A Guide to Quantified Propositional Gödel Logic

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Abstract. Gödel logic is a non-classical logic which naturally turns up in a number of different areas within logic and computer science. By choosing subsets of the unit interval $[0,1]$ as the underlying set of truth-values many different Gödel logics have been defined. Unlike in classical logic, adding propositional quantifiers to Gödel logics in many cases increases the expressive power of the logic, and motivates thorough investigation. In a series of recent papers [8, 7, 6, 5, 4], we have started a research program to investigate quantified Gödel logics in a systematic manner. In this paper, we survey the results obtained so far. In the conclusion, we outline the future directions of this research program.

1 Introduction

In 1932, Gödel [12] introduced a family of finite-valued propositional logics to show that intuitionistic logic does not have a characteristic finite matrix. Dummett [10] later generalized these to an infinite set of truth-values, and showed that the set of its tautologies G_∞ is axiomatized by intuitionistic logic extended by the linearity axiom $(A \supset B) \vee (B \supset A)$. [1] Gödel logic naturally turns up in a number of different areas of logic and computer science. In particular, it was recognized as one of the most important formalizations of fuzzy logic [13].

Propositional Gödel logic can be extended by quantifiers in different ways, in particular by *first-order quantifiers* (universal and existential quantification over object variables) and *propositional* or *"fuzzy" quantifiers* (universal and existential quantification over propositions).

While there is only *one* infinite-valued propositional Gödel logic, *uncountably many* different quantified propositional Gödel logics are induced by different infinite subsets of truth-values over $[0,1]$. Of particular importance are the truth-value sets $V_\infty = [0,1], V_\downarrow = \{0\} \cup \{1/n : n \geq 1\}, V_\uparrow = \{1\} \cup \{1 - 1/n : n \geq 1\}$ and $V_k = \{1\} \cup \{1 - 1/n : n = 1, \ldots, k-1\}$.

In contrast to classical propositional logic, propositional quantification may increase the expressive power of Gödel logics. More precisely, statements about the topological structure of the set of truth-values (taken as infinite subsets of

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[1] Note that Dummett and other authors write **LC** instead of G_∞.

the real interval $[0,1]$) can be expressed using propositional quantifiers [8]. For example, over the truth-value sets $[0, \frac{1}{2}] \cup \{1\}$ and $[0,1]$, we obtain two different quantified propositional Gödel logics, but only one in the framework of first order Gödel logic.

The purpose of this paper is to summarize and relate recent results about quantified propositional Gödel logics [8, 7, 6, 5, 4]. We will consider the logics based on the above-mentioned infinite sets V_∞, $V_\downarrow$ and $V_\uparrow$. Particular emphasis will be put on decidability, axiomatizability, and quantifier elimination. In the conclusion, we outline the major directions for future research, as we envisage them.

2 Gödel Logics

We work in the language of propositional logic containing a countably infinite set $Var = \{p, q, \ldots\}$ of (propositional) variables, the constants $\bot, \top$, as well as the connectives $\wedge, \vee$, and $\supset$. Propositional variables and constants are considered atomic formulas. Uppercase letters will serve as meta-variables for formulas. If $A(p)$ is a formula containing the variable p free, then $A(X)$ denotes the formula with all occurrences of the variable p replaced by the formula X. $Var(A)$ is the set of variables occurring in the formula A. We use the abbreviations $\neg A$ for $A \supset \bot$ and $A \leftrightarrow B$ for $(A \supset B) \wedge (B \supset A)$.

Semantics. The most important form of Gödel logic is defined over the real unit interval $V_\infty = [0,1]$; in a more general framework, the truth-values are taken from a set V such that $\{0,1\} \subseteq V \subseteq [0,1]$. In the case of k-valued Gödel logic $\mathbf{G}_k$, we take $V_k = \{1 - 1/i : i = 1, \ldots, k-1\} \cup \{1\}$. Moreover, we consider the sets $V_\uparrow = \{1 - 1/i : i \geq 1\} \cup \{1\}$ and $V_\downarrow = \{1/n : n \geq 1\} \cup \{0\}$.

