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Fibonacci and Lucas Identities from Toeplitz–Hessenberg Matrices

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Abstract

In this paper, we consider determinants for some families of Toeplitz–Hessenberg matrices having various translates of the Fibonacci and Lucas numbers for the nonzero entries. These determinant formulas may also be rewritten as identities involving sums of products of Fibonacci and Lucas numbers and multinomial coefficients. Combinatorial proofs are provided of several of the determinants which make use of sign-changing involutions and the definition of the determinant as a signed sum over the symmetric group. This leads to a common generalization of the Fibonacci and Lucas determinant formulas in terms of the so-called Gibonacci numbers.

Keywords: Fibonacci sequence; Lucas sequence; Toeplitz–Hessenberg matrix; Trudi's formula; Multinomial coefficient

MSC 2010 No.: 11B39, 15B05

1. Introduction

Let F_n denote the n-th Fibonacci and L_n the n-th Lucas number, both satisfying the recurrence

$$b_n = b_{n-1} + b_{n-2}, \qquad n \geq 2,$$

but with the respective initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ (e.g., see Koshy (2017)). In this paper, we find some new relations involving the Fibonacci and Lucas sequences which arise as determinants of certain families of Toeplitz–Hessenberg matrices.

Formulas relating determinants to Fibonacci and/or Lucas numbers have been an object of recent interest. In some cases, these sequences arise as determinants for certain families of matrices having integer entries, while in other cases these sequences are the actual entries of the matrix whose determinant is being evaluated. For example, Öcal et al. (2005) studied determinantal representations of k-generalized Fibonacci and Lucas numbers and obtained as a result Binet's formulas for these sequences. Janjić (2010) considered a particular type of upper Hessenberg matrix and showed its relationship with a generalization of the Fibonacci numbers (see also related work by Bicknell-Johnson and Spears (1996)). Cereceda (2014) later provided some determinantal representations of the general terms of second and third-order linear recurrent sequences with arbitrary initial conditions and similar work has been done by Kaygısız and Şahin (2012) for Fibonacci-type numbers in conjunction with various Hessenberg matrices. Civciv (2008) studied the determinant of a five-diagonal matrix with Fibonacci entries, while in Tangboonduangjit and Thanatipanonda (2016), determinants of matrices whose entries are powers of the Fibonacci numbers were considered. For further examples of combinatorial determinants, we refer the reader to İpek (2011), İpek and Arı (2014), Jaiswal (1969), and Kılıç and Arıkan (2017).

In this paper, we provide determinant formulas for Toeplitz–Hessenberg matrices whose (i, j)-th lower triangular entries are of the form F_{i-j+c} or $F_{2(i-j)+c}$ for various c. Comparable formulas are then ascertained for the Lucas numbers and multinomial analogues are also discussed. Finally, combinatorial proofs which make use of parity-changing involutions and the definition of the determinant as a signed sum over the symmetric group are provided for several of the identities. Adapting the combinatorial arguments yields general determinant formulas involving the Gibonacci numbers.

2. Toeplitz–Hessenberg matrices and determinants

A lower *Toeplitz–Hessenberg matrix* is a square matrix of the form

$$M_n(a_0, a_1, \ldots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}, \tag{1}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$. This class of matrices has been encountered in various applications (e.g., Merca (2013), Vein and Dale (1999) and references contained therein).

Expanding its determinant, which we will denote by $\det(M_n)$, repeatedly along the first row, we obtain the recurrence

$$\det(M_n) = \sum_{k=1}^{n} (-a_0)^{k-1} a_k \det(M_{n-k}), \tag{2}$$

where $\det(M_0) = 1$, by definition.

To simplify notation, we write $\det(a_1, a_2, \dots, a_n)$ in place of $\det\big(M_n(1, a_1, a_2, \dots, a_n)\big)$.

In the next two sections, we evaluate $\det(a_1, a_2, \dots, a_n)$ in which the entries a_i are various translates of the Fibonacci or Lucas sequences (or of the respective half sequences).

3. Toeplitz–Hessenberg matrices with Fibonacci entries

The following theorem gives the value of $\det(a_1, a_2, \dots, a_n)$ for several Fibonacci entries a_i. Recall that the n-th Pell number P_n is defined recursively by

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2,$$

where $P_0 = 0$ and $P_1 = 1$ (see Koshy (2014)).

Theorem 3.1.

Let $n \geq 1$, except when noted otherwise. Then

$$\det(F_0, F_1, \dots, F_{n-1}) = (-1)^{n-1}, \qquad n \geq 2; \tag{3}$$

$$\det(F_0, F_2, \dots, F_{2n-2}) = (-1)^n (1 - 2^{n-1}); \tag{4}$$

$$\det(F_1, F_2, \dots, F_n) = \frac{1 - (-1)^n}{2}; \tag{5}$$

$$\det(F_1, F_3, \dots, F_{2n-1}) = (-1)^{n-1} 2^{n-2}, \qquad n \geq 2; \tag{6}$$

$$\det(F_2, F_3, \dots, F_{n+1}) = 0, \qquad n \geq 3; \tag{7}$$

$$\det(F_2, F_4, \dots, F_{2n}) = (-1)^{n-1} n; \tag{8}$$

$$\det(F_3, F_4, \dots, F_{n+2}) = 1, \qquad n \geq 2; \tag{9}$$

$$\det(F_3, F_5, \dots, F_{2n+1}) = (-1)^{n-1}, \qquad n \geq 2; \tag{10}$$

$$\det(F_4, F_5, \dots, F_{n+3}) = n + 2; \tag{11}$$

$$\det(F_4, F_6, \dots, F_{2n+2}) = 0, \qquad n \geq 3; \tag{12}$$

$$\det(F_5, F_6, \dots, F_{n+4}) = \frac{(2+\sqrt{2})^{n+2} + (2-\sqrt{2})^{n+2}}{8}; \tag{13}$$

$$\det(F_5, F_7, \dots, F_{2n+3}) = P_{n+2}. \tag{14}$$

Proof:

We will prove formula (13) using induction on n. The other identities may be established in a similar manner, so we omit their proofs for the sake of brevity. Clearly, formula (13) works when $n = 1$ and $n = 2$. Suppose it is true for all $k \leq n - 1$, where $n \geq 3$.

