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Global Questions in the Topology of Singular Spaces

During the last decade, several new tools have been found for the study of the topology of singular spaces. Our object here is to review the part of this work that relates to intersection homology.

When he introduced homology for the study of manifolds, Poincaré made it clear that he was motivated by applications in three directions: analysis (differential equations), algebraic geometry, and group theory ([68], p. 194). Each of these three fields leads to questions about singular spaces as well. After developing some general theory, we will illustrate its usefulness in three sections devoted to applications in these three areas.

The singular spaces that arise in applications are usually complex algebraic (or analytic) varieties. In this report, we will restrict ourselves to methods which apply to these. Even here, desire for unity, limitations of space, and ignorance force the omission of many important subjects. For example, we do not mention the rather well developed theory of characteristic classes and Riemann–Roch (see [73], [59], [9], [10], [30], [31], [32]).

§ 1. Stratifications. Perhaps the primary reason why the study of singular spaces blossomed at this time was the creation of stratification theory [58], [80], [74], [62]. This illuminated the local structure of analytic varieties.

Let X be a complex analytic or algebraic variety of pure complex dimension n. Then X admits a locally finite decomposition into disjoint connected nonsingular analytic subvarieties $\{S_a\}$ of varying dimension called *strata*, which satisfies a *homogeneity condition* along the strata: for any two points p and q on a stratum S_a, there exists a homeomorphism of X to X, preserving all the strata and taking p to q. We denote the codimension of S_a by c_a, i.e. $c_a = n - \dim S_a$. The space Σ which is the union of all S_a such that $c_a > 0$ contains all the singularities of X. The strata $S_a \subset \Sigma$ may be thought of as loci on which X is "uniformly singular".

This homogeneity of X along S_α is guaranteed by the Whitney conditions: (1) the closure $\bar{S}_\alpha$ of S_α is a union of strata and (2) if a sequence of points $b_i \in S_\beta$ and a sequence $a_i \in S_\alpha$ both approach the same point $c \in S_\beta$, then the limit of the secant lines connecting a_i to b_i is contained in the limit of the tangent spaces at a_i, if both limits exist.

Intersection homology

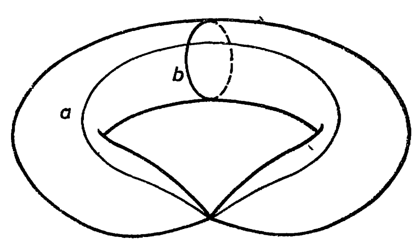
§ 2. Motivation. One of the important properties of the ordinary homology $H_i(M)$ of a compact oriented $2n$ real dimensional manifold M is *Poincaré duality*:

(1) *If $i+j = 2n$, then any pair of homology classes $A \in H_i(M)$ and $B \in H_j(M)$ have representative cycles $a \in A$ and $b \in B$ that intersect in finitely many points.*

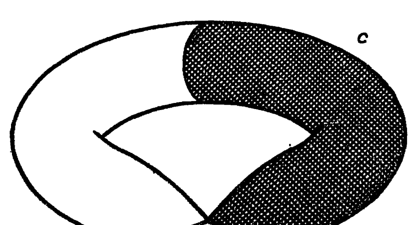
(2) *The number of these intersection points counted according to their multiplicities is independent of the choice of a and b and is denoted $A \cap B$.*

(3) *If M is compact, the bilinear pairing $H_i(M, Q) \times H_j(M, Q) \to Q$ which sends (A, B) to $A \cap B$ is nondegenerate.* (See [53] for the definition of intersection multiplicities.)

An obvious extension of $H_i(M)$ to singular varieties X is the usual homology group $H_i(X)$. Part (2) of the Poincaré duality fails even for the simplest singular space; the nodal cubic.



From the position of a and b in the picture, we would conclude that $A \cap B = 1$. However, b is the boundary of a chain c so $B = 0$; hence $A \cap B = 0$.



Another obvious extension of $H_i(M)$ to singular X would be $H^{2n-i}(X)$ which coincidens with it in the nonsingular case. By viewing $H^{2n-i}(X)$ as traces in X of bigger cycles in an ambient nonsingular space, one may visualize $H^{2n-i}(X)$ as the homology of a complex of chains which satisfy a transversality condition with respect to the singularities of X (see [37]). If X is the nodal cubic, a does not satisfy this condition, and neither does the chain c. So $H^1(X)$ is one-dimensional, generated by B. Property (2) of Poincaré duality holds but (3) fails since no class pairs nontrivially to B.