A *valuation* $v\colon\ Var \to V$ is an assignment of values in V to the propositional variables. It can be extended to formulas using the following truth functions introduced by Gödel [12]:

$$\begin{aligned} v(\bot) &= 0 \\ v(\top) &= 1 \\ v(A \wedge B) &= \min(v(A), v(B)) \end{aligned} \qquad \begin{aligned} v(A \vee B) &= \max(v(A), v(B)) \\ v(A \supset B) &= \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases} \end{aligned}$$

A formula A is a *tautology* over a truth-value set $V \subseteq [0,1]$ if for all valuations $v\colon\ Var \to V$, $v(A) = 1$. The *propositional logics* $\mathbf{G}_\infty$, $\mathbf{G}_\uparrow$, $\mathbf{G}_\downarrow$ and $\mathbf{G}_k$ are the sets of tautologies over the corresponding truth-value sets, e.g., $\mathbf{G}_\infty = \{A : A \text{ a tautology over } V_\infty\}$. We also write $\mathbf{G} \models A$ for $A \in \mathbf{G}$ ($\mathbf{G} \in \{\mathbf{G}_\infty, \mathbf{G}_\uparrow, \mathbf{G}_\downarrow, \mathbf{G}_k\}$).

Equivalent semantics, which stress the close relationship with intuitionistic logic, are provided by linearly ordered Kripke structures [10] and linearly ordered Heyting algebras [14].

Remark 1. It is easy to see that for ordinary propositional Gödel logic, the tautologies coincide for all infinite V.

The abbreviation $A \prec B$ for $(A \supset B) \wedge ((B \supset A) \supset A)$ will be used extensively below. It expresses strict linear order in the sense that

$$v(A \prec B) = \begin{cases} 1 & \text{if } v(A) < v(B) \text{ or } v(B) = 1 \\ \min(v(A), v(B)) & \text{otherwise} \end{cases}$$

Propositional Quantification. In *classical* propositional logic we define $(\exists p)A(p)$ by $A(\bot) \vee A(\top)$ and $(\forall p)A(p)$ by $A(\bot) \wedge A(\top)$. In other words, propositional quantification is semantically defined by the supremum and infimum, respectively, of truth functions (with respect to the usual ordering "$0 < 1$" over the classical truth-values $\{0, 1\}$). This can be extended to Gödel logic by using *fuzzy quantifiers*. Syntactically, this means that we allow formulas $(\forall p)A$ and $(\exists p)A$ in the language. Free and bound occurrences of variables are defined in the usual way. Given a valuation v and $w \in V$, define $v[w/p]$ by $v[w/p](p) = w$ and $v[w/p](q) = v(q)$ for $q \not\equiv p$. The semantics of fuzzy quantifiers is then defined as follows:

$$v((\exists p)A) = \sup\{v[w/p](A) : w \in V\} \qquad v((\forall p)A) = \inf\{v[w/p](A) : w \in V\}$$

When we consider quantifiers, V has to be closed under infima and suprema, since otherwise truth-values for quantified formulas are not defined.

Remark 2. In [8] it was shown that there is an uncountable number of different quantified Gödel logics. This is done by encoding all ω-strings over the alphabet $\{0, 1\}$.

Using the above definitions, it is straightforward to extend the notion of tautologyhood to the new language. We write $\mathbf{G}_\infty^{\mathrm{qp}}$ ($\mathbf{G}_\uparrow^{\mathrm{qp}}$, $\mathbf{G}_\downarrow^{\mathrm{qp}}$, $\mathbf{G}_k^{\mathrm{qp}}$) for the set of tautologies in the extended language over V_∞ ($V_\uparrow, V_\downarrow$, V_k).

To investigate $\mathbf{G}_\uparrow^{\mathrm{qp}}$, we also add the additional unary connective $\circ$ to the language. The truth function for $\circ$ is given by $v(\circ A) = v((\forall p)((p \supset A) \vee p))$. In $\mathbf{G}_\uparrow^{\mathrm{qp}}$, this makes

$$v(\circ A) = \begin{cases} 1 & \text{if } v(A) = 1 \\ 1 - \frac{1}{n+1} & \text{if } v(A) = 1 - \frac{1}{n} \end{cases}$$

We abbreviate $\circ \ldots \circ A$ (n occurrences of $\circ$) by $\circ^n A$. We will show below that every quantified propositional formula is equivalent in $\mathbf{G}_\uparrow^{\mathrm{qp}}$ to a quantifier-free formula, which in general can contain $\circ$. $\circ A$ itself (or the equivalent formula $(\forall p)((p \supset A) \vee p)$), however, is not in general equivalent to a quantifier-free formula not containing $\circ$. Inspection of the truth tables shows that a quantifier-free formula containing only the variable q takes one of 0, $v(q)$, or 1 as its value under a given valuation v, and thus no such formula can define $\circ q$.

