The Atiyah-Singer Index theorem what it is and why you should care

Rafe Mazzeo

October 10, 2002

Before we get started:

At one time, economic conditions caused the closing of several small clothing mills in the English countryside. A man from West Germany bought the buildings and converted them into dog kennels for the convenience of German tourists who liked to have their pets with them while vacationing in England. One summer evening, a local resident called to his wife to come out of the house.

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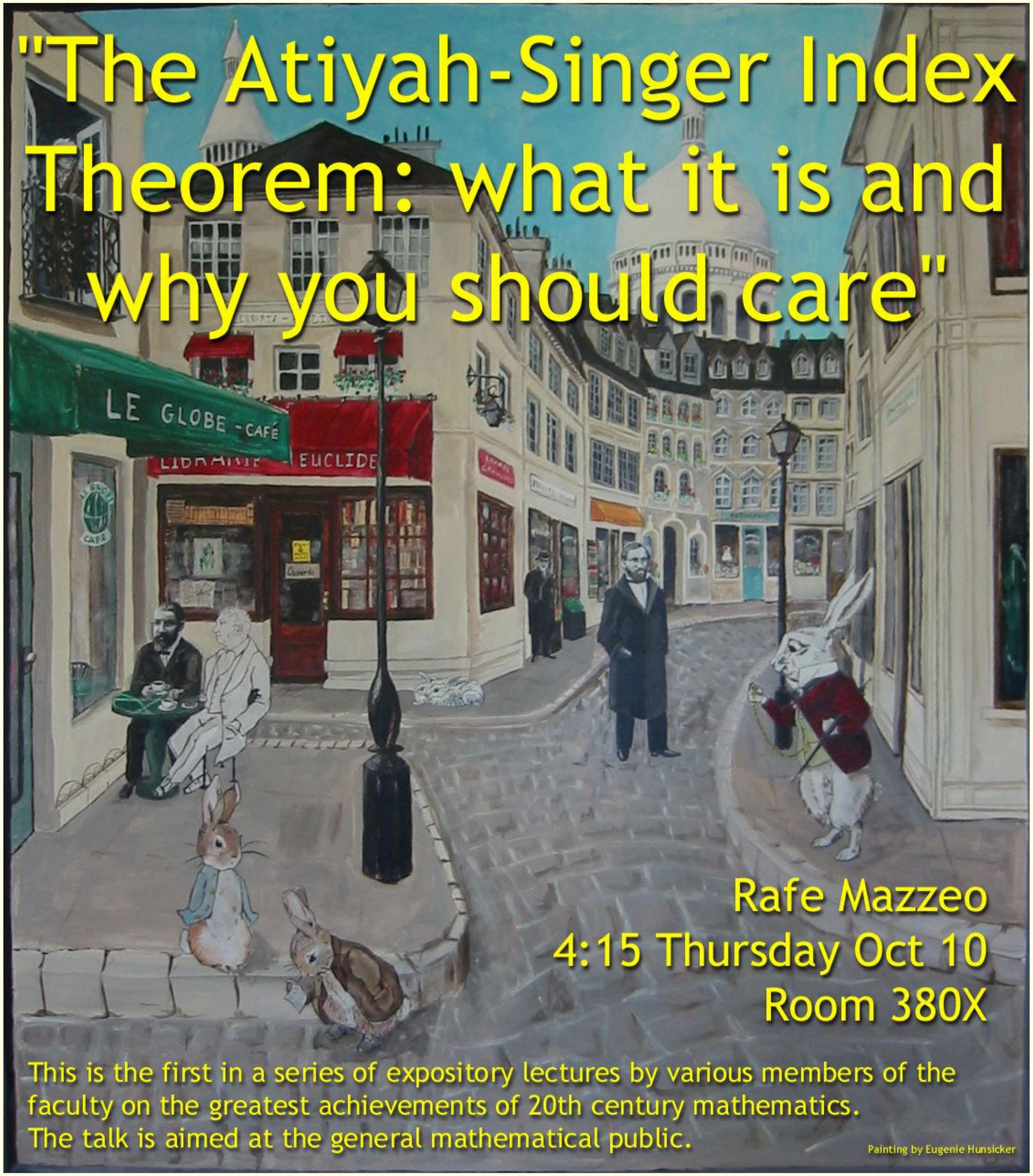
"Just listen!", he urged. "The Mills Are Alive With the Hounds of Munich!"


