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ON A PROBLEM OF I. SCHUR

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TO THE MEMORY OF I. SCHUR

1. Introduction

1. Let $n \geqq 3$, and let Q_n denote the class of polynomials $f(x)$ of degree n satisfying the condition $|f(x)| \leqq 1$ in the interval $-1 \leqq x \leqq +1$. Let $Q_n(x_0)$ denote the subclass of Q_n characterized by the further restriction $f''(x_0) = 0$.

A well-known theorem of A. Markoff[1] states that $|f'(x)| \leqq n^2$ for $-1 \leqq x \leqq +1$ provided that $f(x) \in Q_n$; here $|f'(x)| = n^2$ holds if and only if $x = \pm 1$ and $f(x) = \pm T_n(x)$, where $T_n(x)$ denotes the n^{th} Tchebycheff polynomial. We observe that $T_n(x)$ does not belong to the classes $Q_n(\pm 1)$.

Some years ago I. Schur[2] proved the following interesting theorem: *Let* $-1 \leqq x_0 \leqq +1$, *and let* $f(x)$ *belong to* $Q_n(x_0)$. *Then* $|f'(x_0)| < \frac{1}{2}n^2$. Moreover he showed: *Let* m_n *be the least positive constant (depending only on* n*) such that* $|f'(x_0)| \leqq m_n \cdot n^2$ *for all* $f(x) \in Q_n(x_0)$, *and* x_0 *in* $-1 \leqq x \leqq +1$. *If* $\mu = \limsup_{n\to\infty} m_n$, *then*

$$0.217 \cdots \leqq \mu \leqq 0.465 \cdots. \tag{1.1}$$

Obviously

$$m_n \cdot n^2 = \max_{-1 \leqq x_0 \leqq +1} \max_{f(x) \in Q_n(x_0)} |f'(x_0)|. \tag{1.2}$$

The main purpose of the present note is to determine the constant μ and the polynomial $f(x)$ for which the extremum (1.2) is attained. In terms of the constant m_n, we obtain a bound for the derivative $f'(x)$ of a polynomial $f(x)$ which satisfies the condition that $|f'(x)|$ has a relative maximum at the point x considered.

2. *Let* $u_n(x)$ *be the polynomial of the class* $Q_n(+1)$ *for which* $u_n'(1)$ *is a maximum.* This polynomial $u_n(x) = u_n(x; A_n)$ can be determined from the transcendental equations (2.5), (2.6) and (2.17) of §2 (see below). It is a special case of a remarkable class of polynomials $u_n(x; A)$ considered first by G. Zolotareff[3]

[1] A. Markoff, *On a certain problem of D. I. Mendeleieff* (in Russian), Zapiski Imperatorskoi Akademii Nauk, vol. 62 (1889), pp. 1–24.

[2] I. Schur, *Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall*, Mathematische Zeitschrift, vol. 4 (1919), pp. 271–287.

[3] G. Zolotareff, (a) *On a question concerning a minimum value* (in Russian), Dissertation "pro venia legendi," published in lithographed form, 1868, Oeuvres, vol. 2 (1902), pp. 130–166; (b) *Application of elliptic functions to questions concerning functions which deviate the least from zero* (in Russian), Zapiski Imperatorskoi Akademii Nauk, vol. 30 (1877), Oeuvres, vol. 2, pp. 1–59; (c) *Sur l'application des fonctions elliptiques aux questions de maxima et minima*, Bulletin de l'Académie des Sciences de St.-Pétersbourg, series 3, vol. 24 (1878), pp. 305–310, Mélanges, 5, pp. 419–426, Oeuvres, vol. 1 (1931), pp. 369–374.

playing also a role in the important investigations of W. Markoff.[4] Recently N. Achyeser[5] used polynomials of the Zolotareff type in his investigations on polynomials of least deviation in two disjoint intervals. With the previous notation, our main result is:

THEOREM 1. *The extremum* $m_n \cdot n^2$ *in* (1.2) *is attained for* $x_0 = +1$ *and for the Zolotareff polynomials* $\pm u_n(x)$ [*or for* $x_0 = -1$ *and for* $\pm u_n(-x)$], *provided* n *is sufficiently large. Furthermore*

$$\lim_{n \to \infty} m_n = \mu \tag{1.3}$$

exists and

$$\mu = k^{-2}(1 - E/K)^2 = 0.3124 \cdots , \tag{1.4}$$

where k^2 *is the only root of the transcendental equation*

$$(K - E)^3 + (1 - k^2)K - (1 + k^2)E = 0 \tag{1.5}$$

satisfying the condition $0 < k^2 < 1$. *Here* K *and* E *are the complete elliptic integrals associated with the modulus* k.

A further analysis and discussion of a few special cases furnishes the more informative

THEOREM 2. *If* $n > 3$ *the extremum* $m_n \cdot n^2$ *in* (1.2) *is attained in the cases mentioned in Theorem* 1, *and only in these cases. If* $n = 3$, *it is attained for* $x_0 = 0$ *and for the Tchebycheff polynomials* $\pm T_3(x)$, *and only then.*

In §§2 and 3 of the present paper we first study as a preparation the general polynomials $u_n(x; A)$ of Zolotareff and the special case $u_n(x) = u_n(x; A_n)$ mentioned above. The proof of Theorem 1 is then given in §§4 and 5, and that of Theorem 2 in §§6 and 7. In §8 we consider two problems of Zolotareff in which the polynomials $u_n(x; A)$ were first used; §9 contains another application.

The polynomials of Zolotareff occur in numerous other related extremum problems. They satisfy a simple differential equation by means of which they can be brought in relationship with the multiplication problem of elliptic integrals. In what follows we have tried to reduce the use of elliptic functions to a minimum.[6]

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[4] W. Markoff, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen*, Mathematische Annalen, vol. 77 (1916), pp. 213–258. The Russian original appeared 1892.

[5] N. Achyeser, (a) *Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen*, Bulletin de l'Académie des Sciences de l'URSS, Classe des sciences mathématiques et naturelles, series 7, 1932, pp. 1163–1202; (b) *Über einige Funktionen, die in gegebenen Intervallen am wenigsten von Null abweichen*, Bulletin de la Société Physico-Mathématique de Kazan, series 3, vol. 3 (1928), pp. 1–69.

[6] Zolotareff and Achyeser make extensive use of the theory of elliptic functions; however, W. Markoff does not.

2. On the polynomials of Zolotareff

1. It is a classical fact that there is a unique polynomial $T_n(x)$ of degree n (the n^{th} polynomial of Tchebycheff) having the following property: The curve $y = T_n(x)$, $-1 \leqq x \leqq +1$, consists of n monotonic arcs varying between $+1$ and -1; $T_n(1) = 1$, and $T_n(-1) = (-1)^n$. This polynomial satisfies the differential equation

$$n^2(1 - y^2) = (1 - x^2)y'^2 \tag{2.1}$$

from which follows

$$y = \cos\left\{n \int_1^x (1 - t^2)^{-\frac{1}{2}}\, dt\right\}. \tag{2.2}$$

2. We show that there are infinitely many polynomials y of degree n possessing the following property: The curve y, $-1 \leqq x \leqq +1$, consists of $n - 1$ monotonic arcs varying between $+1$ and -1, $y = 1$ for $x = 1$, and $y = (-1)^{n-1}$ for $x = -1$. Such a curve necessarily has $n - 1$ roots in $-1 \leqq x \leqq +1$ and consequently one more outside this interval. If this additional root is >1, y satisfies a differential equation of the form

$$n^2(1 - y^2) = (1 - x^2)y'^2 \frac{(B - x)(C - x)}{(A - x)^2} \tag{2.3}$$

where $y' = 0$ for $x = A$, $y = 1$ for $x = B$, $y = -1$ for $x = C$, and $1 < A < B < C$. A similar differential equation holds if the additional root of y mentioned above is <-1. (The second case can be obtained from the first one by replacing x by $-x$.)