In the definition of intersection homology, we will restore Poincaré duality to the singular case by placing conditions on how chains meet the singular strata which are less strict than the transversality condition for $H^{2n-i}(X)$. For the nodal cubic, a will not be allowed but c will, so the first intersection homology group is zero.

§ 3. Definition of intersection homology. Let X be an n-dimensional complex analytic variety with a Whitney stratification $\{S_a\}$, and let c_a be the complex codimension of S_a. Denote by $\{C_i(X)\}$, $i \in \mathbf{Z}$, the complex of geometric chains on X. (Several choices for this work equally well. For example, we can take $C_i(X)$ to be the piecewise linear i-chains with respect to some piecewise linear structure on X. An element c of $C_i(X)$ would then be a simplicial chain for some triangulation (depending on c) of X. Another choice would be subanalytic chains [45].)

We define the complex of intersection chains $\{IC_i(X)\}$ to be the subcomplex of $\{C_i(X)\}$ consisting of those chains $c \in C_i(X)$ satisfying the *allowability condition* ([39], [3]):

The chain c intersects each singular stratum $S_a \subset \Sigma$ in a set of (real) dimension less than $i - c_a$ and its boundary ∂c intersects each singular stratum $S_a \subset \Sigma$ in a set of dimension less than $i-1-c_a$. (Note that c_a, the complex codimension of S_a, is half the real codimension.)

For some purposes dimension bounds other than $i-c_a$ and $i-1-c_a$ are useful, and the chains described above are sometimes called middle-perversity intersection chains in the literature. We will not use these other dimension bounds here.

DEFINITION [39]. The ith *intersection homology* group of X, denoted $IH_i(X)$, is the ith homology group of the chain complex $\{IC_i(X)\}$.

If X is nonsingular, then $IH_i(X) = H_i(X)$. In general the groups $IH_i(X)$ have many attributes of more familiar topological functors. They depend only on the topology of S — in particular they are independent of the stratification $\{S_a\}$ used in their definition [4]. If X is compact, they are finitely generated. If $U \subset X$ is open, there are relative groups $IH_i(X, U)$ which fit in the usual long exact sequence and satisfy excision. Over a field F, they satisfy the Kunneth theorem

$$IH_i(X \times Y; F) = \bigoplus_{j+k=i} IH_j(X; F) \otimes IH_k(Y, F).$$

They are not homotopy invariants, but they are both covariant and contravariant functors on a class of maps called placid maps.

DEFINITION. A map of a purely m-dimensional variety Y^m to X^n is *placid* if X can be stratified by strata $\{S_a\}$ so that for each a, $\dim_C f^{-1}(S_a) \leqslant m - c_a$.

If f is placid, the homomorphisms

$$f_*: IH_i(Y) \to IH_i(X) \quad \text{and} \quad f^*: IH_i(X) \to IH_{i+2(m-n)}(Y)$$

are defined as usual, essentially by image cycles and transverse inverse image cycles.

§ 4. The Kähler package. The intersection homology $IH_i(X)$ of a singular algebraic variety X satisfies a large part of the package of special properties of the ordinary homology of a Kähler manifold. These results are all false for $H_i(X)$.

POINCARÉ DUALITY [3]. (1) *If $i+j = 2n$, then any pair of intersection homology classes $A \in IH_i(X)$ and $B \in IH_j(X)$ have representatives $a \in A$ and $b \in B$ that intersect only in $X - \Sigma_n$ and in finitely many points.*

(2) *The number of these intersection points counted according to their multiplicities is independent of the choice of a and b, and is denoted $A \cap B$.*

(3) *If X is compact, the bilinear pairing*

$$IH_i(X; Q) \times IH_j(X; Q) \to Q$$

which sends A, B to $A \cap B$ in nondegenerate.

The corresponding pairing over the integers is not unimodular [38]. For general i and j, there is an intersection pairing $IH_i(X) \times IH_j(X) \to H_{i+j-2n}(X)$ but no ring structure on $IH_*(X)$.