"The Atiyah-Singer Index Theorem: what it is and why you should care"
LE GLOBE -CAFÉ
EUCLIDE
Rafe Mazzeo
4:15 Thursday Oct 10
Room 380X
This is the first in a series of expository lectures by various members of the faculty on the greatest achievements of 20th century mathematics.
The talk is aimed at the general mathematical public.
Painting by Eugenie Hunsicker

Background:

The Atiyah-Singer theorem provides a fundamental link between differential geometry, partial differential equations, differential topology, operator algebras, and has links to many other fields, including number theory, etc.

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The purpose of this talk is to provide some sort of idea about the general mathematical context of the index theorem and what it actually says (particularly in special cases), to indicate some of the various proofs, mention a few applications, and to hint at some generalizations and directions in the current research in index theory.

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10. Getzler rescaling

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11. Application: obstruction to existence of positive scalar curvature metrics

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13. The Atiyah-Patodi-Singer index theorem

The simplest example:

Suppose

$$A : \mathbf{R}^n \longrightarrow \mathbf{R}^k$$

is a linear transformation. The rank + nullity theorem states that

$$\mathsf{dim}\,\mathsf{ran}\,(A) + \mathsf{dim}\,\mathsf{ker}(A) = n.$$

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$$\mathrm{dim}\,\mathrm{ran}\,(A) + \mathrm{dim}\,\mathrm{ker}(A) = n.$$

To prove this, decompose

$$\mathbf{R}^n = K \oplus K', \qquad K = \mathrm{ker}\, A$$

$$\mathbf{R}^k = R \oplus R', \qquad R = \mathrm{ran} A.$$

Then

$$A : K' \longrightarrow R$$

is an isomorphism, and

$$n = \dim K + \dim K', \qquad k = \dim R + \dim R'.$$

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Hence

$$n - k = (\dim K + \dim K') - (\dim R + \dim R')$$

$$= \dim K - \dim R' = \dim \ker A - (\dim \mathbf{R}^k - \dim \mathrm{ran} A).$$

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$$= \dim K - \dim R' = \dim \ker A - (\dim \mathbf{R}^k - \dim \mathrm{ran} A).$$

$$= \mathrm{ind}(A),$$

where by definition, the index of A is defined to be

$$\mathrm{ind}(A) = \dim \ker A - \dim \mathrm{coker} A.$$

The first infinite dimensional example:

Let ℓ^2 denote the space of all square-summable sequences $(a_0, a_1, a_2, \ldots)$.

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Define the left shift and right shift operators

$$L : \ell^2 \longrightarrow \ell^2, \qquad L((a_0, a_1, a_2, \ldots)) = (a_1, a_2, \ldots),$$

and

$$R : \ell^2 \longrightarrow \ell^2, \qquad R((a_0, a_1, a_2, \ldots)) = (0, a_0, a_1, a_2, \ldots).$$

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Then

$$\dim \ker L = 1, \quad \dim \operatorname{coker} L = 0$$

so

$$\operatorname{ind} L = 1.$$

On the other hand

$$\dim \ker R = 1, \qquad \dim \operatorname{coker} R = 0$$

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Notice that

$$\operatorname{ind} L^N = N, \qquad \operatorname{ind} R^N = -N,$$

so we can construct operators with any given index.

Fredholm operators:

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Example:

$$A = \frac{1}{i}\frac{d}{d\theta}$$

acting on

$$H_1 = H^1(S^1) = \{u : \int |u'|^2 \, d\theta < \infty\}, \qquad H_2 = L^2(S^1),$$

$$A(e^{in\theta}) = ne^{in\theta}.$$

Definition:

A is *Fredholm* if $\ker A$ is finite dimensional, $\mathrm{ran} A$ is closed and $\mathrm{coker} A$ is finite dimensional.

If A is Fredholm, then

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Intuition: If A is Fredholm, then

$$H_1 = K \oplus K', \qquad H_2 = R \oplus R',$$

where

$$A : K' \longrightarrow R$$

is an isomorphism and both K and R' are finite dimensional.