Solving the differential equation (2.3), we obtain

$$y = \cos\left\{n \int_1^x (A - t)(B - t)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(1 - t^2)^{-\frac{1}{2}}\, dt\right\}. \tag{2.4}$$

From the properties of y mentioned above we find

$$\int_{-1}^{+1} (A - t)(B - t)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(1 - t^2)^{-\frac{1}{2}}\, dt = (n - 1)\pi/n, \tag{2.5}$$

$$\int_{+1}^{B} (A - t)(B - t)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}}\, dt = 0, \tag{2.6}$$

$$\int_{B}^{C} (t - A)(t - B)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}}\, dt = \pi/n. \tag{2.7}$$

By a well-known application of Cauchy's theorem we see that the sum of the integrals (2.5) and (2.7) is π, so that (2.7) is a consequence of (2.5).

Conversely, if (2.5) and (2.6) hold, an easy discussion (encircling the singular points -1, $+1$, B, C) shows that (2.4) is an analytic function single-valued and regular in the whole finite x-plane. If $x \to \infty$ we find $y = O(|x|^n)$, so that y

must be a polynomial of degree n. Of course it satisfies the differential equation (2.3), and it has all the properties mentioned above.

For later purposes we note that

$$y' = n^2 \frac{(A-1)^2}{(B-1)(C-1)} \quad \text{at} \quad x = 1, \tag{2.8}$$

$$(-1)^n y' = n^2 \frac{(A+1)^2}{(B+1)(C+1)} \quad \text{at} \quad x = -1. \tag{2.9}$$

These values can be obtained from the differential equation (2.3).

3. LEMMA 1. *Of the three quantities A, B, C $(1 < A < B < C)$ satisfying the two transcendental equations* (2.5) *and* (2.6), *any one can be prescribed arbitrarily provided that*

$$A > 1, \text{ or } B > 1, \text{ or } C > c_n = 1 + 2\alpha_n = 1 + 2\tan^2[\pi/(2n)] \tag{2.10}$$

respectively; the two others are then uniquely determined. As A increases monotonically from 1 to $+\infty$, B and C increase likewise from 1 to $+\infty$ and from c_n to $+\infty$, respectively.

Furthermore the values of $y, y', \cdots, y^{(n)}$ for a fixed x not less than one, and the values of $(-1)^n y, (-1)^{n-1}y', \cdots, y^{(n)}$ for a fixed x not greater than -1, are increasing functions of A.

The only exceptions are $y = 1$ for $x = 1$ and $(-1)^n y = -1$ for $x = -1$. In particular, the expressions (2.8) and (2.9) are respectively increasing and decreasing functions of A.

In order to prove this Lemma, let B denote a fixed value, greater than 1, and let C be variable, such that $C > B$; we define $A = A(C)$ by (2.6) so that $1 < A < B$. Then

$$\int_1^B \left(\frac{dA}{dC} - \frac{1}{2}\frac{A-t}{C-t}\right)(B-t)^{-\frac{1}{2}}(C-t)^{-\frac{1}{2}}(t^2-1)^{-\frac{1}{2}}\,dt = 0;$$

hence

$$\frac{dA}{dC} = \frac{1}{2}\frac{A-t_0}{C-t_0}, \quad \text{where} \quad 1 < t_0 < B.$$

Now consider the function $\lambda(C)$ defined by the left-hand member of (2.5), where $A = A(C)$. We find

$$\begin{aligned}\lambda'(C) &= \int_{-1}^{+1} \left(\frac{dA}{dC} - \frac{1}{2}\frac{A-t}{C-t}\right)(B-t)^{-\frac{1}{2}}(C-t)^{-\frac{1}{2}}(1-t^2)^{-\frac{1}{2}}\,dt \\ &= \int_{-1}^{+1} \left(\frac{1}{2}\frac{A-t_0}{C-t_0} - \frac{1}{2}\frac{A-t}{C-t}\right)(B-t)^{-\frac{1}{2}}(C-t)^{-\frac{1}{2}}(1-t^2)^{-\frac{1}{2}}\,dt < 0,\end{aligned}$$

so that $\lambda(C)$ is decreasing. Let $C \to B$; then from (2.6) we see that $A \to B$, so that

$$\lambda(C) \to \int_{-1}^{+1} (1 - t^2)^{-\frac{1}{2}}\, dt = \pi.$$

Since $\lim_{C\to\infty} \lambda(C) = 0$, the equation $\lambda(C) = (n-1)\pi/n$ has precisely one solution.[7]

4. Further let $p(x)$ and $q(x)$ be two special cases of (2.4) corresponding to the values A', B', C' and A'', B'', C'' of A, B, C, respectively. First suppose that $p'(1) < q'(1)$. Considering the polynomial $\delta(x) = p(x) - q(x)$ at the n points in $-1 \leqq x \leqq +1$ at which $p(x) = \pm 1$ and assuming that $\delta(x) \not\equiv 0$, a familiar argument furnishes the existence of $n-1$ distinct points $+1 > \eta_1 > \eta_2 > \cdots > \eta_{n-1} > -1$ such that $\delta'(\eta_1) > 0$, $\delta'(\eta_2) < 0, \cdots$. Furthermore $\delta'(1) < 0$, so that $\delta'(x)$ has $n-1$ roots (that is, all its roots) in $-1 < x < +1$. The same holds for $\delta''(x)$, $\delta'''(x)$, $\cdots$ so that $\delta(x) < 0$, $\delta'(x) < 0$, $\delta''(x) < 0$, $\cdots$ for $x \geqq 1$ [except that $\delta(1) = 0$], and also $(-1)^n\delta(x) < 0$, $(-1)^{n-1}\delta'(x) < 0$, $(-1)^{n-2}\delta''(x) < 0, \cdots$ for $x \leqq -1$ [except that $\delta(-1) = 0$]. From this we easily conclude that the relations $A' < A''$, $B' < B''$, $C' < C''$ hold for the constants corresponding to $p(x)$ and $q(x)$.

If $p'(1) = q'(1)$ the previous argument still holds good [unless $\delta(x) \equiv 0$], except that $\delta'(1) = 0$ so that the roots of $\delta'(x)$ are in $-1 < x \leqq +1$. Consequently $\delta'(x) < 0$ for $x > 1$. Interchanging $p(x)$ and $q(x)$ we obtain $\delta'(x) > 0$, $x > 1$, which is a contradiction; so that in this case $p(x) \equiv q(x)$, $A' = A''$, $B' = B''$, $C' = C''$.

From the previous considerations we conclude that B and C are increasing functions of A. It remains to calculate the limits of B and C as $A \to 1$ and $A \to +\infty$. In the former case, (2.6) shows that $B \to 1$, and from (2.5) we obtain $C \to c_n$ since the equation

$$\int_{-1}^{+1} (1 + t)^{-\frac{1}{2}}(\gamma - t)^{-\frac{1}{2}}\, dt = (n-1)\pi/n$$

has the unique solution $\gamma = c_n$. If $A \to +\infty$ it is obvious that $B \to +\infty$, $C \to +\infty$. This completes the proof of Lemma 1.

5. In what follows we denote the polynomial (2.4) [for which (2.5) and (2.6) hold] by $y = u_n(x; A)$. We note that, from (2.4) and (2.10),

$$\begin{aligned} u_n(x; +1) &= \lim_{A\to 1} u_n(x; A) \\ &= \cos\left\{n \int_1^x (1+t)^{-\frac{1}{2}}(c_n - t)^{-\frac{1}{2}}\, dt\right\} = -T_n\left(\frac{x - \alpha_n}{1 + \alpha_n}\right). \end{aligned} \tag{2.11}$$

Hence $u_n'(+1; +1) = 0$. Also

$$u_n''(+1; +1) = -(1 + \alpha_n)^{-2}T_n''[\cos(\pi/n)] = -\tfrac{1}{4}n^2 \cot^2[\pi/(2n)]. \tag{2.12}$$

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[7] These considerations require only slight modifications if we replace the right-hand members in (2.5) and (2.6) by $\nu\pi/n$ and $\pi - \nu\pi/n$, $1 \leqq \nu \leqq n-1$. The resulting polynomials have been used for various purposes by Achyeser; see loc. cit.