Let $D_n = \det(F_5, F_6, \ldots, F_{n+4})$. Using recurrence (2), we then have

$$\begin{aligned}
D_n &= \sum_{j=1}^{n} (-1)^{j-1} F_{j+4} D_{n-j} \\
&= \sum_{j=1}^{n} (-1)^{j-1} \left(F_{j+3} + F_{j+2}\right) D_{n-j} \\
&= F_4 D_{n-1} + \sum_{j=2}^{n} (-1)^{j-1} F_{j+3} D_{n-j} + F_3 D_{n-1} - F_4 D_{n-2} + \sum_{j=3}^{n} (-1)^{j-1} F_{j+2} D_{n-j} \\
&= 3D_{n-1} - \sum_{j=1}^{n-1} (-1)^{j-1} F_{j+4} D_{n-j-1} + 2D_{n-1} - 3D_{n-2} + \sum_{j=1}^{n-2} (-1)^{j+1} F_{j+4} D_{n-j-2} \\
&= 3D_{n-1} - D_{n-1} + 2D_{n-1} - 3D_{n-2} + D_{n-2} \\
&= 4D_{n-1} - 2D_{n-2} \\
&= 4 \cdot \frac{(2+\sqrt{2})^{n+1} + (2-\sqrt{2})^{n+1}}{8} - 2 \cdot \frac{(2+\sqrt{2})^{n} + (2-\sqrt{2})^{n}}{8} \\
&= \frac{(2+\sqrt{2})^{n+2} + (2-\sqrt{2})^{n+2}}{8}.
\end{aligned}$$

Consequently, formula (13) is true in the n case and thus, by induction, it holds for all positive integers. ∎

Note that formula (5) above is well-known (for example, see Merca (2014) and Goy (2016) as well as a result of Macfarlane (2010) having (5) as a special case).

4. Toeplitz–Hessenberg matrices with Lucas entries

Next, we investigate the Lucas counterparts of some of the results from Theorem 3.1.

Theorem 4.1.

Let $n \geq 1$, except when noted otherwise. Then

$$\det(L_0, L_1, \ldots, L_{n-1}) = \frac{5 \cdot 2^n - 2(-1)^n}{6}; \tag{15}$$

$$\det(L_0, L_2, \ldots, L_{2n-2}) = \frac{10 - (-2)^n}{6}; \tag{16}$$

$$\det(L_1, L_2, \ldots, L_n) = (-1)^{\lfloor \frac{n}{2} \rfloor} \cdot \frac{3+(-1)^n}{2}; \tag{17}$$
$$\det(L_1, L_3, \ldots, L_{2n-1}) = \frac{(1+3i)(-1-i)^n + (1-3i)(-1+i)^n}{4}; \tag{18}$$
$$\det(L_2, L_3, \ldots, L_{n+1}) = 5 \cdot 2^{n-2}, \qquad n \geq 2; \tag{19}$$
$$\det(L_2, L_4, \ldots, L_{2n}) = \frac{5-(-1)^n}{2}; \tag{20}$$
$$\det(L_3, L_4, \ldots, L_{n+2}) = 5 \cdot 2^{n-1} - 1; \tag{21}$$
$$\det(L_3, L_5, \ldots, L_{2n+1}) = 5, \qquad n \geq 2, \tag{22}$$

where $i = \sqrt{-1}$ and $\lfloor \alpha \rfloor$ is the floor of α.

Proof:

We will prove (18) using induction on n; the others may be shown inductively by comparable arguments. When $n = 1$ and $n = 2$, the formula holds. Now assume (18) holds for all $k \leq n-1$, where $n \geq 3$.

Let $D_n = \det(L_1, L_3, \ldots, L_{2n-1})$. Using (2) and the well-known formula $L_{2k} = \sum_{s=1}^{k} L_{2s-1} + 2$ (see, e.g., Koshy (2017)), we then have

$$\begin{aligned}
D_n &= \sum_{k=1}^{n} (-1)^{k-1} L_{2k-1} D_{n-k} \\
&= L_1 D_{n-1} + \sum_{k=2}^{n} (-1)^{k-1} \big(L_{2k-2} + L_{2k-3}\big) D_{n-k} \\
&= D_{n-1} + \sum_{k=1}^{n-1} (-1)^k L_{2k} D_{n-k-1} - \sum_{k=1}^{n-1} (-1)^{k-1} L_{2k-1} D_{n-k-1} \\
&= D_{n-1} + \sum_{k=1}^{n-1} (-1)^k \left(\sum_{s=1}^{k} L_{2s-1} + 2 \right) D_{n-k-1} - D_{n-1} \\
&= \sum_{k=1}^{n-1} \sum_{s=1}^{k} (-1)^k L_{2s-1} D_{n-k-1} + 2 \sum_{k=1}^{n-1} (-1)^k D_{n-k-1} \\
&= \sum_{s=1}^{n-1} \sum_{k=1}^{n-s} (-1)^{k+s-1} L_{2k-1} D_{n-k-s} + 2 \sum_{k=2}^{n} (-1)^{k-1} D_{n-k} \\
&= \sum_{s=1}^{n-1} (-1)^s D_{n-s} + 2 \left(\sum_{k=1}^{n-1} (-1)^{k-1} D_{n-k} - D_{n-1} + (-1)^{n-1} D_0 \right) \\
&= \sum_{s=1}^{n-1} (-1)^{s-1} D_{n-s} - 2D_{n-1} + 2(-1)^{n-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=1}^{n-1}(-1)^{s-1}\frac{(1+3i)(-1-i)^{n-s}+(1-3i)(-1+i)^{n-s}}{4} \\
&\quad - 2\,\frac{(1+3i)(-1-i)^{n-1}+(1-3i)(-1+i)^{n-1}}{4} + 2(-1)^{n-1} \\
&= \frac{(-1)^{n-1}}{4}\Bigg((1+3i)(1+i)^n\sum_{s=1}^{n-1}\frac{1}{(1+i)^s} + (1-3i)(1-i)^n\sum_{s=1}^{n-1}\frac{1}{(1-i)^s} \\
&\quad - (1+3i)(1-i)(1+i)^n - (1-3i)(1+i)(1-i)^n + 8\Bigg) \\
&= \frac{(1+3i)(-1-i)^n+(1-3i)(-1+i)^n}{4}.
\end{aligned}
$$

Thus, it follows by induction that the formula is true for all positive integers. ■

5. Multinomial extensions

Next we focus on multinomial extensions of Theorems 3.1 and 4.1. To this end, we first present the multinomial version of formula (2) (see Muir (1960)).