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Recall that $A : H_1 \to H_2$ is closed if

$$\text{Graph}(A) = \{(h, Ah) : h \in \text{domain}(A)\}$$

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is a closed subspace of $H_1 \oplus H_2$.

(This is always the case if $H_1 = H_2 = L^2$ and A is a differential operator.)

Basic Fact:

If A_t is a one-parameter family of Fredholm operators (depending continuously on t), then $\mathrm{ind}(A_t)$ is independent of t.

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Intuition:

$$A_t = tI - K, \qquad t > 0, \qquad K \quad \text{finite rank, symmetric}$$

Then A_t gains a nullspace and a cokernel of the same dimension every time t crosses an eigenvalue of K.

The Toeplitz index theorem:

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There is an orthogonal projection

$$\Pi : L^2(S^1) \longrightarrow H.$$

Suppose $f(\theta)$ is a continuous (complex-valued) function on S^1. Define the Toeplitz operator

$$T_f : H \longrightarrow H,$$

$$T_f(a) = \Pi M_f \Pi,$$

where M_f is multiplication by f (bounded operator on $L^2(S^1)$).

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Theorem: The Toeplitz operator T_f is Fredholm if and only if f is nowhere 0. In this case, its index is given by the winding number of f (as a map $S^1 \to \mathbf{C}^*$).

Part of proof: We guess that $T_{1/f}$ is a good approximation to an inverse for T_f, so we compute:

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$$= I_H + \Pi [M_f, \Pi] M_{1/f} \Pi.$$

The final term here is a compact operator! Obvious when f has a finite Fourier series; for the general case, approximate any continuous function by trigonometric polynomials.

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First step: Let N be the winding number of f. Then there is a continuous family f_t, where each $f_t : S^1 \to \mathbf{C}^*$ is continuous, such that $f_0 = f$, $f_1 = e^{iN\theta}$.

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$$\Pi M_{e^{iN\theta}} \Pi : \sum_{n=0}^{\infty} a_n e^{in\theta} \longrightarrow$$

$$\begin{cases} \sum_{n=0}^{\infty} a_{n-N} e^{in\theta} & \text{if } N < 0 \\ \sum_{n=N}^{\infty} a_{n-N} e^{in\theta} & \text{if } N \geq 0 \end{cases}$$

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This corresponds to L^N or R^N, respectively. Thus

$$\mathrm{ind}(T_f) = \text{winding number of } f$$

Elliptic operators

Let M be a smooth compact manifold of dimension n.

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In any coordinate chart, a differential operator P on M has the form

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

where

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The symbol of P is the homogeneous polynomial

$$\sigma_m(P)(x;\xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha, \qquad \xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}.$$

If P is a system (i.e. acts between sections of two different vector bundles E and F over M), then the coefficients $a_\alpha(x)$ lie in $\mathrm{Hom}(E_x, F_x)$.

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P is called *elliptic* if

$$\sigma_m(P)(x;\xi) \neq 0 \quad \text{for } \xi \neq 0,$$

(or is invertible as an element of $\mathrm{Hom}(E_x, F_x)$)

Fundamental Theorem: If P is an elliptic operator of order m acting between sections of two vector bundles E and F on a compact manifold M, then

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defines a(n unbounded) Fredholm operator.

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Question: Compute $\mathrm{ind}(P)$.

One expects this to be possible because the index is invariant under such a large class of deformations.

Interesting elliptic operators:

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Nope: the Laplacian is self-adjoint,

$$\langle \Delta_0 u, v \rangle = \langle u, \Delta_0 v \rangle,$$

and so

$$\ker \Delta_0 = \operatorname{coker} \Delta_0 = \mathbf{R}, \quad \text{hence} \quad \operatorname{ind}(\Delta_0) = 0.$$

The same is true for

$$\Delta_k : L^2\Omega^k(M) \longrightarrow L^2\Omega^k(M)$$

$$\Delta_k = d_{k-1}d_k^* + d_{k+1}^* d_k$$

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In fact, the Hodge theorem states that

$$\ker \Delta_k = \operatorname{coker} \Delta_k$$

$$= \mathcal{H}^k(M) \qquad \text{(the space of harmonic forms)},$$

and this is isomorphic to the singular cohomology, $H^k_{\mathrm{sing}}(M, \mathbf{R})$.