Further, let $A \to +\infty$ so that $B \to +\infty$ and $C \to +\infty$. From (2.5)

$$(A - t_1)(B - t_1)^{-\frac{1}{2}}(C - t_1)^{-\frac{1}{2}} = (n-1)/n$$

where t_1 is a suitably chosen number between 0 and 1. Hence $A(BC)^{-\frac{1}{2}} \to 1 - 1/n$, so that, from (2.4),

$$u_n(x; +\infty) = \lim_{A \to \infty} u_n(x; A) = T_{n-1}(x). \tag{2.13}$$

Hence

$$u_n'(+1; +\infty) = (n-1)^2; \tag{2.14}$$

$$u_n''(+1; +\infty) = \tfrac{1}{3}n(n-1)^2(n-2). \tag{2.15}$$

Therefore, as A increases from 1 to $+\infty$, $u_n'(+1; A)$ increases from 0 to $(n-1)^2$, and $u_n''(+1; A)$ increases from the negative value (2.12) to the positive value (2.15), corresponding respectively to $A = 1$ and $A = +\infty$. There is precisely one value of A, $A = A_n$, for which $u_n''(+1; A_n) = 0$. We denote the corresponding values of B and C by B_n and C_n. In §§4 and 5 we shall prove that the function $u_n(x; A_n)$ furnishes the solution of I. Schur's problem formulated above, provided n is sufficiently large.

From the differential equation (2.3) we obtain

$$\begin{aligned} u_n''(+1; A) = \tfrac{1}{3}n^2 \frac{(A-1)^2}{(B-1)(C-1)} \Bigg\{ n^2 \frac{(A-1)^2}{(B-1)(C-1)} \\ - 1 - 2\left(\frac{2}{A-1} - \frac{1}{B-1} - \frac{1}{C-1}\right)\Bigg\}, \end{aligned} \tag{2.16}$$

so that the condition $u_n''(+1; A) = 0$ is equivalent to

$$n^2 \frac{(A-1)^2}{(B-1)(C-1)} = 1 + 2\left(\frac{2}{A-1} - \frac{1}{B-1} - \frac{1}{C-1}\right). \tag{2.17}$$

The transcendental equations (2.5), (2.6) and (2.17) determine the constants $A = A_n$, $B = B_n$, $C = C_n$ uniquely. These constants depend only on n.

The polynomial $y = u_n(x; A_n)$ is completely determined by the following conditions: The curve y, $-1 \leq x \leq +1$, consists of $n - 1$ monotonic arcs varying between $+1$ and -1, $y = 1$ for $x = 1$, $y = (-1)^{n-1}$ for $x = -1$ and $y'' = 0$ for $x = 1$.

3. The limiting process $n \to \infty$

1. First we prove the following

LEMMA 2. *The constants A_n, B_n, C_n defined by the transcendental equations (2.5), (2.6), (2.17) satisfy*

$$\begin{aligned} \lim_{n\to\infty} n^2(A_n - 1) = a^2/2, \qquad \lim_{n\to\infty} n^2(B_n - 1) = b^2/2, \\ \lim_{n\to\infty} n^2(C_n - 1) = c^2/2 \end{aligned} \tag{3.1}$$

where $0 < a < b < c$. The numerical values of a, b, c are given in (3.17).

By Lemma 1

$$\liminf_{n\to\infty} n^2(C_n - 1) \geqq \pi^2/2$$

and from (2.17)

$$n^2 \frac{A_n - 1}{C_n - 1} \geqq \frac{2}{A_n - 1} - \frac{2}{C_n - 1},$$

$$n^4(A_n - 1)^2 + 2n^2(A_n - 1) \geqq 2n^2(C_n - 1), \tag{3.2}$$

so that

$$\liminf_{n\to\infty} n^2(A_n - 1) \geqq (1 + \pi^2)^{\frac{1}{2}} - 1. \tag{3.3}$$

The same inequality holds if we replace A_n by B_n.

On the other hand, let us assume that $n^2(C_n - 1) \to +\infty$ for a proper subsequence $n = n_\nu$ as $\nu \to \infty$; then, from (3.2), $n^2(A_n - 1) \to +\infty$, so that $n^2(B_n - 1) \to +\infty$. Therefore, by (2.17), for the same subsequence $n = n_\nu$,

$$\frac{(A_n - 1)^2}{(B_n - 1)(C_n - 1)} \to 0, \tag{3.4}$$

Now let ω be a fixed positive number; for large n, from (2.5),

$$\begin{aligned} \pi &= n \int_{-1}^{+1} (1 - t^2)^{-\frac{1}{2}}\, dt \{1 - (A_n - t)(B_n - t)^{-\frac{1}{2}}(C_n - t)^{-\frac{1}{2}}\} \\ &> n \int_0^{\omega/n} d\varphi \{1 - (A_n - \cos\varphi)(B_n - \cos\varphi)^{-\frac{1}{2}}(C_n - \cos\varphi)^{-\frac{1}{2}}\} \qquad (3.5) \\ &= \int_0^{\omega} d\psi \{1 - (A_n - \cos(\psi/n))(B_n - \cos(\psi/n))^{-\frac{1}{2}}(C_n - \cos(\psi/n))^{-\frac{1}{2}}\}. \end{aligned}$$

Since

$$\begin{aligned} &(A_n - \cos(\psi/n))(B_n - \cos(\psi/n))^{-\frac{1}{2}}(C_n - \cos(\psi/n))^{-\frac{1}{2}} \\ &\qquad \leqq (A_n - 1)(B_n - 1)^{-\frac{1}{2}}(C_n - 1)^{-\frac{1}{2}}\left\{1 + \frac{1 - \cos(\psi/n)}{A_n - 1}\right\} \end{aligned}$$

and since for $n = n_\nu$, as $\nu \to \infty$,

$$\frac{n^2(1 - \cos(\psi/n))}{n^2(A_n - 1)} \to 0$$

uniformly in ψ, for $0 \leqq \psi \leqq \omega$, we find $\pi \geqq \omega$. This is a contradiction if we choose $\omega > \pi$. Thus we have proved that the points of accumulation of the sequences $n^2(A_n - 1)$, $n^2(B_n - 1)$, $n^2(C_n - 1)$ are positive and finite.

2. Now let $n = n_\nu$ be a subsequence for which the limits (3.1) exist, where

$0 < a \leqq b \leqq c < +\infty$. From (2.5), (2.6) and (2.7) we shall derive $a < b < c$ and

$$\int_0^\infty \{1 - (a^2 + u^2)(b^2 + u^2)^{-\frac{1}{2}}(c^2 + u^2)^{-\frac{1}{2}}\}\, du = \pi, \tag{3.6}$$

$$\int_0^b (a^2 - u^2)(b^2 - u^2)^{-\frac{1}{2}}(c^2 - u^2)^{-\frac{1}{2}}\, du = 0, \tag{3.7}$$

$$\int_b^c (u^2 - a^2)(u^2 - b^2)^{-\frac{1}{2}}(c^2 - u^2)^{-\frac{1}{2}}\, du = \pi. \tag{3.8}$$

Also from (2.17) by the same limiting process ($n = n_\nu$, $\nu \to \infty$),

$$\frac{a^4}{b^2 c^2} - 4\left(\frac{2}{a^2} - \frac{1}{b^2} - \frac{1}{c^2}\right) = 0. \tag{3.9}$$

Instead of (3.7) we can show more precisely

$$\begin{cases} n \displaystyle\int_1^{A_n} (A_n - t)(B_n - t)^{-\frac{1}{2}}(C_n - t)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}}\, dt \\ \qquad\qquad \to \displaystyle\int_0^a (a^2 - u^2)(b^2 - u^2)^{-\frac{1}{2}}(c^2 - u^2)^{-\frac{1}{2}}\, du, \\ n \displaystyle\int_{A_n}^{B_n} (t - A_n)(B_n - t)^{-\frac{1}{2}}(C_n - t)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}}\, dt \\ \qquad\qquad \to \displaystyle\int_a^b (u^2 - a^2)(b^2 - u^2)^{-\frac{1}{2}}(c^2 - u^2)^{-\frac{1}{2}}\, du, \end{cases} \tag{3.10}$$

the two limits being the same.