Lemma 5.1. (Trudi's formula)

Let M_n be the matrix defined in (1). Then

$$
\det(M_n) = \sum_{\substack{t_1,\ldots,t_n\ge 0\\ t_1+2t_2+\cdots+nt_n=n}} (-a_0)^{n-T_n} p_n(t) a_1^{t_1} a_2^{t_2}\cdots a_n^{t_n}, \tag{23}
$$

where $T_n = t_1+\cdots+t_n$ and $p_n(t) = \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ is the multinomial coefficient.

In particular, when $a_0 = 1$, formula (23) is known as *Brioschi's formula* (see Muir (1960)). For example,

$$
\begin{aligned}
\det(M_4) &= (-1)^0\binom{4}{4,0,0,0}a_1^4 + (-1)^1\binom{3}{2,1,0,0}a_1^2a_2 \\
&\quad + (-1)^2\binom{2}{1,0,1,0}a_1a_3 + (-1)^2\binom{2}{0,2,0,0}a_2^2 + (-1)^3\binom{1}{0,0,0,1}a_4 \\
&= a_1^4 - 3a_1^2a_2 + 2a_1a_3 + a_2^2 - a_4.
\end{aligned}
$$

Trudi's formula (23), coupled with Theorem 3.1 above, yields the following result.

Theorem 5.2.

Let $n \geq 1$, except when noted otherwise. Then

$$\sum_{\substack{t_1,\dots,t_{n-1}\geq 0\\ 2t_1+\cdots+nt_{n-1}=n}} (-1)^{T_{n-1}} p_{n-1}(t) F_1^{t_1} F_2^{t_2} \cdots F_{n-1}^{t_{n-1}} = -1, \quad n \geq 2;$$

$$\sum_{\substack{t_1,\dots,t_{n-1}\geq 0\\ 2t_1+\cdots+nt_{n-1}=n}} (-1)^{T_{n-1}} p_{n-1}(t) F_2^{t_1} F_4^{t_2} \cdots F_{2n-2}^{t_{n-1}} = 1 - 2^{n-1};$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_1^{t_1} F_2^{t_2} \cdots F_n^{t_n} = \frac{(-1)^n - 1}{2}; \tag{24}$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_1^{t_1} F_3^{t_2} \cdots F_{2n-1}^{t_n} = -2^{n-2}, \quad n \geq 2;$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_2^{t_1} F_4^{t_2} \cdots F_{2n}^{t_n} = -n;$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_2^{t_1} F_3^{t_2} \cdots F_{n+1}^{t_n} = 0, \quad n \geq 3; \tag{25}$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_3^{t_1} F_4^{t_2} \cdots F_{n+2}^{t_n} = (-1)^n, \quad n \geq 2;$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_3^{t_1} F_5^{t_2} \cdots F_{2n+1}^{t_n} = -1, \quad n \geq 2; \tag{26}$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_4^{t_1} F_5^{t_2} \cdots F_{n+3}^{t_n} = (-1)^n (n+2);$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_4^{t_1} F_6^{t_2} \cdots F_{2n+2}^{t_n} = 0, \quad n \geq 3; \tag{27}$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_5^{t_1} F_6^{t_2} \cdots F_{n+4}^{t_n} = \frac{(-2-\sqrt{2})^{n+2} + (-2+\sqrt{2})^{n+2}}{8};$$

$$\sum_{\substack{t_1,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) F_5^{t_1} F_7^{t_2} \cdots F_{2n+3}^{t_n} = (-1)^n P_{n+2}, \quad n \geq 2,$$

where $T_n = t_1 + \cdots + t_n$ and $p_n(t) = \binom{t_1+\cdots+t_n}{t_1,\dots,t_n}$.

For example, it follows from formulas (25), (26) and (27) that

$$F_2^4 - 3F_2^2F_3 + 2F_2F_4 + F_3^2 - F_5 = 0,$$
$$F_3^5 - 4F_3^3F_5 + 3F_3^2F_7 + 3F_3F_5^2 - 2F_3F_9 - 2F_5F_7 + F_{11} = 1,$$

$$F_4^6 - 5F_4^4F_6 + 4F_4^3F_8 + 6F_4^2F_6^2 + 2F_4F_{12} - F_6^3 + 2F_6F_{10} + F_8^2 - F_{14} = 0,$$

respectively. Note that formula (24) was stated without proof in Goy (2017).

The next theorem gives analogous results for the Lucas numbers.

Theorem 5.3.

Let $n \geq 1$, except when noted otherwise. Then

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_0^{t_1} L_1^{t_2} \cdots L_{n-1}^{t_n} = \frac{5(-2)^n-2}{6}; \tag{28}$$

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_0^{t_1} L_2^{t_2} \cdots L_{2n-2}^{t_n} = \frac{10(-1)^n-2^n}{6};$$

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_1^{t_1} L_2^{t_2} \cdots L_{n}^{t_n} = (-1)^{\lfloor \frac{n}{2} \rfloor} \cdot \frac{1+3(-1)^n}{2};$$

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_1^{t_1} L_3^{t_2} \cdots L_{2n-1}^{t_n} = \frac{(1+3i)(1+i)^n+(1-3i)(1-i)^n}{4};$$

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_2^{t_1} L_3^{t_2} \cdots L_{n+1}^{t_n} = 5(-2)^{n-2}, \quad n \geq 2;$$

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_2^{t_1} L_4^{t_2} \cdots L_{2n}^{t_n} = \frac{5(-1)^n-1}{2}; \tag{29}$$

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_3^{t_1} L_4^{t_2} \cdots L_{n+2}^{t_n} = (-1)^n(5\cdot 2^{n-1}-1);$$

$$\sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{T_n} p_n(t) L_3^{t_1} L_5^{t_2} \cdots L_{2n+1}^{t_n} = 5(-1)^n, \quad n \geq 2,$$

where $T_n = t_1 + \cdots + t_n$ and $p_n(t) = \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}$.