Similarly

$$D = d + d^* : L^2\Omega^*(M) \longrightarrow L^2\Omega^*(M),$$

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The Gauss-Bonnet operator:

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The Gauss-Bonnet operator:

$$D_{\mathrm{GB}} : L^2\Omega^{\mathrm{even}} \longrightarrow L^2\Omega^{\mathrm{odd}}$$

$$\ker D_{\mathrm{GB}} = \bigoplus_k \mathcal{H}^{2k}(M),$$

$$\mathrm{coker} D_{\mathrm{GB}} = \bigoplus_k \mathcal{H}^{2k+1}(M),$$

and so

$$\operatorname{ind} D_{\mathsf{GB}} = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M) = \chi(M),$$

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the Euler characteristic of M.

Note that $\operatorname{ind}(D_{\mathsf{GB}}) = 0$ when $\dim M$ is odd.

This is not the index theorem, but only a Hodge-theoretic calculation of the index of D_{GB} in terms of something familiar.

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One can think of this integral either as a differential geometric quantity, or else as a characteristic number (the evaluation of the Euler class of TM on the fundamental homology class $[M]$).

The signature operator:

$$D_{\mathsf{sig}} = d + d^* : L^2\Omega^+(M) \longrightarrow L^2\Omega^-(M)$$

where

$$\Omega^\pm(M) \text{ is the } \pm \text{ eigenspace of an involution}$$

$$\tau : \Omega^*(M) \longrightarrow \Omega^*(M), \qquad \tau^2 = 1.$$

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When $\dim M = 4k$, then

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which is a basic topological invariant, the signature of M.

In all other cases, $\mathrm{ind} D_{\mathsf{sig}} = 0$.

The Hirzebruch signature theorem states that

$$\operatorname{sign}(M) = \int_M \mathcal{L}(p),$$

where $\mathcal{L}(p)$ is the L-polynomial in the Pontrjagin classes.

This is another ‘curvature integral’, and also a characteristic number. (It is a homotopy invariant of M.)

The (generalized) Dirac operator:

Let $E^{\pm}$ be two vector bundles over (M, g), and suppose that there is a (fibrewise) multiplication

$$\mathrm{cl} : TM \times E^{\pm} \to E^{\mp}, \qquad \text{Clifford multiplication}$$

which satisfies

$$\mathrm{cl}(v)\mathrm{cl}(w) + \mathrm{cl}(w)\mathrm{cl}(v) = -2\langle v, w\rangle \mathrm{Id}.$$

Let $E = E^{+} \oplus E^{-}$.

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Let $E = E^{+} \oplus E^{-}$.

Let $\nabla : \mathcal{C}^{\infty}(M;E) \to \mathcal{C}^{\infty}(M;E \otimes T^{*}M)$ be a connection.

Now define

$$D : L^2(M; E) \longrightarrow L^2(M; E)$$

by

$$D = \sum_{i=1}^{n} \mathsf{cl}(e_i)\nabla_{e_i},$$

where $\{e_1, \ldots, e_n\}$ is any local orthonormal basis of TM.

The Clifford relations show that

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where $\mathcal{A}$ is a homomorphism.

In other words,

"D^2 is the Laplacian up to a term of order zero."

In certain cases (when M is a spin manifold: $w_2 = 0$), there is a distinguished Dirac operator D acting between sections of the spinor bundle S over M.

One can then ‘twist’ this basic Dirac operator

$$D : L^2(M; S) \longrightarrow L^2(M; S)$$

by tensoring with an arbitrary vector bundle E, to get the twisted Dirac operator

$$D_E = D \otimes \mathbf{1} \oplus \mathsf{cl}(\mathbf{1} \otimes \nabla^E) : L^2(M : S \otimes E) \longrightarrow L^2(M; S \otimes E).$$

The index theorem for twisted Dirac operators states that

$$\mathrm{ind}(D_E) = \int_M \hat{A}(TM)\mathrm{ch}(E).$$

The integrand is again a ‘curvature integral’; it is the product of the $\hat{A}$ genus of M and the Chern character of the bundle E.

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Both D_{GB} and D_{sig} are twisted Dirac operators, and both $\mathrm{Pf}(R)$ and $\mathcal{L}(p)$ are the product of $\hat{A}(TM)$ and $\operatorname{ch}(E)$ for appropriate bundles E.