First, (3.9) is obvious and this equation shows that $a = b = c$ is impossible. In case $a < b = c$ both formulas (3.10) follow easily [writing $t = 1 + u^2/(2n^2)$]; but the first limit is finite and the second one turns out to be $+\infty$, which is a contradiction. In case $a = b < c$ the same formulas can be easily established again, but the first limit is positive whereas the second one is 0 [since $\max\{(t - A_n)(C_n - t)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}}\}$, $A_n \leqq t \leqq B_n$, is bounded]. Therefore $a < b < c$.

Now (3.7) and (3.8) follow directly, and (3.6) can also be easily obtained. However (3.6) follows also from (3.8) by applying Cauchy's theorem to

$$f(z) = 1 - (a^2 - z^2)(b^2 - z^2)^{-\frac{1}{2}}(c^2 - z^2)^{-\frac{1}{2}}$$

integrated along the half-circle $|z| = R$, $\Re z \geqq 0$ and along the segment $\Re z = 0$, $-R \leqq \Im z \leqq +R$, $R \to +\infty$.

3. Substituting $u^2 = b^2 \sin^2 \varphi$ in (3.7) and $u^2 = c^2 - (c^2 - b^2)\sin^2\varphi$ in (3.8) we find

$$\int_0^{\pi/2} (a^2 - b^2 \sin^2 \varphi)(c^2 - b^2 \sin^2 \varphi)^{-\frac{1}{2}}\, d\varphi = 0, \tag{3.11}$$

$$(3.12)\qquad \int_0^{\pi/2} \{c^2 - a^2 - (c^2 - b^2)\sin^2\varphi\}\{c^2 - (c^2 - b^2)\sin^2\varphi\}^{-\frac{1}{2}}\,d\varphi = \pi.$$

Using the standard notation these equations can be written in the form

$$(3.13)\qquad (1 - a^2/c^2)K = E, \qquad cE' - (a^2/c)K' = \pi$$

where the complete elliptic integrals K and E belong to the modulus $k = b/c$. Eliminating a^2/c^2 we find

$$(3.14)\qquad E/K + (E' - \pi/c)/K' = 1.$$

Comparing this with the classical equation[8]

$$(3.15)\qquad EK' + E'K - KK' = \pi/2$$

we obtain $c = 2K$. Hence

$$(3.16)\qquad a^2 = 4K(K - E), \qquad b = 2kK, \qquad c = 2K.$$

The relation (3.9) furnishes the transcendental equation (1.5) of Theorem 1 (see §1) for the modulus k. This equation has precisely one root as k^2 goes from 0 to 1 [which shows that the limits (3.1) exist as $n \to \infty$ unrestrictedly]. Indeed, differentiating the left-hand member of (1.5) with respect to k^2,[9] we have

$$\tfrac{3}{2}E\{k'^{-2}(K - E)^2 - 1\},$$

where k' is the complementary modulus. The expression in the curly bracket increases with k^2, as the well-known power series expansion of K and E shows; it is negative for small k^2 and positive as k^2 approaches 1. Therefore the left-hand member of (1.5) first decreases and then increases; but for $k^2 = 0$ it is zero and for $k^2 \to 1 - 0$ it tends to $+\infty$. This establishes Lemma 2.

Using the tables of Milne-Thomson[10] we find

$$(3.17)\qquad \begin{aligned} &k^2 = 0.84\cdots, \quad a^2 = 11.4055\cdots, \quad b = 4.3245\cdots, \\ &\qquad\qquad\qquad c = 4.7185\cdots, \quad a^4/b^2c^2 = 0.3124\cdots. \end{aligned}$$

We also note that (2.4) implies that

$$(3.18)\qquad \lim_{n\to\infty} u_n(\cos(z/n); A_n) = \cos\left\{\int_0^z (a^2 + u^2)(b^2 + u^2)^{-\frac{1}{2}}(c^2 + u^2)^{-\frac{1}{2}}\,du\right\}$$

uniformly in z, for all complex z such that $|z| \leqq R$.

4. Another limiting formula important for the proof of Theorem 1, is

LEMMA 3. *Let $A = A_n'$ be a sequence of values such that $A_n' - 1 = o(n^{-2})$.*

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[8] See for instance, E. T. Whittaker and G. N. Watson, *A course of Modern Analysis*, Fourth edition, 1935, p. 520.

[9] See Whittaker-Watson, loc. cit. p. 521.

[10] L. M. Milne-Thomson, *Ten-figure table of the complete elliptic integrals K, K', E, E' and a table of* $1/\vartheta_3^2(0\mid\tau)$, $1/\vartheta_3^2(0\mid\tau')$, Proceedings of the London Mathematical Society, series 2, vol. 33 (1932), pp. 160–164.

Denoting the corresponding values of B and C determined from the equations (2.5) *and* (2.6) *by B_n' and C_n', respectively, we have*

$$\lim_{n\to\infty} u_n(\cos(z/n); A_n') = -\cos\{(\pi^2 + z^2)^{\frac{1}{2}}\}. \tag{3.19}$$

The last equation holds uniformly in z, for all complex z such that $|z| \leqq R$.

We note that (3.19) arises from (3.18) on writing $a = b = 0$, $c = \pi$.

For the proof we use an argument similar to that of Part 1. Let ω be fixed, $\omega > 0$; we find [see (3.5)]

$$\pi > \int_0^\omega d\psi\{1 - (A_n' - \cos(\psi/n))(B_n' - \cos(\psi/n))^{-\frac{1}{2}}(C_n' - \cos(\psi/n))^{-\frac{1}{2}}\}. \tag{3.20}$$

Assuming for a certain subsequence $n = n_\nu$, $\nu \to \infty$, that the limits

$$\lim n^2(B_n' - 1) = \beta, \qquad \lim n^2(C_n' - 1) = \gamma$$

exist, we have $\beta \geqq 0$, $\gamma \geqq \pi^2/2$. Thus we conclude from (3.20)

$$\pi \geqq \int_0^\omega d\psi\{1 - (\psi^2/2)(\beta + \psi^2/2)^{-\frac{1}{2}}(\gamma + \psi^2/2)^{-\frac{1}{2}}\},$$

so that

$$\pi \geqq \int_0^\infty d\psi\{1 - (\psi^2/2)(\beta + \psi^2/2)^{-\frac{1}{2}}(\gamma + \psi^2/2)^{-\frac{1}{2}}\}. \tag{3.21}$$

Now

$$\pi = \int_0^\infty d\psi\{1 - \psi(\pi^2 + \psi^2)^{-\frac{1}{2}}\}; \tag{3.22}$$

consequently (3.21) and (3.22) involve a contradiction, unless $\beta = 0$, $\gamma = \pi^2/2$.

Further

$$\begin{aligned} &u_n(\cos(z/n); A_n') \\ &\quad = \cos\left\{\int_0^z (A_n' - \cos(\psi/n))(B_n' - \cos(\psi/n))^{-\frac{1}{2}}(C_n' - \cos(\psi/n))^{-\frac{1}{2}}\, d\psi\right\}. \end{aligned} \tag{3.23}$$

Now let $0 < \epsilon < \pi < R$ and $|z| = R$. Then

$$\begin{aligned} &\int_0^\epsilon (A_n' - \cos(\psi/n))(B_n' - \cos(\psi/n))^{-\frac{1}{2}}(C_n' - \cos(\psi/n))^{-\frac{1}{2}}\, d\psi \\ &\qquad \leqq \int_0^\epsilon (A_n' - \cos(\psi/n))^{\frac{1}{2}}(C_n' - \cos(\psi/n))^{-\frac{1}{2}}\, d\psi \to \int_0^\epsilon \psi(\pi^2 + \psi^2)^{-\frac{1}{2}}\, d\psi \end{aligned}$$

as $n \to \infty$; the last integral is arbitrarily small with ϵ. Integrating from ϵ to z, we can assume that $\psi \neq 0, \pm\pi$ on the path of integration; and the assertion follows immediately from (3.23) for $n \to \infty$.