For example, formulas (28) and (29) yield

$$L_0^4 - 3L_0^2L_1 + 2L_0L_2 + L_1^2 - L_3 = 13,$$
$$L_2^5 - 4L_2^3L_4 + 3L_2^2L_6 + 3L_2L_4^2 - 2L_2L_8 - 2L_4L_6 + L_{10} = 3,$$

respectively.

6. Combinatorial proofs

In the proofs of this section, we will employ the combinatorial interpretation for the determinant of an $n \times n$ matrix $A = (a_{i,j})$ as a signed sum over the symmetric group $\mathcal{S}_n$ given by

$$\det(A) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathrm{sgn}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}, \tag{30}$$

where sgn(σ) denotes the *sign* of the permutation σ. See, e.g., Benjamin and Cameron (2005), Benjamin et al. (2007) and Benjamin and Shattuck (2007) for combinatorial proofs of determinants of various types of matrices whose entries are discrete sequences. Note that in the case of a Toeplitz–Hessenberg matrix A, the only permutations that contribute to the sum in (30) are those in which each cycle of the disjoint cycle decomposition comprises a set of consecutive integers in increasing order (where it is understood that the smallest element is the first in each cycle). For if σ is not of this form, then $\sigma(j) > j + 1$ for some j, which implies that the entry $a_{j,\sigma(j)}$ of A is zero and thus the corresponding product in (30) is zero.

Note that a permutation α of $[n] = \{1, 2, \ldots, n\}$ whose disjoint cycles consist of consecutive integers as described is synonymous with a composition of n, upon identifying the cycle lengths as parts (where cycles are assumed to be arranged in increasing order of smallest elements). So if A is Toeplitz–Hessenberg, then each part of size i in the corresponding composition, which we will also denote by α, is weighted by a_i (assuming each superdiagonal entry of A is 1). Also, the sign of the composition α is the same as that of the corresponding permutation and is given by $(-1)^{n-\mu(\alpha)}$, where $\mu(\alpha)$ denotes the number of parts of α. Thus, the sum in (30) may be regarded in the case when A is Toeplitz–Hessenberg of size n and having superdiagonal entries 1 as a signed sum over weighted compositions α, where the sign is defined by $(-1)^{n-\mu(\alpha)}$ and the weight of α is the product of the weights of its individual parts where a part of size i is assigned the weight a_i for $i \geq 1$.

Recall that F_{n+1} gives the number of linear square-and-domino tilings of length n, where squares and dominos are 1×1 and 1×2 pieces, respectively, and are considered indistinguishable. Denote by $\mathcal{T}_n$ the set of linear tilings of length n if $n \geq 1$, with $\mathcal{T}_0$ consisting of the empty tiling. Throughout, we will regard members of $\mathcal{T}_n$ as sequences of squares (s's) and dominos (d's) such that the number of s's plus twice the number of d's equals n. Note that the tiling interpretation for F_n was used by Benjamin and Quinn (2003) to explain combinatorially many of the identities from Vajda (1989).

Proofs of Theorem 3.1, Identities (3), (5), (7) and (9).

Proof of* (9)*:

We first describe a combinatorial interpretation for $\det(A)$, where A has (i, j)-th entry F_{i+3-j} if $j \leq i + 1$ and 0 otherwise. Define

$$\mathcal{A}_r = \{(a_1, t_1), \ldots, (a_r, t_r) : \sum_{i=1}^{r} a_i = n,\ a_i > 0 \text{ and } t_i \in \mathcal{T}_{a_i+1} \text{ for all } i\}$$

for $1 \leq r \leq n$ and let $\mathcal{A} = \cup_{r=1}^{n} \mathcal{A}_r$. Let the sign of $\lambda \in \mathcal{A}_r$ be $(-1)^{n-r}$. By the definition of the determinant, $\det(A)$ gives the sum of the signs of all members of $\mathcal{A}$. Note that, equivalently, the sign of $\lambda \in \mathcal{A}_r$ is given by $(-1)^{\sum_{i=1}^{r} |t_i|}$, where $|t_i|$ denotes the length of the tiling t_i.

Let $\mathcal{A}' \subseteq \mathcal{A}$ consist of those $\lambda = (a_1, t_1), \ldots, (a_m, t_m)$ for any m such that either (i) $a_m \geq 2$, with the tiling t_m ending in s, or (ii) $a_m = 1$, with $t_m = s^2$. If (i) occurs, then replace (a_m, t_m) with the two pairs $(a_m - 1, t_m - s), (1, s^2)$, whereas if (ii) occurs, then we reverse this operation. Note that $n \geq 2$ implies $m \geq 2$ in case (ii) and that this mapping defines a sign-changing involution of $\mathcal{A}'$.

Now let $\lambda \in \mathcal{A} - \mathcal{A}'$, with $\lambda \neq (a_1, t_1), \ldots, (a_n, t_n)$ where $a_1 = \cdots = a_n = 1$ and $t_1 = \cdots = t_n = d$. Let r denote the largest index such that $(a_r, t_r) \neq (1, d)$. Suppose either (I) $a_r \geq 2$ and t_r ends in d, or (II) $a_r \geq 1$ and t_r ends in s. Note that in case (II), we have (a_r, t_r) followed by $(a_{r+1}, t_{r+1}) = (1, d)$, for otherwise λ would belong to $\mathcal{A}'$. If (I) occurs, then replace (a_r, t_r) with $(a_r - 1, t_r - d + s), (1, d)$, and if (II) occurs, then reverse the operation. This defines a sign-changing involution of $\mathcal{A} - \mathcal{A}'$ except for the excluded element, which has sign 1. Combining with the previous involution of $\mathcal{A}'$ implies $\det(A) = 1$ if $n \geq 2$. ∎

Identities* (3)*,* (5) *and* (7)*.