There is a general index theorem for arbitrary elliptic operators P (which are not necessarily of order 1), acting between sections of two bundles E and F. This takes the form

$$\mathrm{ind}(P) = \int_M \mathcal{R}(P, E, F).$$

The right hand side here is a characteristic number for some bundle associated to E, F and the *symbol* of P.

By Chern-Weil theory, this can be written as a curvature integral, but this is less natural now.

Intermission:

One day at the watering hole, an elephant looked around and carefully surveyed the turtles in view. After a few seconds thought, he walked over to one turtle, raised his foot, and KICKED the turtle as far as he could. (Nearly a mile) A watching hyena asked the elephant why he did it? "Well, about 30 years ago I was walking through a stream and a turtle bit my foot. Finally I found the S.O.B and repaid him for what he had done to me." "30 years!!! And you remembered...But HOW???"

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The original proofs

The index is computable because it is invariant under many big alterations and behaves well with respect to many natural operations:

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An elaboration of this same strategy may be used to reduce the computation of the index of a general elliptic operator P to that of a twisted Dirac operator on certain very special manifolds (products of complex projective spaces).

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2. Replace the twisted Dirac operator on the manifold M by a twisted Dirac operator on a new manifold M' such that $M - M' = \partial W$. The index stays the same!

3. Check that the two sides are equal on ‘enough’ examples, products of complex projective spaces (these generate the cobordism ring). This requires the multiplicativity of the index.

A simplification:

Instead of the final two steps, it is in many ways preferable to compute the index of twisted Dirac operators on an arbitrary manifold M directly.

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From now on I'll focus only on the index theorem for twisted Dirac operators.

The heat equation approach

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Note that if $D = d + d^*$ on Ω^{even} or Ω^+, then $D = d + d^*$ on Ω^{odd} or Ω^-, so

$$D^*D = (d + d^*)^2 = dd^* + d^*d, \qquad DD^* = dd^* + d^*d$$

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The initial value problem for the heat equation is

$$\left(\frac{\partial}{\partial t} + P\right) u = 0 \qquad \text{on } M \times [0, \infty),$$

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The solution is given by the heat kernel H

$$u(x,t) = (Hu_0)(x,t) = \int_M H(x,y,t)\phi(y)\, dV_y.$$

Let $\{\phi_j, \lambda_j\}$ be the eigendata for P, i.e.

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$$H(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y)$$

because

$$u_0 = \sum_{j=0}^{\infty} c_j \phi_j(x) \Rightarrow \int_M \left(\sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y) \right) \sum_{j'=0}^{\infty} c_{j'} \phi_{j'}(y) \, dV_y$$

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Spectral cancellation

Let $E(\lambda) = \{\phi : D^*D\phi = \lambda\phi\}$ and $F(\mu) = \{\psi : DD^*\phi = \mu\phi\}$. Then

$$\text{for } \lambda \neq 0, \qquad D : E(\lambda) \longrightarrow F(\lambda)$$

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is an isomorphism.

Proof:

$$D^*D\phi = \lambda\phi \Rightarrow D(D^*D\phi) = (DD^*)(D\phi) = \lambda(D\phi)$$

so λ is an eigenvalue for DD^*.

Also need to check that $D\phi \neq 0$:

$$D\phi = 0 \Rightarrow 0 = ||D\phi||^2 = \langle D^* D\phi, \phi \rangle \Rightarrow \phi \in \ker D.$$

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Conclusion The heat kernel traces are given by the sums

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In particular,

$$\mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*}) \qquad \text{is independent of } t!$$

Localization as $t \searrow 0$

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On the other hand, along the diagonal

$$H(x, x, t) \sim \sum_{j=0}^{\infty} a_j t^{-n/2+j}.$$

The coefficients a_j are called the *heat trace invariants*.