LEFSCHETZ HYPERPLANE THEOREM [4], [41]. *Let X^n be a closed subvariety of complex projective m-space CP^m and let $H^{m-1} \subset CP^m$ be a generic hyperplane. Then*

$$j_*: IH_i(X \cap H; Z) \to IH_i(X; Z)$$

is an isomorphism for $i < n-1$ *and is surjective for* $i = n-1$, *where* $j: X \cap H \hookrightarrow X$ *is the (placid) inclusion.*

HARD LEFSHETZ THEOREM [27], [12]. *Let* X *be a closed subvariety of* CP^m. *Then intersecting with a generic hyperplane* $H \subset CP^m$ *induces a mapping* $\bigcap [H]: IH_i(X) \to IH_{i-2}(X)$, *and for all* k *the iterated map*

$$IH_{n+k}(X; Q) \xrightarrow{(\cap [H])^k} IH_{n-k}(X; Q)$$

is an isomorphism.

CONJECTURE [26]. *If* X *is compact and projective, then* $IH_k(X; C)$ *has a pure Hodge decomposition*

$$IH_k(X; C) = \bigoplus_{i+j=k} IH_{i,j}(X)$$

with the following properties:

(a) $IH_{i,j}(X) = \overline{IH_{j,i}(X)}$;

(b) *If* $f: Y^m \to X^n$ *is placid,* $f_* IH_{i,j}(Y) \subset IH_{i,j}(X)$ *and* $f^* IH_{i,j}(X) \subset IH_{i+m-n, j+m-n}(Y)$;

(c) *The usual relations with the Lefschetz map* $\bigcap [H]$ *and the duality pairing* $A \cap B$ *hold; in particular the Hodge index theorem is valid.*

§ 5. Interpretation of intersection homology. Since the thrust of the last section (and the next two) is that $IH_*(X)$ behaves in many circumstances exactly like the ordinary homology of a nonsingular variety, one might well ask if $IH_*(X)$ is in fact the homology of an associated non-singular variety. The answer is: Sometimes.

A *small* resolution $\tilde{X} \to X$ is a resolution for which X can be stratified by strata $\{S_a\}$ such that if $p \in S_a$, then $\dim f^{-1}(p) < \frac{1}{2} c_a$, where c_a is the codimension of S_a in X. If $\pi: \tilde{X} \to X$ is a small resolution, then π induces an isomorphism $H_i(\tilde{X}) \cong IH_i(X)$ (see [40]).

This observation leads to the following fanciful question: In such a situation, how much of the topology of $\tilde{X}$ can be read from that of X? More precisely, what invariants I can be defined for all singular spaces so that, whenever a small resolution $\tilde{X}$ exists, $I(X) = I(\tilde{X})$? The list includes $H_*(\tilde{X})$ with its intersection bilinear form, the Wu class in $H_*(\tilde{X})$ [38], and the Chern numbers of $\tilde{X}$ for the signature, the Euler characteristic, and the arithmetic genus. But a bound on such speculation is provided by the fact that X may have two different small resolutions $\tilde{X}_1$ and $\tilde{X}_2$. Examples exist where the intersection ring structures of $\tilde{X}_1$ and

$\tilde{X}_2$ are not isomorphic and the Chern classes are not correspondingly placed. (This shows that there can be no natural ring structure on intersection homology.)

§ 6. Stratified Morse theory. We present here what appears to be the correct analogue for singular analytic varieties of Morse theory for manifolds [41]. Suppose X is embedded in a smooth complex analytic variety M and is Whitney stratified by $\{S_\alpha\}$. For each stratum S_α, define the *conormal space* $C(S_\alpha)$ to be the closure in T^*M, the cotangent bundle of M, of the set of cotangent vectors which lie over S_α and annihilate all tangent vectors in TS_α, the tangent space to S_α. An *X-critical point* of a smooth function $f\colon M \to \boldsymbol{R}$ is a critical point $p \in S_\alpha$ of the restriction of f to some stratum S_α. The *critical value* of f at a critical point p is $f(p)$. A smooth function $f\colon M \to \boldsymbol{R}$ is called *Morse* for X if

(a) The restriction f/S_α is Morse for all $S_\alpha \subset X$.

(b) If $p \in S_\alpha$ is an X-critical point, $df(p) \notin C(S_\beta)$ for any $\beta \neq \alpha$.