We proceed with similar notation as before. Let B, C and D be $n \times n$ matrices whose respective entries are $b_{i,j}$, $c_{i,j}$ and $d_{i,j}$ which are zero if $j > i + 1$ and 1 if $j = i + 1$. Suppose $b_{i,j} = F_{i-j}$, $c_{i,j} = F_{i+1-j}$ and $d_{i,j} = F_{i+2-j}$ if $j \leq i$. Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ denote the collections of pairs $(a_1, t_1), (a_2, t_2), \ldots,$ where $(a_1, a_2, \ldots)$ is a composition of n and t_i are tilings of length $a_i - 2$, $a_i - 1$ and a_i for all i, respectively, with any number of pairs possible. Note that in the case of $\mathcal{B}$, it is understood that $a_i \geq 2$ for all i. Define the sign in each case to be $(-1)^{n-r}$, where r denotes the number of pairs. Then $\det(B)$, $\det(C)$ and $\det(D)$ are seen to give the respective sum of signs of all members of $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, respectively.

To establish the result in each case, we define an involution on the underlying set. We first consider $\det(B)$. Let $\lambda^* \in \mathcal{B}$ be given by $\lambda^* = (n, s^{n-2})$. Let $\lambda \in \mathcal{B} - \{\lambda^*\}$ be given by $\lambda = (a_1, t_1), \ldots, (a_r, t_r)$, where $a_i \geq 2$ and t_i has length $a_i - 2$ for all i. If λ is such that $a_r \geq 4$ and t_r contains a d and ends in exactly j s's where $j \geq 0$, then replace the pair (a_r, t_r) with the two pairs $(a_r - (j + 2), t_r - ds^j), (j + 2, s^j)$. We reverse this action if the last part is ≥ 2 with the last tiling consisting of all squares, whence there is a predecessor pair since we are excluding λ^*. This mapping defines a sign-changing involution of $\mathcal{B} - \{\lambda^*\}$, with λ^* having sign $(-1)^{n-1}$, which implies the formula for $\det(B)$ in (3) for $n \geq 2$.

To establish the formula for $\det(C)$, let $\lambda^* \in \mathcal{C}$ be given by $\lambda^* = (n, d^{(n-1)/2})$ if n is odd. We define an involution on $\mathcal{C} - \{\lambda^*\}$ if n is odd and on all of $\mathcal{C}$ if n is even. Let $\lambda \in \mathcal{C}$ be given by $\lambda = (a_1, t_1), \ldots, (a_r, t_r)$, where $|t_i| = a_i - 1$ for all i. If $a_r \geq 2$ and t_r contains at least one square and ends in sd^j for some $j \geq 0$, then replace (a_r, t_r) with $(a_r - (2j + 1), t_r - sd^j), (2j + 1, d^j)$. We reverse this action if the tiling in the last pair consists of a (possibly empty) string of dominos. Note that there is a predecessor to the last pair in the latter case if n is odd since we are excluding λ^*. As the sign of λ^* is positive, the formula for $\det(C)$ given in (5) follows.

Finally, to show (7), we assume $n \geq 3$ and let $\lambda = (a_1, t_1), \ldots, (a_r, t_r) \in \mathcal{D}$. If $a_r \geq 3$ and t_r ends in d, then replace (a_r, t_r) with $(a_r - 2, t_r - d), (2, d)$, and vice versa, if the tiling in the last pair is a single domino. If $a_r \geq 2$ and t_r ends in s, then replace (a_r, t_r) with $(a_r - 1, t_r - s), (1, s)$, and vice versa, if the tiling in the final pair is a single square. Note that $n \geq 3$ implies that the reverse operations can be performed in each case. Combining the two mappings yields a sign-changing involution of $\mathcal{D}$, which gives (7). ■

Proof of Theorem 3.1, Identities (4)**,** (8)**,** (10) **and** (12)**.**

Proof of (4)***:***

Let $\mathcal{E} = \{(a_1, t_1), (a_2, t_2), \ldots : a_1 + a_2 + \cdots = n,\ a_i \geq 2$ and $t_i \in \mathcal{T}_{2a_i-3}$ for all $i\}$. Let $\lambda \in \mathcal{E}$ have sign $(-1)^{n-r}$, where r denotes the number of pairs of λ. Then $\det(E)$ gives the sum of the signs of all members of $\mathcal{E}$, where $E = M_n(1, F_0, \ldots, F_{2n-2})$. To show (4), we will define an involution on $\mathcal{E}$ whose survivors each have sign $(-1)^{n-1}$ and have cardinality $2^{n-1} - 1$.

Given $m \geq 2$, let $\mathcal{T}^*_{2m-3} \subseteq \mathcal{T}_{2m-3}$ comprise those tilings τ for which the leftmost run of s within τ is of odd length, with all other runs of s of even length. Let $\mathcal{E}' \subseteq \mathcal{E}$ comprise those members consisting of a single pair (n, τ), where $\tau \in \mathcal{T}^*_{2n-3}$. Note that each member of $\mathcal{E}'$ has sign $(-1)^{n-1}$. To show $|\mathcal{E}'| = 2^{n-1} - 1$, we represent τ sequentially as $\tau = \tau_1 \tau_2 \cdots \tau_{n-1}$, where $\tau_i = d$, s or s^2 with s occurring exactly once and prior to any occurrences of s^2. By concatenating the various pieces represented by the τ_i, one recreates the tiling τ which one may verify indeed belongs to $\mathcal{E}'$. Note that then there are $2^{n-1} - 1$ such τ as they correspond to binary sequences of length $n - 1$ where the possibility that τ equals d^{n-1} is excluded.