3 Decidability

In this section we show that quantified propositional Gödel logics over the sets $V_\uparrow$, $V_\downarrow$ and V_∞ are decidable.

For $V_\uparrow$ and $V_\downarrow$ this is achieved by a reduction to S1S, the monadic theory of one successor which was shown to be decidable by Büchi [9]. More precisely, S1S is the set of second-order formulas in the language with second-order quantification restricted to monadic set variables X, Y, ... with one unary function $'$ (successor) which are true in the model $\langle \omega, ' \rangle$.

Theorem 3. *[8] Validity in* $\mathbf{G}_\downarrow^{\mathrm{qp}}$ *is decidable.*

Proof. We identify a truth-value $1/n$ with the infinite binary sequence $0^{n-1}1^\omega$, and 0 with 0^ω. Since we consider validity, we may without loss of generality assume that all variables in a Gödel logic formula are quantified.

The following translation associates with each Gödel logic formula ϕ an S1S formula ϕ^x with one open first order variable x such that ϕ^x expresses the infinite binary sequence which encodes the truth-value of ϕ:

$$\begin{aligned}
p^x &= X_p(x) \\
\bot^x &= X_\bot(x) \\
\top^x &= (\forall z)(z = z) \\
(B \wedge C)^x &= B^x \wedge C^x \\
(B \to C)^x &= (\exists x)(B^x \wedge \neg C^x) \to C^x \\
(\forall a)B^x &= (\forall A)[(\forall z)(A(z) \to A(x') \to B^x] \\
(\exists a)B^x &= (\exists A)[(\forall z)(A(z) \to A(x') \wedge B^x]
\end{aligned}$$

It follows immediately from the definitions that ϕ is a tautology in $\mathbf{G}_\downarrow^{\mathrm{qp}}$ iff $\forall x \phi^x$ is true in S1S.

Theorem 4. *[4] Validity in* $\mathbf{G}_\uparrow^{\mathrm{qp}}$ *is decidable.*

Sketch of Proof: Suppose ϕ is a quantified propositional formula, and B is a formula in the language of S1S with only x free. Let $TV(B(x))$ abbreviate $(\forall z)(B(z') \supset B(z))$. We define ϕ^x by:

$$\begin{aligned}
(B \supset C)^x &= (\forall y)(B^y \supset C^y) \vee (\exists y)(B^y \wedge \neg C^y) \wedge C^x \\
(\forall p)B^x &= (\forall X_p)(TV(X_p(x)) \supset B^x) \\
(\exists p)B^x &= (\exists X_p)(TV(X_p(x)) \wedge B^x)
\end{aligned}$$

The remaining cases are defined as in the proof of the previous theorem.

Consider the following reduction:

$$\Phi(\phi) = (\forall X_\bot)((\forall x)\neg X_\bot(x) \supset (\forall x)\phi^x)$$

The idea behind this is to correlate truth-values in $V_\uparrow$ with subsets of ω which are closed under predecessor, i.e., predicates in

$$TV = \{P \subseteq \omega : \text{if } n \in P \text{ then } m \in P \text{ for all } m \leq n\}.$$

Under this correlation, 1 corresponds to ω, and $1-1/n$ corresponds to $\{1, \ldots, n\}$. One can prove that a formula ϕ is a tautology in $\mathbf{G}_\uparrow^{\mathrm{qp}}$ iff $S1S \models \Phi(\phi)$.

Let us now turn to G_∞. Since V_∞ is dense, a reduction to S1S appears to be difficult to obtain. However, as we shall see in Section 5, there is an effective quantifier elimination procedure for G_∞, and therefore, G_∞ turns out to be decidable. Note that a similar argument can also be used to obtain an alternative proof for decidability of $\mathbf{G}_\uparrow^{\mathrm{qp}}$.