(c) The critical values of f are distinct.

The set of Morse functions for X is open and dense in the C^∞ topology; Morse functions are C^0 structurally stable on X [67]. For any number $s \in \boldsymbol{R}$ we denote by $X_{<s}$ the set of $x \in X$ such that $f(x) < s$. If $p \in S_\alpha$ is a critical point for the Morse function f, we define the Morse index λ_p for f at p to be $c_\alpha +$ (the Morse index for $f | S_\alpha$ at p).

THEOREM. *There exists a unique set of abelian groups A_α one for each stratum S_α, such that for any proper Morse function $f\colon M \to \boldsymbol{R}$*

(a) *if the interval $[s, t)$ contains no critical values, then $IH_i(X_{<t}, X_{<s}; \boldsymbol{Z}) = 0$ for all i;*

(b) *if the interval $[s, t)$ contains the critical value v of one critical point $p \in S_\alpha$, and λ_p is the Morse index of f at p, then*

$$IH_i(X_{<t}, X_{<s}; \boldsymbol{Z}) = \begin{cases} 0 & \text{for } i \neq \lambda_p, \\ A_\alpha & \text{for } i = \lambda_p. \end{cases}$$

There is no analogous notion of a Morse index for ordinary homology replacing intersection homology since $H_i(X_{<t}, X_{<s})$ may be nonzero for several i, even if $[s, t)$ contains only one critical value.

The groups A_α are very difficult to calculate, but they are important since they arise in a number of other contexts. If m_α denotes the rank of the free part of A_α, the algebraic cycle in T^*M

$$\mathrm{ch}(X) = \sum_\alpha m_\alpha C(S_\alpha)$$

is called the *characteristic variety* of X. It will play a role in § 7 and § 11 below.

§ 7. Lefschetz fixed point theorem. If $f: X \to X$ is a placid self-map, for example a self-homeomorphism, then the intersection homology Lefschetz number $IL(f) = \sum(-1)^i \operatorname{trace}\left(f_*: IH_i(X, Q) \to IH_i(X; Q)\right)$ has an expression which is localized at the fixed point set of f. More precisely, for each connected component K of the fixed point set there is a Lefschetz index $IL(f, K)$ determined by the local behavior of f near K, such that the sum of the $IL(f, K)$ over all connected components K is $IL(f)$. We give two formulas for $IL(f, K)$, both analogues of classical formulas for manifolds.

First we treat continuous placid self-maps $f: X \to X$ and give a result in the framework of [76]. Given any non-singular point $p \in X$, the cycle $[p \times X] \in X \times X$ satisfies the allowability conditions of § 2 and therefore lies in the intersection homology group $IH_{2n}(X \times X)$.

THEOREM [42]. *Let U_Δ and U_f be open regular neighborhoods of the diagonal and the graph of f respectively in $X \times X$. Then there are unique intersection homology classes with compact support $[\Delta] \in IH_{2n}(U_\Delta; Q)$ and $[f] \in IH_{2n}(U_f; Q)$ such that $[\Delta] \cap [p \times X] = 1$* and *$[f] \cap [p \times X] = 1$ for all non-singular $p \in X$. For these classes, we have*

$$IL(f) = [\Delta] \cap [f].$$

We may choose U_Δ and U_f so small that $U_\Delta \cap U_f$ has a unique connected component containing each connected component $\Delta \cap K \times K$ of $\Delta \cap X$ graph f. In this way $[\Delta] \cap [f]$ may be considered as a sum of contributions from each K, and we obtain a formula for $IL(f, K)$ as well. For example, if p is an isolated fixed point of f, then $IL(f, p)$ is the linking number of $L \cap [\Delta]$ and $L \cap [f]$ in $L =$ the link of $p \times p$ in $X \times X$. (One can verify that linking numbers exist for disjoint $(2n-1)$-dimensional intersection homology cycles in L.)

Second we treat integrals of vector fields and give a generalization of the Hopf index formula. Suppose X is embedded in a smooth m-dimensional complex variety M, and let $v: X \to TM$ be a vector field on X, i.e. a (possibly discontinuous) section of TM defined only over X, such that if $p \in S_\alpha$, $v(p) \in TS_\alpha$. We suppose that v can be integrated (i.e. there is a one-parameter family $f_t: X \to X$ of self-maps of X such that f_0 is the identity and $\partial/\partial t\, f_t(p) = v(f_t(p))$) and that the fixed points of f_t are exactly the zeros of v for $t \in (0, 1]$. "Controlled vector fields" (see [62]) provide a rich supply of such v. Choose a continuous section s: $M \to T^*M$

of the cotangent bundle to M with the property that if $p \in X$ and $v(p) \neq 0$ then $s(p)v(p) > 0$.