We now define an involution of $\mathcal{E} - \mathcal{E}'$. Let $(a_1, t_1), \ldots, (a_r, t_r) \in \mathcal{E} - \mathcal{E}'$. First suppose $a_r \geq 4$ and $t_r \in \mathcal{T}_{2a_r-3} - \mathcal{T}^*_{2a_r-3}$. Consider within t_r the first s of the rightmost run of odd length. Then at least one s occurs within t_r to the left of this s, for otherwise t_r would belong to $\mathcal{T}^*_{2a_r-3}$. Thus, t_r may be decomposed as $t_r = \alpha s d^j s \beta$, where α is arbitrary, $j \geq 1$ and all runs of s in β are of even length. Note that α is nonempty since it must contain an s, by parity considerations. Let β have length 2ℓ where $\ell \geq 0$ and let $x = j + \ell + 1$; note that $x \geq 2$. If $\lambda \in \mathcal{E} - \mathcal{E}'$ with (a_r, t_r) as described, then replace the pair (a_r, t_r) with $(a_r - x, \alpha), (x, d^{j-1} s \beta)$. Now suppose (a_r, t_r) within λ is such that $t_r \in \mathcal{T}^*_{2a_r-3}$. Then $r \geq 2$, lest λ belongs to $\mathcal{E}'$. In this case, we reverse the previous operation by adding a_r to a_{r-1} and appending the tiling sdt_r of length $2a_r$ to t_{r-1}. Note that $t_{r-1} s d t_r$ belongs to $\mathcal{T}_{2m-3} - \mathcal{T}^*_{2m-3}$ where $m = a_{r-1} + a_r$ since the leftmost run of s is either of even length or of odd length with at least one other run of s having odd length. Combining the two mappings then yields the desired involution of $\mathcal{E} - \mathcal{E}'$, which implies the formula for $\det(E)$ given in (4). ■

Identities (8) **and** (10)**.**

To show (8) and (10), let $\mathcal{F}$ and $\mathcal{G}$ be the same as $\mathcal{E}$ above except that $\mathcal{T}_{2a_i-3}$ is replaced by $\mathcal{T}_{2a_i-1}$ and $\mathcal{T}_{2a_i}$, respectively, in the definition. Let $F = M_n(1, F_2, \ldots, F_{2n})$ and $G = M_n(1, F_3, \ldots, F_{2n+1})$. Then $\det(F)$ and $\det(G)$ give the sum of the signs of all members of $\mathcal{F}$ and of $\mathcal{G}$, respectively.

To complete the proof of (8), first let $\mathcal{F}' \subseteq \mathcal{F}$ consist of those pairs for which $r = 1$ (i.e., $a_1 = n$) and $t_1 = d^i s^{2n-2i-1}$ for some $0 \leq i \leq n-1$. Then $|\mathcal{F}'| = n$, with each member of $\mathcal{F}'$ having sign $(-1)^{n-1}$. Suppose that $\lambda = (a_1, t_1), \ldots, (a_r, t_r) \in \mathcal{F} - \mathcal{F}'$ such that t_r is of the form $\alpha d s^{2m}$ for some α and $m \geq 0$. Note that $\alpha \neq \varnothing$ since it must contain an s. In this case, we replace the final pair $(a_r, \alpha d s^{2m})$ with the two pairs $(a_r - m - 1, \alpha), (m+1, s^{2m+1})$. We reverse this operation if $r \geq 2$ and the tiling in the final pair consists of all squares. Now suppose $t_r = \beta s d^j s^{2m+1}$ for some β (necessarily $\neq \varnothing$) and $j \geq 1, m \geq 0$. In this case, we replace $(a_r, \beta s d^j s^{2m+1})$ with $(n-m-j-1, \beta), (m+j+1, d^j s^{2m+1})$, where we reverse this operation if the final tiling is of the form $d^j s^{2m+1}$ with $r \geq 2$. Combining the two previous mappings yields a sign-changing involution of $\mathcal{F} - \mathcal{F}'$ which implies the formula for $\det(F)$ given in (8).

To complete the proof of (10), we define an involution on $\mathcal{G}$ as follows. First, consider replacing a final pair (a_r, t_r) of the form $(m, \alpha d)$ where $\alpha \neq \varnothing$ (which implies $m \geq 2$) with $(m-1, \alpha), (1, d)$, and reversing this if $(a_r, t_r) = (1, d)$. Note that in the latter case, $n \geq 2$ implies $r \geq 2$. On the other hand, if the final pair has the form $(m, \beta s d^i s)$, where $\beta \neq \varnothing$ and $i \geq 0$, then we replace it by $(m-i-1, \beta), (i+1, s d^i s)$, and vice versa, if possible (i.e., if $r \geq 2$). Combining the two mappings yields a sign-changing involution of $\mathcal{G} - \{\lambda^*\}$, where $\lambda^* = (n, s d^{n-1} s)$. This implies $\det(G) = (-1)^{n-1}$ if $n \geq 2$, as desired. ∎

Identity (12)**.**

Let $\mathcal{H}$ be the same as $\mathcal{E}$ above except that $\mathcal{T}_{2a_i-3}$ is replaced by $\mathcal{T}_{2a_i+1}$. We define a sign-changing involution on $\mathcal{H}$ where $n \geq 3$ in several steps as follows. Suppose $\lambda = (a_1, t_1), \ldots, (a_r, t_r) \in \mathcal{H}$. If $t_r = \alpha d$ where $|\alpha| \geq 3$, then replace (a_r, t_r) by $(a_r - 1, \alpha), (1, sd)$, and vice versa, if the last tiling is sd. If $t_r = \beta s^2$ where $|\beta| \geq 3$, then replace by $(a_r - 1, \beta), (1, s^3)$, and vice versa, if the last tiling is s^3. If (a_r, t_r) is such that $t_r = \gamma s d^i s$ where $i \geq 1$ and $|\gamma| \geq 3$, then replace (a_r, t_r) by $(a_r - i - 1, \gamma), (i+1, s^2 d^i s)$, which we reverse if the final tiling has the appropriate form and there is a predecessor pair.