4. Proof of Theorem 1

In what follows, the symbols Q_n, $Q_n(x_0)$ defined in §1 are used.

1. LEMMA 4. *Suppose* $-1 \leqq x_0 \leqq +1$, *and let* $f_0(x)$ *be a polynomial of the class* $Q_n(x_0)$ *for which* $\max |f'(x_0)|$, $f(x) \in Q_n(x_0)$, *is attained. Then* $|f_0(x)|$ *assumes its maximum* 1 *at least n times in* $-1 \leqq x \leqq +1$.

The proof follows the usual lines. Let $f_0'(x_0) > 0$ and let us suppose that the assertion of Lemma 4 does not hold. Denote by $x_1, x_2, \cdots, x_l$; $l < n$, the distinct values in $-1 \leqq x \leqq +1$ for which $|f_0(x_\nu)| = 1$ and write $\omega(x) = \prod_{\nu=1}^{l}(x - x_\nu)$. If $-1 < x_0 < +1$ we have $x_0 \neq x_\nu$ [otherwise $f_0'(x_0)$ would be 0]. However if $x_0 = \pm 1$ we may have $x_0 = x_\nu$, in which case $\omega(x_0) = 0$ but $\omega'(x_0) \neq 0$.

We form the polynomial

$$(4.1) \qquad r(x) = -\sum_{\nu=1}^{l} \operatorname{sgn} f_0(x_\nu) \frac{\omega(x)}{\omega'(x_\nu)(x - x_\nu)} + \omega(x)\{a(x - x_0) + b\}$$

and want to determine the constants a and b such that $r'(x_0) > 0$, $r''(x_0) = 0$; this can certainly be done provided the linear equations

$$a\omega(x_0) + b\omega'(x_0) = G,$$

$$2a\omega'(x_0) + b\omega''(x_0) = H$$

have a determinant $\neq 0$. Now $\omega(x_0)\omega''(x_0) - 2\{\omega'(x_0)\}^2 \neq 0$ is obvious if $\omega(x_0) = 0$ (cf. above); but if $\omega(x_0) \neq 0$,

$$\frac{\omega''(x_0)}{\omega(x_0)} - 2\left\{\frac{\omega'(x_0)}{\omega(x_0)}\right\}^2 = -\sum_{\nu=1}^{l}(x_0 - x_\nu)^{-2} - \left\{\frac{\omega'(x_0)}{\omega(x_0)}\right\}^2 < 0.$$

Obviously $r(x)$ is of degree $l + 1 \leqq n$ and we find for sufficiently small $\epsilon > 0$ that $|f_0(x) + \epsilon r(x)| \leqq 1$ in $-1 \leqq x \leqq +1$; hence $f_0(x) + \epsilon r(x)$ belongs to $Q_n(x_0)$. On the other hand $f_0'(x_0) + \epsilon r'(x_0) > f_0'(x_0)$ which is a contradiction. This proves Lemma 4.

2. Let the extremum (1.2) be attained for the value x_0 and for $f(x) = f_0(x)$, $f_0(x) \in Q_n(x_0)$. Then $f_0''(x_0) = 0$, and $f_0(x)$ possesses the property formulated in Lemma 4. Further we show that $f_0'''(x_0) \neq 0$. By Lemma 4, $|f_0(x)|$ attains its relative maximum 1 in $-1 < x < +1$ for at least $n - 2$ distinct points for which $f_0'(x) = 0$. Since $f_0''(x)$ vanishes an odd number of times between two consecutive roots of $f_0'(x)$, we find that $f_0''(x)$ has precisely one simple root between two consecutive roots of $f_0'(x)$, and these roots of $f_0''(x)$ are maximum points of $|f_0'(x)|$. The number of these maximum points is at least $n - 3$. If x_0 is one of these points, we must have $f_0'''(x_0) \neq 0$. If x_0 is different from these maximum points (whose number in this case is $n - 3$), then we must have again $f_0'''(x_0) \neq 0$, and thus there is a relative maximum of $|f_0'(x)|$ at $x = x_0$.

If we assume that $f_0'(x_0) > 0$ then $f_0''(x_0) = 0$, $f_0'''(x_0) < 0$, so that $f_0'(x)$ has a relative maximum at $x = x_0$.

Now we distinguish various cases.

(a) $x_0 = \pm 1$.

Let $x_0 = +1$ and let us denote an extremum polynomial of our problem by $u_n(x)$, $u_n'(1) > 0$, $u_n''(1) = 0$. As we showed before, $u_n'(x)$ has at least $n - 2$ and $u_n''(x)$ at least $n - 3$ distinct roots in $-1 < x < +1$. Since $u_n''(1) = 0$, we find that $n - 2$ is the precise number of roots of $u_n'(x)$ in $-1 < x < +1$. Consequently $| u_n(-1) | = | u_n(+1) | = 1$; and, since $u_n'(1) > 0$, we find $u_n(1) = 1$, $u_n(-1) = (-1)^{n-1}$.

Thus the curve $y = u_n(x)$, $-1 \leqq x \leqq +1$, consists of $n - 1$ monotonic arcs varying between $+1$ and -1, and $u_n(1) = 1$, $u_n(-1) = (-1)^{n-1}$, $u_n'(1) > 0$, $u_n''(1) = 0$.

Hence from the last remark of §2 we conclude that $u_n(x)$ is identical with the polynomial $u_n(x; A_n)$ defined there.

Consequently, under the assumption $x_0 = \pm 1$, the extremum polynomials of our problem are $\pm u_n(x; A_n)$ and $\pm u_n(-x; A_n)$, respectively. The asymptotic value of $| u_n'(1; A_n) |$ is $a^4 b^{-2} c^{-2} \cdot n^2$ [see (3.17)].

(b) $-1 < x_0 < +1$, and there exists a polynomial $g(x)$ of Q_n for which $| g'(x_0) | > | f_0'(x_0) |$. Suppose $f_0'(x_0) > 0$, $g'(x_0) > 0$.

Consider the polynomial $h_\epsilon(x) = f_0(x) + \epsilon\{g(x) - f_0(x)\}$, $0 < \epsilon < 1$. Obviously $h_\epsilon(x) \epsilon Q_n$; furthermore $h_\epsilon'(x_0) > f_0'(x_0)$. For sufficiently small ϵ there is a root of $h_\epsilon''(x)$ in the neighborhood of x_0, x_0' say, and $h_\epsilon'(x)$ attains a positive relative maximum at $x = x_0'$. We evidently have

$$h_\epsilon'(x_0') \geqq h_\epsilon'(x_0) > f_0'(x_0)$$

which shows that $f_0(x)$ cannot be the extremum polynomial.

5. Proof of Theorem 1 (continued)

The remaining case requires a more elaborate discussion. This case is:

(c) $-1 < x_0 < +1$ and $f_0(x)$ is the polynomial in Q_n with the maximum value of $f'(x_0)$.

Then W. Markoff has shown[11] that $f_0(x)$ must be one of the polynomials

$$\begin{gathered} \pm T_n(x), \qquad \pm T_{n-1}(x), \qquad \pm T_n\left(\frac{x-\alpha}{1+\alpha}\right), \qquad \pm T_n\left(\frac{x+\alpha}{1+\alpha}\right), \\ \pm u_n(\pm x; A) \end{gathered} \tag{5.1}$$

where $0 < \alpha < \alpha_n = \tan^2[\pi/(2n)]$, and $u_n(x; A)$ are the Zolotareff polynomials defined and discussed above. As $n \to \infty$, the largest relative maximum of $| T_n'(x) |$ in $-1 < x < +1$ is asymptotically $M \cdot n^2$ where $-M$ is the minimum of $\sin \theta/\theta$ for real θ, that is $M = 0.2172 \cdots$. Comparing this result with the asymptotic value of $u_n'(1; A_n)$, that is with $a^4 b^{-2} c^{-2} \cdot n^2$ [see (3.17)], we see that for large values of n the four first types in (5.1) can be excluded.

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[11] Loc. cit. p. 249.

