1}\mathfrak{I}(\supset)(0,\inf_{s_1'}\mathfrak{I}[R_1(s_1')])\\ &= \sup_{s_1}\mathfrak{I}(\supset)(0,0)\\ &= 1\end{aligned}$$

The crucial part in the above calculation is that $\mathfrak{I}[R_1(s_1)]$ is equal to 0 under *all* interpretations into a truth value set in $G_{\downarrow k}$

$$\begin{aligned}\mathfrak{I}[R_1(s_1)] &= \mathfrak{I}[\forall x_2\ldots\forall x_n P(s_1,x_2,\ldots,x_n)]\\ &= \inf_{s_2}\ldots\inf_{s_n}\mathfrak{I}[P(s_1,s_2,\ldots,s_n)]\\ &= 0\end{aligned}$$

because all the truth value sets from $G_{\downarrow k}$ have a Cantor-Bendixon rank of $k < n$.

Therefor the truth value of $L_{1,n}(P)$ is the truth value of $\forall x_1 R_1(x_1)$ and the truth value of the left side of the implication of A_n is bound by the same value from above.

$$\begin{aligned}\mathfrak{I}[L_{1,n}(P)] &= \mathfrak{I}[\forall x_1 R_1(x_1)]\\ \mathfrak{I}[\bigwedge_{i=1}^{n} L_{i,n}(P)] &\le \mathfrak{I}[L_{1,n}(P)] = \mathfrak{I}[\forall x_1 R_1(x_1)]\end{aligned}$$

Finally the truth value of the right side of A_n is also $\mathfrak{I}[\forall x_1 R_1(x_1)]$ and we obtain

$$\mathfrak{I}[A_n] = 1$$

which proves that for I_n the second part of the theorem is true.

The extension to all other models in this subclass is trivial to prove using the isomorphism and the note above on the truth value of $\mathfrak{I}[R_1(s_1)]$, thus we are finished. □

4 Conclusions

These separation results are a first step within the program of connecting topological and order theoretic properties of underlying truth value sets with logical properties. Further directions in the research should lead to a finer granulation of discernible Gödel logics. The next step in this direction will be the extension of these results too general infima/suprema combinations, which needs a new language of expressing order properties of countable set, since the classification after topological types $\tau = (\lambda, n)$ is to general. Finally this research should lead to an answer whether there are uncountable many first order Gödel logics or not.

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