Let $\mathcal{H}'$ denote the subset of $\mathcal{H}$ for which the preceding (composite) involution is not defined. That is, $\mathcal{H}'$ consists of those members of $\mathcal{H}$ such that $(a_r, t_r) = (i, d^i s)$ for some $1 \leq i \leq n$, together with $(a_1, t_1) = (n, s^2 d^{n-1} s)$. Let $\mathcal{H}^* \subseteq \mathcal{H}'$ consist of those members such that all tilings t_i are of the form $d^j s$ for some $1 \leq j \leq n$, except for the first tiling t_1, which can also be sd or s^3 or have the form $s^2 d^j s$ where $j < n$. If $\rho \in \mathcal{H}' - \mathcal{H}^*$, identify the largest index $i > 1$ such that either $t_i \neq d^j s$ for any j, where if no such index exists, then $i = 1$ and $t_1 \neq sd, s^3, d^j s, s^2 d^j s$ for any $j \geq 1$. We then treat (a_i, t_i) as we did the final pair (a_r, t_r) above and apply one the operations described, noting that if $i = 1$, an operation is used where no predecessor is needed.

Finally, members $\tau = (a_1, t_1), \ldots, (a_r, t_r) \in \mathcal{H}^*$ are uniquely determined by the lengths of the t_i for $i > 1$ together with the form assumed by t_1. Thus, members of $\mathcal{H}^*$ are synonymous with compositions $x_1 + \cdots + x_r = n$ such that the first part x_1 may be marked in two or three ways depending on whether $x_1 > 1$ or $x_1 = 1$. Define an involution on $\mathcal{H}^*$ by replacing x_r with $x_r - 1, 1$ if $x_r > 1$ and $r \geq 2$, and reversing this action if $x_r = 1$ with $r \geq 3$. This pairs all members of $\mathcal{H}^*$ except those of the form $x_1 = n$ or $x_1 = n-1, 1$. Since $n \geq 3$, there are exactly four members of $\mathcal{H}^*$ of either form and they come in two pairs whose members have opposite sign. Combining all of the above mappings then yields the desired sign-changing involution of $\mathcal{H}$, which implies (12). ■

Remark 6.1.

A combinatorial proof may be given for (11) that is comparable to the one above for (12). We were unable, however, to find a combinatorial proof of the formula for

$$\det(F_1, F_3, \ldots, F_{2n-1}) = (-1)^{n-1}2^{n-2},$$

where $n \geq 2$.

It is possible to extend the arguments above for (7), (10) and (12) to the Lucas case.

Proof of Theorem 4.1, Identities (17)**,** (20) **and** (22)**.**

Proof of (17) ***and*** (20)***:***

Let $\mathcal{J}$ be the same as $\mathcal{D}$ in the proof of (7) except that an initial domino may be marked (denoted by d') within any of the tilings t_i. We apply the same involution to $\mathcal{J}$ as was applied to $\mathcal{D}$ above upon considering the rightmost pair not of the form $(2, d')$. Thus, the set of survivors are those members of $\mathcal{J}$ of the form $(a, \alpha), (2, d'), \ldots, (2, d')$, where $a = 1$ and $\alpha = s$ if n is odd and $a = 2$ and $\alpha = d$ or d' if n is even. If n is odd, then the sign of the sole survivor is given by $(-1)^{(n-1)/2}$, whereas both survivors have sign $(-1)^{n/2}$ when n is even. Combining the odd and even cases yields formula (17).

Let $\mathcal{K}$ be as $\mathcal{G}$ above in the proof of (10) except that an initial domino may be marked. Then the involution defined on $\mathcal{G}$ can also be applied to $\mathcal{K}$ if one assumes that the pair $(1, d)$ (where the domino is unmarked) is moved in the first case. We then apply the involution if possible to members of $\mathcal{K}$ whose final pair is $(1, d')$ by considering the rightmost pair not of this form. The set of survivors are those $\lambda \in \mathcal{K}$ having the form $(b, \beta), (1, d'), \ldots, (1, d')$, where $(b, \beta) = (1, d)$, $(1, d')$ or $(i + 1, sd^i s)$ for some $0 \leq i \leq n - 1$. Let $K = M_n(1, L_2, \ldots, L_{2n})$. If n is even, then the sum of the signs of λ having the third stated form is zero, which implies $\det(K) = 2$ in this case. If n is odd, then this sum of signs is 1, which implies $\det(K) = 3$. Combining the even and odd cases gives (20). ■

Identity (22)**.**

Let $\mathcal{L}$ ($\mathcal{L}', \mathcal{L}^*$, resp.) be the same as $\mathcal{H}$ ($\mathcal{H}', \mathcal{H}^*$, resp.) in the proof of (12) above except that now initial dominos may be marked. Since the mappings defined above on $\mathcal{H} - \mathcal{H}'$ and $\mathcal{H}' - \mathcal{H}^*$ do not entail moving an initial domino, they may also be applied to $\mathcal{L} - \mathcal{L}'$ and $\mathcal{L}' - \mathcal{L}^*$, respectively. Note that in addition to the forms described above for members of $\mathcal{H}^*$, pairs within $\lambda \in \mathcal{L}^*$ can also equal $(j, d'd^{j-1}s)$ for any $j \geq 1$. We thus can represent λ by a composition $x_1 + \cdots + x_r = n$, where x_1 is marked in one of three or four ways depending on whether $x_1 > 1$ or $x_1 = 1$, respectively, with each x_i for $i > 1$ either marked or unmarked. Define an involution on $\mathcal{L}^*$ by replacing x_r with $x_r - 1, 1$ if $x_r > 1$ and $r \geq 2$, and vice versa, if $x_r = 1$ and $r \geq 3$, where the 1 is unmarked in both instances (which corresponds to $t_r = ds$). Let $1'$ denote a marked non-initial 1. Then the survivors of the involution are those of the form

$$i + x + 1' + \cdots + 1' = n, \tag{31}$$

where $x = 1$ or $1'$ and the initial i is marked in three possible ways if $i > 1$ (corresponding to a pair of the form $(i, d^i s)$, $(i, d'd^{i-1}s)$ or $(i, s^2 d^{i-1}s)$) and in four possible ways if $i = 1$ (as $(1, sd)$ is also possible).