As W. Markoff has further shown[12], $f_0(x) = \pm u_n(x; A)$ if and only if (a) x_0 belongs to certain open intervals in $-1 < x < +1$, and (b):

$$\text{(5.2)} \qquad \frac{d}{dx}\left\{\frac{(1 - x^2)u_n'(x; A)}{x - A}\right\} = 0 \qquad \text{at} \quad x = x_0.$$

Since $u_n'(x_0 ; A) \neq 0$, and $u_n''(x_0 ; A) = 0$, the latter-mentioned condition implies that

$$\text{(5.3)} \qquad x_0 = A - (A^2 - 1)^{\frac{1}{2}}, \qquad A - 1 = (1 - x_0)^2/(2x_0),$$

so that $0 < x_0 < +1$. Now we distinguish again two cases:

(c′) $0 < x_0 \leqq (1 - 16n^{-2})^{\frac{1}{2}}$. According to S. Bernstein's theorem

$$\text{(5.4)} \qquad | u_n'(x ; A) | \leqq n(1 - x^2)^{-\frac{1}{2}} \leqq n^2/4.$$

(c″) $(1 - 16n^{-2})^{-\frac{1}{2}} < x_0 < 1$. Then $A - 1 = A_n' - 1 = O(n^{-4})$. Now we assume that this case occurs for an infinite number of values of n, and we write $x_0 = \cos(z_0/n)$; then z_0 is bounded. From Lemma 3 we conclude that

$$\text{(5.5)} \qquad \lim n^{-2} u_n' (\cos (z/n); A_n') = -\frac{\sin\{(\pi^2 + z^2)^{\frac{1}{2}}\}}{(\pi^2 + z^2)^{\frac{1}{2}}}.$$

The maximum of the absolute value of the last expression for real z is $M = 0.2172 \cdots$ so that this case can be also eliminated.

The assumption $f_0(x) = \pm u_n(-x; A)$ can be dealt with similarly.

Thus for large n only Case (a) remains. This completes the proof of Theorem 1.

6. Proof of Theorem 2

1. First we consider again the case (c) defined in §5 and let x_0 belong to one of the open intervals in $-1 \leqq x \leqq +1$ in which the maximum of $f'(x_0), f(x) \in Q_n$, is attained for the Zolotareff polynomial $f(x) = u_n(x; A)$. [The argument is similar for $-u_n(x; A)$ or $\pm u_n(-x; A)$.] Then $f(x) = u_n(x; A) = f_0(x)$, where $f_0(x)$ has the same meaning as in §§4 and 5, so that $f_0(x) \in Q_n(x_0)$; that is, $f_0''(x_0) = 0$. We have $f_0'(x_0) > 0$, $f_0'''(x_0) < 0$.

By an important theorem of W. Markoff[13], to every positive ϵ correspond values x_1 such that[14]

(α) $0 < |x_1 - x_0| < \epsilon$;

(β) if $f_1(x) = u_n(x; A')$ denotes the polynomial of Q_n for which $f'(x_1)$ becomes a maximum, then

$$\text{(6.1)} \qquad f_1'(x_1) > f_0'(x_0).$$

[12] Loc. cit. pp. 233–246.

[13] Loc. cit. p. 257.

[14] In fact, a whole half-neighborhood of x_0 satisfies this condition.

Now if ϵ is sufficiently small, $f_1''(x)$ will have a root, say x_1', in the neighborhood of x_0; we can assume that $-1 < x_1' < +1$. Also $f_1'''(x_1') < 0$, so that $f_1'(x)$ has a relative maximum at $x = x_1'$; hence

$$(6.2) \qquad f_1'(x_1') \geqq f_1'(x_1) > f_0'(x_0),$$

which shows that $f_0(x)$ can not be the solution of our problem.

This argument leaves as the only possibilities for $f_0(x)$ either the Zolotareff polynomials $\pm u_n(x; A_n)$ with $x_0 = \pm 1$, or the Tchebycheff polynomials $\pm T_n(x)$.

2. Let D_n be the largest root of $u_n(x; A_n)$, $B_n < D_n < C_n$. Using the convexity of $u_n(x; A_n)$ for $x > 1$, we deduce

$$(6.3) \qquad D_n - B_n > C_n - D_n .$$

Further we make use of a theorem of I. Schur on the largest roots of the derivatives of an algebraic equation with only real roots.[15] Applying this theorem to $u_n(x; A_n)$ we obtain

$$(6.4) \qquad D_n - A_n \leqq A_n - 1$$

so that

$$2(A_n - 1) \geqq D_n - 1 > \tfrac{1}{2}(B_n - 1 + C_n - 1) > \{(B_n - 1)(C_n - 1)\}^{\frac{1}{2}}.$$

Hence, from (2.8),

$$(6.5) \qquad u_n'(1; A_n) > n^2/4.$$

3. On the other hand we show that

$$(6.6) \qquad |\, T_n'(x)\,| \leqq n^2/4 \qquad \text{if} \qquad T_n''(x) = 0$$

provided $n \geqq 5$ (with equality only if $n = 5$). Incidentally, I. Schur has proved (6.6) for all large n.[16]

Let φ be a root of the equation $\tan n\varphi = n \tan \varphi$, $0 < \varphi < \pi/2$. Then the assertion is

$$(6.7) \quad n \left| \frac{\sin n\varphi}{\sin \varphi} \right| = n^2(n^2 \sin^2 \varphi + \cos^2 \varphi)^{-\frac{1}{2}} \leqq n^2/4, \qquad \sin \varphi \geqq \left(\frac{15}{n^2 - 1}\right)^{\frac{1}{2}}.$$

It is sufficient to show this for the largest root $x_n = \cos \varphi_n$ of $T_n''(x)$; that is, for the smallest positive value φ_n, $\pi < n\varphi_n < 3\pi/2$, satisfying the equation above.

The function

$$(6.8) \qquad h(\psi) = \frac{\tan n\psi}{n \tan \psi}$$

[15] I. Schur, *Zwei Sätze über algebraische Gleichungen mit lauter reellen Wurzeln*, Journal für die reine und angewandte Mathematik, vol. 144 (1914), pp. 75–88.

[16] I. Schur, loc. cit.[2], p. 277.

increases from 0 to $+\infty$ as ψ increases from π/n to $3\pi/(2n)$. Let z be the smallest positive root of the equation $\tan z = z$, $\pi < z < 3\pi/2$. Since

$$h(z/n) = \frac{z}{n \tan (z/n)} < 1, \tag{6.9}$$

we have $\varphi_n > z/n$, so that (6.7) follows from

$$\sin (z/n) \geqq \left(\frac{15}{n^2 - 1}\right)^{\frac{1}{2}}. \tag{6.10}$$

Since $n \sin (z/n)$ increases and $n^2/(n^2 - 1)$ decreases as n increases, the last inequality will be proved for $n \geqq 6$ if we prove it for $n = 6$. But

$$\sin (z/6) \geqq (3/7)^{\frac{1}{2}} = 0.6546 \cdots, \tag{6.11}$$

since[17] $z = 4.4934 \cdots$ and $\sin (z/6) = 0.6808 \cdots$.

In the case $n = 5$ we have

$$T_5''(x) = 320x^3 - 120x, \quad x_5 = \cos \varphi_5 = (3/8)^{\frac{1}{2}}, \quad \sin \varphi_5 = (5/8)^{\frac{1}{2}}. \tag{6.12}$$

Comparing (6.5) and (6.6) we obtain $\pm u_n(\pm x; A_n)$ as the only eligible extremum polynomials [and $x_0 = \pm 1$ as the points at which the extremum is obtained] provided $n \geqq 5$.

7. Proof of Theorem 2 (continued)

The previous result holds also for $n = 4$, as a direct discussion shows; however, it fails for $n = 3$.