4 Axiomatizations

All the calculi we consider are based on the following set of axioms:

I1	$A \to (B \to A)$	I6	$B \to (A \vee B)$
I2	$(A \wedge B) \to A$	I7	$(A \wedge \neg A) \to B$
I3	$(A \wedge B) \to B$	I8	$(A \to \neg A) \to \neg A$
I4	$A \to (B \to (A \wedge B))$	I9	$\bot \to A$
I5	$A \to (A \vee B)$	I10	$A \to \top$

I11 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
I12 $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

These axioms, together with the rule of modus ponens, define the system IPC that is sound and complete for intuitionistic propositional logic. The system GL is obtained by adding to IPC the linearity axiom

$$\text{Lin} \quad (A \supset B) \vee (B \supset A).$$

which expresses the linearity of the ordering of truth-values or, equivalently, of the states of Kripke models.

Theorem 5. *[10] The system* GL *is sound and complete for* $\mathbf{G}_\infty$.

Turning to quantified propositional logic, a natural system $\mathsf{IPC}^{\mathrm{qp}}$ to start with is obtained by adding to IPC the following two axioms (see [11]):

$$\supset\exists \quad A(C) \supset (\exists p)A(p) \qquad\qquad \supset\forall \quad (\forall p)A(p) \supset A(C)$$

and the rules:

$$\frac{A(p) \supset B^{(p)}}{(\exists p)A(p) \supset B^{(p)}}\text{R}\exists \qquad\qquad \frac{B^{(p)} \supset A(p)}{B^{(p)} \supset (\forall p)A(p)}\text{R}\forall$$

where for any formula B, the notation $B^{(p)}$ indicates that p does not occur free in B, i.e., p is a (propositional) *eigenvariable*.
The system $\mathsf{QG}_\infty^{\mathrm{qp}}$ is obtained by taking all above-mentioned axioms and rules plus the following two axioms:

Or-Shift $\forall x(A^{(x)} \vee B) \;\to\; A^{(x)} \vee \forall x B$
Density $\forall x(A^{(x)} \to x \vee x \to B^{(x)}) \;\to\; (A^{(x)} \to B^{(x)})$

Theorem 6. *[8] The system* $\mathsf{QG}_\infty^{\mathrm{qp}}$ *is sound and complete for* $\mathbf{G}_\infty^{\mathrm{qp}}$.

It was shown in [8] that instances of the quantifier axioms, where the formula X is quantifier free, suffice for the completeness of the calculus.

Let $\mathsf{QG}^{\mathrm{qp}}_{\uparrow}$ be the system obtained by adding to $\mathsf{IPC}^{\mathrm{qp}}$ the axioms (Lin),

$$\forall\vee \qquad (\forall p)[A \vee B(p))] \supset [A \vee (\forall p)B(p)]$$

where $p \notin A$, and the following:

G1	$\bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B)$	G4	$(A \supset \bigcirc B) \supset ((A \supset C) \vee (C \supset B))$
G2	$A \prec \bigcirc A$	G5	$(A \leftrightarrow \bot) \vee (\exists p)(A \leftrightarrow \bigcirc p)$
G3	$(\bigcirc A \supset \bigcirc B) \supset ((A \supset B) \vee \bigcirc B)$	G6	$(A \prec B) \supset (\bigcirc A \supset B)$

Theorem 7. *[4] The system* $\mathsf{QG}^{\mathrm{qp}}_{\uparrow}$ *is sound for and complete for* $\mathbf{G}^{\mathrm{qp}}_{\uparrow}$*.*

Remark 8. The density axiom is not valid in $\mathbf{G}^{\mathrm{qp}}_{\uparrow}$. On the other hand, it is easy to see that $v(\bigcirc A) = v(A)$ in V_∞, and hence axiom (G2) is not valid in $\mathbf{G}^{\mathrm{qp}}_{\infty}$. Thus neither of $\mathbf{G}^{\mathrm{qp}}_{\infty}$ and $\mathbf{G}^{\mathrm{qp}}_{\uparrow}$ is included in the other. This is in contrast to the situation in propositional entailment and first-order logic, where V_∞ defines the smallest Gödel logic and is included in all others.

An Analytic Calculus for $\mathbf{G}^{\mathrm{qp}}_{\infty}$

In [5] an analytic calculus for $\mathbf{G}^{\mathrm{qp}}_{\infty}$ has been defined. This calculus uses hypersequents, a simple and natural generalization of Gentzen's (ordinary) sequents (see, e.g., [2] for an overview).