They are local, i.e. they can be written as

$$a_j = \int_M p_j(x)\, dV_x,$$

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If P is a geometric differential operator, then the p_j are universal functions of the metric and its covariant derivatives up to some order.

$$\mathrm{ind}(D) = \mathrm{Tr} e^{-tD^*D} - \mathrm{Tr} e^{-tDD^*}$$

$$= \sum_{j=0}^{\infty} a_j(D^*D) t^{-n/2+j} - \sum_{j=0}^{\infty} a_j(DD^*) t^{-n/2+j}$$

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The difficulty:

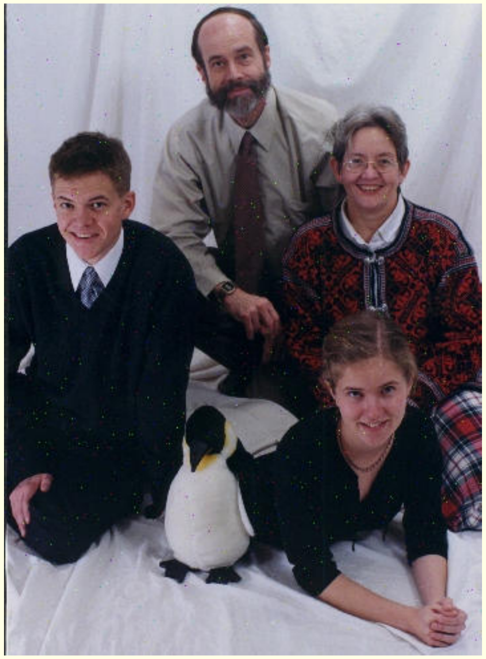
The term $a_{n/2}$ is high up in the asymptotic expansion, and so *very* hard to compute explicitly!

V.K. Patodi did the computation explicitly for a few basic examples (Gauss-Bonnet, etc.). He discovered a miraculous cancellation which made the computation tractable.

His computation was extended somewhat by

Atiyah-Bott-Patodi (1973)

P. Gilkey gave another proof based on geometric invariant theory, i.e. the idea that these universal polynomials are $SO(n)$-invariant, in an appropriate sense, and hence must be polynomials in the Pontrjagin forms. Then, you calculate the coefficients by checking on lots of examples ....



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he defined a rescaling of the Clifford algebra and the Clifford bundles in terms of the variable t, which as the effect that the 'index term' $a_{n/2}$ occurs as the *leading* coefficient!

The first application: (non)existence of metrics of positive scalar curvature

Suppose (M^{2n}, g) is a spin manifold. Thus the Dirac operator D acts between sections of the spinor bundles $S^{\pm}$.

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The Weitzenböck formula states that

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Corollary: (Lichnerowicz) If M admits a metric g of positive scalar curvature, then $\ker D = \mathrm{coker} D = 0$. Hence $\mathrm{ind} D = 0$, and so the $\hat{A}$-genus of M vanishes.

The Gromov-Lawson conjecture states that a slight generalization of this condition (the vanishing of the corresponding integral characteristic class, and suitably modified to allow odd dimensions) is necessary and sufficient for the existence of a metric of positive scalar curvature (at least when $\pi_1(M) = 0$).

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Proved (when M simply connected) if the dimension is ≥ 5 (Stolz).

Another application: computing dimensions of moduli spaces:

Suppose N is a nonlinear elliptic operator such that the solutions of $N(u) = 0$ correspond to some interesting geometric objects, e.g.

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If the linearization $L = DN|_{u_0}$, the implicit function theorem implies that

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Best situation (unobstructed deformation theory) is when L surjective, so that

$$\text{ind} L = \dim \ker L.$$

This is computable, and is the local dimension of solution space.

Example:

Let M^4 be compact and E a rank 2 bundle over M with charge $c_2(E) = k \in \mathbf{Z}$. Then the moduli space $\mathcal{M}_k$ of all self-dual Yang-Mills instantons of charge k ‘should be’ (and is, for generic choice of background metric) a smooth manifold, with

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Freedom to choose generic metric means one can always assume one is in the unobstructed situation.

The index formula for manifolds with boundary

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$$D = \mathsf{cl}(e_n) \left(\frac{\partial}{\partial x_n} + A \right),$$

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$$A : L^2(Y; S) \longrightarrow L^2(Y; S)$$

is *self-adjoint*!

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Weyl eigenvalue asymptotics for D implies that $\eta(s)$ is holomorphic for $\mathrm{Re}(s) > n$.

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The APS index formula is then a sort of transgression formula, connecting the index in even dimensions and the eta invariant in odd dimensions.