1. We have for $n = 4$:

$$T_4(x) = 8x^4 - 8x^2 + 1, \quad T_4'(x) = 32x^3 - 16\, x, \quad T_4''(x) = 96x^2 - 16, \tag{7.1}$$

so that, with the same notation as before, $x_4 = 6^{-\frac{1}{2}}$ and

$$| T_4'(x_4) | = (16/3)(2/3)^{\frac{1}{2}} = 4.3546 \cdots. \tag{7.2}$$

On the other hand, let us denote by y_1 and y_2, the values of x for which the relative extrema of $u_4(x; A_4)$ in $-1 \leqq x \leqq +1$ are attained; thus $-1 < y_1 < y_2 < +1$, say. Then

$$u_4(x; A_4) = 1 - \lambda(1 - x)(B_4 - x)(y_1 - x)^2 \tag{7.3}$$

must satisfy the following conditions:

$$(7.4) \quad \begin{cases} (\alpha): \quad u_4(-1; A_4) = -1, & \lambda(B_4 + 1)(y_1 + 1)^2 = 1, \\ (\beta): \quad u_4(y_2; A_4) = -1, & \lambda(1 - y_2)(B_4 - y_2)(y_1 - y_2)^2 = 2, \\ (\gamma): \quad u_4'(y_2; A_4) = 0, & \dfrac{1}{y_2 - 1} + \dfrac{1}{y_2 - B_4} + \dfrac{2}{y_2 - y_1} = 0, \\ (\delta): \quad u_4''(1; A_4) = 0, & 2B_4 + y_1 = 3. \end{cases}$$

[17] See, for instance, E. Jahnke-F. Emde, Funktionentafeln, 1933, p. 30.

Hence $B_4 < 2$. Let

$$(7.5)\qquad \begin{aligned} 1 - y_2 = h(B_4 - 1), \qquad B_4 - y_2 = (h + 1)(B_4 - 1),\\ y_2 - y_1 = (2 - h)(B_4 - 1); \end{aligned}$$

then (γ) becomes:

$$(7.6)\qquad \frac{1}{h} + \frac{1}{h+1} + \frac{2}{h-2} = 0; \quad \text{i.e.,} \quad h = (1 + (33)^{\frac{1}{2}})/8 = 0.8430\cdots.$$

Further, writing $v(x) = x(x + 1)(x - 2)^2$, we obtain from (α) and (β)

$$(7.7)\qquad v\left(\frac{2}{B_4 - 1}\right) = v(h).$$

Since $v(x) = v(h)$ has h as a double root, it can be reduced to a quadratic equation giving

$$(7.8)\qquad \frac{2}{B_4 - 1} = 3/2 - h + \tfrac{1}{2}(10h + 5)^{\frac{1}{2}} = 2.4893\cdots.$$

Now

$$(7.9)\qquad u_4'(1; A_4) = \lambda(B_4 - 1)(1 - y_1)^2 = 4\,\frac{\dfrac{2}{B_4 - 1}}{v\left(\dfrac{2}{B_4 - 1}\right)} = 4.7881\cdots.$$

Comparison of this value with (7.2) furnishes $u_4(x; A_4)$ as the solution.

2. Finally in the case $n = 3$,

$$(7.10)\qquad \begin{aligned} T_3(x) = 4x^3 - 3x, \qquad T_3'(x) = 12x^2 - 3, \qquad T_3''(x) = 24x,\\ x_3 = 0, \qquad |\,T_3'(x_3)\,| = 3. \end{aligned}$$

On the other hand,

$$(7.11)\qquad u_3(x; A_3) = 1 - \lambda(1 - x^2)(B_3 - x)$$

with a relative minimum at $x = y_1$, $-1 < y_1 < +1$, satisfies the following conditions:

$$(7.12)\qquad \begin{aligned} &(\alpha)\colon u_3(y_1\,; A_3) = -1, && \lambda(1 - y_1^2)(B_3 - y_1) = 2,\\ &(\beta)\colon u_3'(y_1\,; A_3) = 0, && 3y_1^2 - 2B_3y_1 - 1 = 0,\\ &(\gamma)\colon u_3''(1; A_3) = 0, && B_3 = 3, \end{aligned}$$

so that

$$(7.13)\qquad \begin{cases} y_1 = 1 - 2\cdot 3^{-\frac{1}{2}}, \qquad \lambda = 3^{\frac{3}{2}}/8,\\ u_3(x; A_3) = 1 - 3^{\frac{3}{2}}(1 - x^2)(3 - x)/8, \quad u_3'(1; A_3) = 3^{\frac{3}{2}}/2 < 3. \end{cases}$$

This completes the proof of Theorem 2.

8. Two problems of Zolotareff

1. The previous considerations permit a very simple approach to the following interesting theorem of Zolotareff:[18]

THEOREM 3. *Let σ be a given positive number and $f(x)$ an arbitrary polynomial of degree n of the form*

$$f(x) = x^n - \sigma x^{n-1} + \cdots . \tag{8.1}$$

Then $\max |f(x)|$, $-1 \leqq x \leqq +1$, *is minimized if and only if*

(a) $f(x) = \text{const. } u_n(x; A)$ *provided* $\sigma \geqq n\alpha_n$,

(b) $f(x) = 2^{1-n}(1 + \sigma/n)^n T_n\left(\dfrac{x - \sigma/n}{1 + \sigma/n}\right)$ *provided* $0 < \sigma \leqq n\alpha_n$.

Here $u_n(x; A)$ denotes the polynomial (2.4); *and in case* (a) $A = A(\sigma)$ *is a uniquely determined function which increases monotonically from 1 to $+\infty$ as σ increases from $n\alpha_n$ to $+\infty$; $\alpha_n = \tan^2 [\pi/(2n)]$.*

A corresponding result holds for negative σ, obtained by replacing $f(x)$ by $(-1)^n f(-x)$. For $\sigma = 0$ the extremum is given by Tchebycheff's polynomial.

From (2.4) we obtain, for $x > C$,

$$\begin{aligned} -u_n(x; A) &= Rx^n - Sx^{n-1} + \cdots \\ &= \cosh\left\{n \int_C^x (t - A)(t - B)^{-\frac{1}{2}}(t - C)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}}\, dt\right\} \\ &= \cosh\left\{n (\log x - \log C) \right. \\ &\qquad + n \int_C^\infty [(t - A)(t - B)^{-\frac{1}{2}}(t - C)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}} - t^{-1}]\, dt \\ &\qquad \left. - n \int_x^\infty [(t - A)(t - B)^{-\frac{1}{2}}(t - C)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}} - t^{-1}]\, dt\right\}; \end{aligned}$$

so that, as $x \to +\infty$,

$$\begin{aligned} -u_n(x; A) = \tfrac{1}{2}(x/C)^n \exp&\left\{n \int_C^\infty [(t - A)(t - B)^{-\frac{1}{2}}(t - C)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}} - t^{-1}]\, dt \right. \\ &\left. - n \int_x^\infty [(\tfrac{1}{2}(B + C) - A)t^{-2} + O(t^{-3})]\, dt\right\} + O(x^{-n}). \end{aligned}$$

Consequently

$$R = \tfrac{1}{2}C^{-n} \times$$
$$\exp\left\{n \int_C^\infty [(t - A)(t - B)^{-\frac{1}{2}}(t - C)^{-\frac{1}{2}}(t^2 - 1)^{-\frac{1}{2}} - t^{-1}\, dt\right\} > 0, \tag{8.2}$$
$$S/R = n\{\tfrac{1}{2}(B + C) - A\} > 0,$$

so that R and S are continuous functions of A.

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[18] Loc. cit.[3] (a), (b), (c).

From the results of Lemma 1,

$$\left(\frac{d}{dx}\right)^n u_n(x; A) = -n!R, \tag{8.3}$$

is an increasing function of A. Let $A_1 < A_2$, and let R_1, S_1, R_2, S_2 be the corresponding values of R and S, $R_1 > R_2$. Considering $R_1^{-1}u_n(x; A_1) - R_2^{-1}u_n(x; A_2)$ at the extremum points of $u_n(x; A_2)$ in $-1 \leqq x \leqq +1$ we see that it cannot be of degree $n - 2$, so that $S_1/R_1 \neq S_2/R_2$. Hence S/R is monotonic. Its minimum value is attained for

$$u_n(x; +1) = -T_n\left(\frac{x - \alpha_n}{1 + \alpha_n}\right),$$

so that min $(S/R) = n\alpha_n$. Its maximum value is attained for $u_n(x; +\infty) = T_{n-1}(x)$, so that max $(S/R) = +\infty$.