Definition 9. *A* hypersequent *is a structure of the form* $\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ *where each* $\Gamma_i \Rightarrow \Delta_i$ *is a sequent called a* component *of the hypersequent.*

We consider sequents and hypersequents as *multisets* of formulas and *multisets* of sequents, respectively.

In [1] Avron defined the hypersequent calculus GLC for $\mathbf{G}_\infty$. GLC is simply obtained by adding to the hypersequent calculus for intuitionistic logic the so called communication rule (*com*) allowing to prove the linearity axiom. The calculus HQGL for $\mathbf{G}^{\mathrm{qp}}_{\infty}$ is obtained by augmenting GLC with suitable rules for introducing propositional quantifiers as well as the (*tt*) rule allowing to prove the density axiom.

More precisely, axioms and rules of HQGL are as follows:

Axioms and Cut rule:

$$\bot \Rightarrow, \quad \Rightarrow \top, \quad A \Rightarrow A \qquad\qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A \quad \mathcal{H} \mid A, \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow B}\ (cut)$$

Internal structural rules:

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow C}{\mathcal{H} \mid A, \Gamma \Rightarrow C}\ (iw \Rightarrow) \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow}{\mathcal{H} \mid \Gamma \Rightarrow A}\ (\Rightarrow iw) \qquad \frac{\mathcal{H} \mid A, A, \Gamma \Rightarrow C}{\mathcal{H} \mid A, \Gamma \Rightarrow C}\ (ic \Rightarrow)$$

External structural rules:

$$\frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow C}\ (ew) \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow C \mid \Gamma \Rightarrow C}{\mathcal{H} \mid \Gamma \Rightarrow C}\ (ec) \qquad \frac{\mathcal{H} \mid \Gamma_1, \Gamma_1' \Rightarrow A_1 \quad \mathcal{H} \mid \Gamma_2, \Gamma_2' \Rightarrow A_2}{\mathcal{H} \mid \Gamma_1', \Gamma_2' \Rightarrow A_1 \mid \Gamma_1, \Gamma_2 \Rightarrow A_2}\ (com)$$

Logical rules:

$$\frac{\mathcal{H} \mid A_1, \Gamma \Rightarrow C \quad \mathcal{H} \mid A_2, \Gamma \Rightarrow C}{\mathcal{H} \mid A_1 \vee A_2, \Gamma \Rightarrow C}\ (\vee \Rightarrow) \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A_i}{\mathcal{H} \mid \Gamma \Rightarrow A_1 \vee A_2}\ (\Rightarrow \vee_i)\ i \in \{1,2\}$$

$$\frac{\mathcal{H} \mid A_i, \Gamma \Rightarrow C}{\mathcal{H} \mid A_1 \wedge A_2, \Gamma \Rightarrow C}\ (\wedge \Rightarrow_i)\ i \in \{1,2\} \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A \quad \mathcal{H} \mid \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \wedge B}\ (\Rightarrow \wedge)$$

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow A \quad \mathcal{H} \mid B, \Gamma \Rightarrow C}{\mathcal{H} \mid A \to B, \Gamma \Rightarrow C}\ (\to\Rightarrow) \qquad \frac{\mathcal{H} \mid A, \Gamma \Rightarrow B}{\mathcal{H} \mid \Gamma \Rightarrow A \to B}\ (\Rightarrow\to)$$

Rules for propositional quantifiers:

$$\frac{\mathcal{H} \mid A(X), \Gamma \Rightarrow C}{\mathcal{H} \mid (\forall p)A(p), \Gamma \Rightarrow C}\ (\forall \Rightarrow) \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A(a)}{\mathcal{H} \mid \Gamma \Rightarrow (\forall p)A(p)}\ (\Rightarrow \forall)$$

$$\frac{\mathcal{H} \mid A(a), \Gamma \Rightarrow C}{\mathcal{H} \mid (\exists p)A(p), \Gamma \Rightarrow C}\ (\exists \Rightarrow) \qquad \frac{\mathcal{H} \mid \Gamma \Rightarrow A(X)}{\mathcal{H} \mid \Gamma \Rightarrow (\exists p)A(p)}\ (\Rightarrow \exists)$$

The (tt) rule:

$$\frac{\mathcal{H} \mid \Pi \Rightarrow a \mid a, \Gamma \Rightarrow C}{\mathcal{H} \mid \Pi, \Gamma \Rightarrow C}\ (tt)$$

In the above rules formula X is required to be *quantifier free* and the propositional variable a is subject to the usual eigenvariable condition; i.e., it must not occur freely in the lower hypersequent.