Now suppose n is even. Assume for now that i is marked in one of the first three possible ways if $i = 1$. Then compositions in (31) starting with $i + x$ where i is odd and $x = 1$ pair up with those starting with $i + x$ where $i < n$ is even, together with the case $i = n$. Also, compositions starting with $i + 1'$ where $i > 1$ is odd pair up with those starting $i + 1'$ where $i < n$ is even. This leaves only compositions of the form $1 + 1' + \cdots + 1'$ in which the initial 1 is marked in one of three ways. Note that this corresponds to members of $\mathcal{L}^*$ of the form $(a, \alpha), (1, d's), \ldots, (1, d's)$, where $a = 1$ and $\alpha = ds$, $d's$ or s^3. Finally, if $i = 1$ and is marked in the fourth way, then there are two additional possible compositions corresponding to the members of $\mathcal{L}^*$ given by $(1, sd), (b, \beta), (1, d's), \ldots, (1, d's)$, where $b = 1$ and $\beta = ds$ or $d's$. Thus, we get five unpaired members of $\mathcal{L}^*$ altogether, each having positive sign, which implies (22) for n even. If $n \geq 3$ is odd, then a similar analysis shows that the same five compositions in (31) are again unpaired, which implies (22) for n odd and completes the proof. ∎

7. Gibonacci generalization

In conclusion, we state a generalization of the determinant formulas above. Let G_n denote the *Gibonacci* number (for example, see Benjamin and Quinn (2003) or Vajda (1989)) defined by the recurrence

$$G_n = G_{n-1} + G_{n-2}, \qquad n \geq 2,$$

with $G_1 = a$ and $G_0 = b$ where a and b are arbitrary (they may even be regarded as indeterminates). Note that F_n corresponds to the case of G_n when $a = 1$ and $b = 0$, while L_n to the case when $a = 1$ and $b = 2$. Extending the combinatorial arguments from the prior section yields the following Gibonacci determinant formulas.

Theorem 7.1.

If $n \geq 1$, then

$$\det(G_1, G_2, \ldots, G_n) = x_n, \tag{32}$$

where x_n is defined recursively by

$$x_n = (a-1)x_{n-1} - (b-1)x_{n-2}, \qquad n \geq 3,$$

with initial conditions $x_1 = a$ and $x_2 = a^2 - a - b$, and

$$\det(G_2, G_4, \ldots, G_{2n}) = y_n, \tag{33}$$

where y_n is defined by

$$y_n = (a + b - 3)y_{n-1} + (b - 1)y_{n-2}, \qquad n \geq 3,$$

with $y_1 = a + b$ and $y_2 = (a + b)^2 - 3a - 2b$. Furthermore, we have

$$\det(G_3, G_5, \ldots, G_{2n+1}) = y_n^*, \quad n \geq 1, \tag{34}$$

where y_n^* is defined by

$$y_n^* = (2a + b - 3)y_{n-1}^* + (a - 1)y_{n-2}^*, \qquad n \geq 3,$$

with $y_1^* = 2a + b$ and $y_2^* = (2a + b)^2 - 5a - 3b$.

Remark 7.2.

Note that (32) is seen to reduce to formulas (5) and (17) when $(a, b) = (1, 0)$ and $(1, 2)$, respectively, while (33) reduces to (8) and (20) for these same values of a and b. When $a = 1$ in (33), one has the explicit formula

$$\det(G_2, G_4, \ldots, G_{2n}) = (b + 1)(b - 1)^{n-1} - \frac{(b - 1)^{n-1} + (-1)^n}{b}, \qquad n \geq 1.$$

Furthermore, note that (32) yields the Fibonacci identities (3), (7), (9), (11) and (13) when $(a, b) = (0, 1)$, $(1, 1)$, $(2, 1)$, $(3, 2)$ and $(5, 3)$, respectively. Also, the Lucas identities (15), (19) and (21) correspond to the $(a, b) = (2, -1)$, $(3, 1)$ and $(4, 3)$ cases of (32). Formula (33) reduces to (4), (6), (10), (12) and (14) when $(a, b) = (1, -1)$, $(0, 1)$, $(1, 1)$, $(2, 1)$ and $(3, 2)$, respectively, while (16), (18) and (22) correspond to the $(a, b) = (-1, 3)$, $(2, -1)$ and $(3, 1)$ cases. In general, taking $a = F_{c+1}$ and $b = F_c$ in (32) and (33) yields expressions for $\det(F_{1+c}, F_{2+c}, \ldots, F_{n+c})$ and $\det(F_{2+c}, F_{4+c}, \ldots, F_{2n+c})$, where c is arbitrary, with a similar remark applying to translates of the Lucas sequence. Finally, observe that (34) may be obtained from (33) by substituting $a + b$ for a and a for b, or can be shown directly by extending the combinatorial proof above for (22). When $a = 1$ in (34), one has

$$\det(G_3, G_5, \ldots, G_{2n+1}) = (b^2 + b - 1)(b - 1)^{n-2}, \qquad n \geq 2.$$

Taking $b = -1$, 0, 1 and 2 in the last formula yields identities (6), (10), (12) and (22), respectively.

8. Conclusion

In this paper, we have found determinant formulas for several families of Toeplitz–Hessenberg matrices having various translates of the Fibonacci and Lucas numbers for the non-zero entries. This extends an earlier result when $a_i = F_i$ for all i. In Theorem 3.1, we found determinant formulas where the entries were translates of the Fibonacci sequence or of just the even or odd subsequence. Some comparable results are given in Theorem 4.1 for the Lucas sequence. A common Gibonacci generalization of the results in Theorems 3.1 and 4.1 whereby arbitrary initial conditions are allowed is found in Theorem 7.1 via a combinatorial approach. The determinant formulas in all of these results may also be expressed (see Theorems 5.2 and 5.3) equivalently as multi-sum identities involving multinomial coefficients and a product of the terms of the sequence in question.

Combinatorial proofs were provided for most of the determinant identities which made use of involutions and the formal definition of the determinant. Such proofs in several instances allow one to see relationships between the various identities that may not be readily apparent in their algebraic derivation. Moreover, combinatorial proofs (especially in the Lucas case) point the way to the Gibonacci generalization of the identities in Theorems 3.1 and 4.1 and also provide a means by which to prove it. Further work on the enumerative aspects of the determinants of Toeplitz–Hessenberg matrices whose entries are combinatorial sequences is forthcoming and will focus on sequences which satisfy various linear recurrences with constant coefficients.

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