Now let $f(x)$ be a polynomial of the form (8.1), and let $\sigma \geqq n\alpha_n$. Then there exists a definite polynomial $u_n(x; A)$, $A = A(\sigma) \geqq 1$, for which $S/R = \sigma$ so that

$$d(x) = f(x) + R^{-1}u_n(x; A) \tag{8.4}$$

is of degree $n - 2$. Let max $|f(x)| \leqq R^{-1}$, $-1 \leqq x \leqq +1$. Then the polynomial (8.4) is alternately $\geqq 0$ and $\leqq 0$ at the points at which $u_n(x; A) = \pm 1$. Unless $d(x) \equiv 0$ this gives $n - 1$ distinct points at which $d'(x)$ is alternately >0 and < 0, and hence $n - 2$ roots for $d'(x)$ which is impossible.

2. The argument is similar in the other case, $0 < \sigma < n\alpha_n$, since the polynomial

$$2^{1-n}(1 + \sigma/n)^n T_n\left(\frac{x - \sigma/n}{1 + \sigma/n}\right) = x^n - \sigma x^{n-1} + \cdots \tag{8.5}$$

assumes its maximum modulus $2^{1-n}(1 + \sigma/n)^n$ precisely n times in $-1 \leqq x \leqq +1$.

Replacing $-R^{-1}u_n(x; A)$ in (8.4) by the left-hand side of (8.5), we obtain the desired result.

3. Another theorem of Zolotareff is the following[19]:

THEOREM 4. *Let x_0, y_0, be arbitrary real numbers, of which $x_0 > 1$, and let $f(x)$ be an arbitrary polynomial of degree n satisfying the conditions*

$$f(x) = x^n + \cdots, \qquad f(x_0) = y_0. \tag{8.6}$$

Then max $|f(x)|$, $-1 \leqq x \leqq +1$, *is a minimum if and only if $f(x)$ is one of the polynomials*

$$\begin{aligned} &-R^{-1}u_n(x; A), \qquad 2^{1-n}(1 + \alpha)^n T_n\left(\frac{x - \alpha}{1 + \alpha}\right), \\ &2^{1-n}(1 + \alpha)^n T_n\left(\frac{x + \alpha}{1 + \alpha}\right), \qquad (-1)^{n-1}R^{-1}u_n(-x; A). \end{aligned} \tag{8.7}$$

[19] Loc. cit.[3] (b), p. 27, (c), p. 371.

Here $A \geqq 1$, $0 \leqq \alpha \leqq \alpha_n = \tan^2 [\pi/(2n)]$ *are certain numbers uniquely determined by* x_0 *and* y_0.

The values of the polynomials (8.7) at $x = x_0$ increase

from $-\infty$ to $2^{1-n}(1 + \alpha_n)^n T_n\left(\dfrac{x_0 - \alpha_n}{1 + \alpha_n}\right) = \beta$

as A decreases from $+\infty$ to $+1$;

from β to $2^{1-n}T_n(x_0)$ as α decreases from α_n to 0,

from $2^{1-n}T_n(x_0)$ to $2^{1-n}(1 + \alpha_n)^n T_n\left(\dfrac{x_0 + \alpha_n}{1 + \alpha_n}\right) = \beta'$

as α increases from 0 to α_n;

from β' to $+\infty$ as A increases from 1 to $+\infty$,

respectively. These facts determine for a given y_0 the extremum polynomial $f_0(x)$ in question. Indeed, consider the difference $f(x) - f_0(x)$ at the points in $-1 \leqq x \leqq +1$ at which $f_0(x) = \pm 1$, and in addition at $x = x_0$. Since this difference is alternately $\geqq 0$ and $\leqq 0$ at these $n + 1$ points, the usual argument gives $n - 1$ distinct roots for its derivative [unless $f(x) \equiv f_0(x)$], which is impossible.

4. The problem defined by the condition

$$f^{(k)}(x_0) = y_0 \tag{8.8}$$

where $1 \leqq k \leqq n - 1$, $x_0 > 1$, and y_0 is arbitrary, can be treated in a similar manner. For $k = n - 1$ we obtain the first problem dealt with above.

9. A further application

The previous considerations furnish another property of the polynomials $u_n(x; A)$ of Zolotareff which play a role in the interesting investigations of W. Markoff [see[4]].

1. We prove the following application of Lemma 1:

THEOREM 5. *Let*

$$1 > x_1 > z_1 > x_2 > z_2 > \cdots > x_{n-2} > z_{n-2} > x_{n-1} > -1 \tag{9.1}$$

be the values of x *characterized by the conditions*

$$u_n(x_\nu\,; A) = 0, \qquad \nu = 1, 2, \cdots, n-1, \tag{9.2}$$

$$u_n'(z_\nu\,; A) = 0, \qquad \nu = 1, 2, \cdots, n-2; \tag{9.3}$$

then the functions $x_\nu = x_\nu(A)$ *and* $z_\nu = z_\nu(A)$ *increase as* A *increases.*[20]

The roots x_ν of $u_n(x; A)$ satisfy the equation

$$\int_{x_\nu}^{1} (A - t)(B - t)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(1 - t^2)^{-\frac{1}{2}}\,dt = (\nu - \tfrac{1}{2})\pi/n, \tag{9.4}$$

$$\nu = 1, 2, \cdots, n-1.$$

[20] Concerning x_ν, see W. Markoff, loc. cit. p. 242. The largest root $D = D(A)$ also increases, as can be concluded from the result of §2, No.[4].

We can assume that $A = A(\rho)$, $B = B(\rho)$, $C = C(\rho)$ are increasing functions of a parameter $\rho, \rho > 0$, all these functions having continuous derivatives. Then $x_\nu = x_\nu(\rho)$ and

$$
\begin{aligned}
&(A - x_\nu)(B - x_\nu)^{-\frac{1}{2}}(C - x_\nu)^{-\frac{1}{2}}(1 - x_\nu^2)^{-\frac{1}{2}} x_\nu'(\rho) \\
&= \int_{x_\nu}^1 \frac{d}{d\rho}\{(A - t)(B - t)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(1 - t^2)^{-\frac{1}{2}}\}\, dt \\
&= \int_{x_\nu}^1 (B - t)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(1 - t^2)^{-\frac{1}{2}}\, dt \left\{A'(\rho) - \tfrac{1}{2}B'(\rho)\frac{A - t}{B - t} - \tfrac{1}{2}C'(\rho)\frac{A - t}{C - t}\right\} \\
&> \left\{A'(\rho) - \tfrac{1}{2}B'(\rho)\frac{A + 1}{B + 1} - \tfrac{1}{2}C'(\rho)\frac{A + 1}{C + 1}\right\}\int_{x_\nu}^1 (B - t)^{-\frac{1}{2}}(C - t)^{-\frac{1}{2}}(1 - t^2)^{-\frac{1}{2}}\, dt \\
&> 0
\end{aligned}
$$

since the expression (2.9) increases with ρ.

The assertion about z_ν can be proved in a similar manner.

2. The assertion about z_ν follows also from the following general remark. Suppose the roots of an algebraic equation are real and distinct, and that they are increasing functions of a parameter; then the same holds for the roots of the derivative. Indeed, using the notation above:

$$\frac{1}{z_\nu - x_1} + \frac{1}{z_\nu - x_2} + \cdots + \frac{1}{z_\nu - x_n} = 0, \qquad x_n = D.$$

[Here $x_n = D$ denotes the only root of $u_n(x; A)$ which is > 1.] Differentiating this relation,

$$\sum_{\mu=1}^{n} \frac{z_\nu' - x_\mu'}{(z_\nu - x_\mu)^2} = 0,$$

so that $x_\mu' > 0$ implies $z_\nu' > 0$.

Repeated application of this argument shows that the roots of all derivatives $u_n^{(k)}(x; A)$ increase as A increases.

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