Theorem 10. *[5]* HQGL *is sound and complete for* $\mathbf{G}_\infty^{\mathrm{qp}}$.

Theorem 11. *[5] If a hypersequent H is derivable in* HQGL *then it is derivable in* HQGL *without using the cut rule.*

Note that a variant of the rule (tt) was used in [18] to axiomatize first-order Gödel logic over V_∞ (called "intuitionistic fuzzy logic" by Takeuti and Titani). Takano [17] later showed that this rule is in fact redundant in the calculus for first-order Gödel logic over V_∞ by referring to arguments already present in A. Horn's [14]. However, an instance of the rule turned out to be *essential* to obtain a complete (Hilbert-style) axiomatization of QGL [8]. Analogously, in HQGL the (tt) rule is essential to derive instances of the density axiom. On the other hand, this rule renders proof search in HQGL rather problematic. Therefore it is useful to know for which fragments of $\mathbf{G}_\infty^{\mathrm{qp}}$ the (tt) rule (or a variant thereof) is actually needed to find a proof. One has

Theorem 12. *[5] The (tt) rule is redundant in the calculus* HQGLm *obtained from* HQGL *by dropping the rules $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$.*

5 Normal Forms and Elimination of Quantifiers

In this section we recall normal form results for $\mathbf{G}_\infty$ and $\mathbf{G}_\uparrow^{\mathrm{qp}}$. These results are crucial in the proofs of quantifiers elimination for $\mathsf{QG}_\infty^{\mathrm{qp}}$ and $\mathsf{QG}_\uparrow^{\mathrm{qp}}$.

For $\mathbf{G}_\infty$, a normal form similar to the disjunctive normal form of classical logic was introduced in [7] (see also [3]). This so-called *chain normal form* is based on the fact that, in a sense, the truth-value of a formula only depends on the ordering of the variables occurring in the formula induced by the valuation under consideration. The chain normal form can then be constructed by enumerating all such orderings (using $\prec$ and $\leftrightarrow$ to encode the ordering) in a manner similar to the way one constructs a disjunctive normal form by enumerating all possible truth-value assignments.

More precisely, let $V = \{v_1, \dots, v_n\}$ be a set of propositional variables. Then a $\prec$-*chain* over V is a formula of the form

$$(0 \bowtie_0 v_{\pi(1)}) \wedge (v_{\pi(1)} \bowtie_1 v_{\pi(2)}) \wedge \cdots \wedge (v_{\pi(n-1)} \bowtie_{n-1} v_{\pi(n)}) \wedge (v_{\pi(n)} \bowtie_n 1)$$

such that π is a permutation of $\{1, \dots, n\}$ and $\bowtie_i$ is either $\equiv$ or $\prec$.

Every $\prec$-chain describes an order type of the variables V. For a formula ϕ, let ϕ^ζ denote the value of ϕ under an evaluation which has the same order type as described by ζ.

Theorem 13. *[7] Let ϕ be a formula in propositional Gödel logic, and $V = var(\phi)$. Then ϕ is equivalent to a formula $\bigvee_{\zeta \in C(V)} \zeta \wedge v_\zeta$ such that $v_\zeta \in V \cup \{0, 1\}$.*

This result allows to show:

Theorem 14. *[8] For every formula ϕ there exists a quantifier-free formula ψ such that $\mathsf{QG}_\infty^{\mathrm{qp}} \vdash \phi \equiv \psi$.*

As a corollary we have

- the system $\mathsf{QG}_\infty^{\mathrm{qp}}$ is complete for $\mathsf{QG}_\uparrow^{\mathrm{qp}}$ (Theorem 6)
- validity in $\mathbf{G}_\infty^{\mathrm{qp}}$ is decidable.
- $\mathbf{G}_\infty$ has interpolation [16], and in fact uniform interpolation [7].
- $\mathbf{G}_\infty^{\mathrm{qp}}$ has uniform interpolation with quantifier-free interpolants.

The notion of chain normal form can be extended as follows in order to deal with the $\circ$ connective of $\mathbf{G}_\uparrow^{\mathrm{qp}}$.

Definition 15. *A formula A of $\mathsf{QG}_\uparrow^{\mathrm{qp}}$ is in* $\circ$-normal form *if it is quantifier-free and for all subformulas $\circ B$ of A, $B \in \{\bot, \top\} \cup Var$ or $B \equiv \circ B'$.*

Proposition 16. *[4] Let A be a quantifier-free formula of $\mathsf{QG}_\uparrow^{\mathrm{qp}}$. Then there exists a formula A' of $\mathsf{QG}_\uparrow^{\mathrm{qp}}$ in $\circ$-normal form such that $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A \leftrightarrow A'$.*

By the previous proposition we can always push the $\circ$ in front of atomic subformulas, so we only need to consider orderings of subformulas of the form $\circ^j B$ with B atomic. Let Γ be a finite subset of $\{\circ^j p, \circ^j \bot : p \in \mathit{Var}, j \in \omega\} \cup \{\top\}$ and $\top, \bot \in \Gamma$.

Definition 17. *A* $\circ$-chain *over* Γ *is an expression of the form*

$$(S_1 \star_1 S_2) \wedge \cdots \wedge (S_{n-1} \star_{n-1} S_n)$$

such that $\Gamma = \{S_1, \ldots, S_n\}$*,* $S_1 \equiv \bot$*,* $S_n \equiv \top$*, and* $\star_i \in \{\leftrightarrow, \prec\}$*, for all* $i = 1, \ldots, n$*.*

Definition 18. *Let* A *be a quantifier free formula in* $\circ$*-normal form,* Γ_A *be the set of all subformulas of* A *of the form* $\circ^j p, \circ^k \bot, \top$*,* $\Gamma \supseteq \Gamma_A$*, and* $C(\Gamma)$ *the set of all possible* $\circ$*-chains over* Γ*. Then*

$$\bigvee_{C \in C(\Gamma)} C \wedge A^C$$

is the $\circ$-chain normal form *for* A *over* Γ*.*

Theorem 19. *[4] Let* A *and* Γ *be as above, and* A' *be the* $\circ$*-chain normal form for* A *over* Γ*. Then* $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash A \leftrightarrow A'$*.*

Theorem 20. *[4] For every closed formula* A *of* $\mathsf{QG}^{\mathrm{qp}}_{\uparrow}$ *there exists a variable-free formula* A^{qf} *such that* $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash A \leftrightarrow A^{\mathrm{qf}}$*.*

As a corollary we have

- the system $\mathsf{QG}^{\mathrm{qp}}_{\uparrow}$ is complete for $\mathbf{G}^{\mathrm{qp}}_{\uparrow}$ (Theorem 7)
- an alternative proof that validity in $\mathbf{G}^{\mathrm{qp}}_{\uparrow}$ is decidable
- $\mathbf{G}^{\mathrm{qp}}_{\uparrow}$ is the intersection of all finite-valued quantified propositional Gödel logics $\mathbf{G}^{\mathrm{qp}}_{k}$.

6 Conclusions

We believe that we now have some initial understanding of the structure of the class of all quantified propositional Gödel logics, and of their relation to the topological and order-type properties of the underlying truth-value sets. A lot of interesting questions remain to be settled. The following three questions are currently under investigation:

- The logic $\mathbf{G}^{\mathrm{qp}}_{\downarrow}$ [6] is the logic of linearly ordered well-founded Kripke structures, and is in the same relation to the temporal logic of "always" as intuitionistic logic is to S4. By Theorem 3, $\mathbf{G}^{\mathrm{qp}}_{\downarrow}$ is decidable. The next step will be to provide an axiomatization for $\mathbf{G}^{\mathrm{qp}}_{\downarrow}$ and to prove by quantifier elimination that it is complete.

– It can be shown that the intersection of *all* quantified propositional Gödel logics is not a quantified propositional Gödel logic (in contrast to ordinary propositional Gödel logics and first-order Gödel logics.) We intend to show that the intersection is in fact axiomatized by the axiom system for $\mathbf{G}_\infty^{\mathrm{qp}}$ without the Density axiom.
– Finally, we are looking for a set-theoretic characterization of those truth-value sets V, over which Quantified Propositional Gödel Logics are well-behaved, i.e., those which admit recursive axiomatizations and decidability. A natural candidate for this characterization is V having finite Cantor-Bendixson rank [15].

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