The image of s is a $2m$-cycle with closed supports $[s(M)]$ in T^*M. Another natural $2m$-cycle in T^*M is $\mathrm{ch}(X)$, defined at the end of § 6. Now given a connected component K of the fixed point set of f_1, pick an open set $U \subset M$ containing K but no other fixed points. The cycles $[s(M)]$ and $\mathrm{ch}(X)$ restrict to cycles $[s(U)]$ and $\mathrm{ch}(X, U)$ with closed support in $T^*(U)$. One can check that the condition $s(p)v(p) > 0$ for $p \in U - K$ guarantees that the intersection of the supports of $[s(U)]$ and $\mathrm{ch}(X, U)$ is compact. Therefore the intersection number $[s(U)] \cap \cap \mathrm{ch}(X, U)$ is well defined.

THEOREM. $IL(f, K) = [s(U)] \cap \mathrm{ch}(X, U)$.

Applying this theorem to the zero vector field, we get

COROLLARY [29]. *If $I\chi$ is the intersection homology Euler characteristic of X and $Z \subset T^*M$ is the zero section, $I\chi = [Z] \cap \mathrm{ch}(X)$.*

§ 8. Enter sheaf theory. The functor which assigns to each open set U in X the group $IC_i^{cl}(U)$ of intersection chains on U with closed support is a *sheaf*. This observation is key for axiomatic characterization of intersection homology as well as for most of its applications.

Because of the numbering conventions prevalent in sheaf theory, we define $\mathbf{IC}^i(U)$ to be $IC_{2n-i}^{cl}(U)$; if $U' \subset U$ then the map $\mathbf{IC}^i(U) \to \mathbf{IC}^i(U')$ is just restriction of geometric chains. Then $\mathbf{IC}^i$ is a sheaf because the allowability conditions of § 2 are local. The boundary map gives a map of sheaves $\delta: \mathbf{IC}^i \to \mathbf{IC}^{i+1}$ such that $\delta \circ \delta = 0$, so we have a complex of sheaves $\mathbf{IC}^\cdot$. As with any complex of sheaves, we can apply several cohomological functors. First is the cohomology sheaf functor $\mathbf{H}^i\mathbf{IC}^\cdot = \mathrm{Ker}\, \delta: \mathbf{IC}^i \to \mathbf{IC}^{i+1} / \mathrm{Im}\, \delta: \mathbf{IC}^{i-1} \to \mathbf{IC}$. This is a sheaf whose stalk $\mathbf{H}^i(\mathbf{IC}^\cdot)_p$ at p is $IH_{2n-i}(X, X-p)$. Second is the hypercohomology functor $\boldsymbol{H}^i(X, \mathbf{IC}^\cdot)$. Since $\mathbf{IC}^i$ is soft, for all i, the group $\boldsymbol{H}^i(X, \mathbf{IC}^\cdot)$ may be computed as the global section cohomology $\ker \delta \mathbf{IC}^i(X) \to \mathbf{IC}^{i+1}(X) / \mathrm{Im}\, \delta: \mathbf{IC}^{i-1}(X) \to \mathbf{IC}^i(X)$. Hence $\boldsymbol{H}^i(X, \mathbf{IC}^\cdot) = IH_{2n-i}^{cl}(X)$ which we also denote $IH^i(X)$. Third is hypercohomology with compact support $IH_c^i(X, \mathbf{IC}^\cdot)$ which for the same reason is $IH_{2n-i}(X)$.

If $\mathbf{P}^\cdot$ and $\mathbf{Q}^\cdot$ are two complexes of sheaves, a *quasi-isomorphism from $\mathbf{P}^\cdot$ to $\mathbf{Q}^\cdot$* is a diagram of complexes $\mathbf{P}^\cdot \xleftarrow{p} \mathbf{R}^\cdot \xrightarrow{q} \mathbf{Q}^\cdot$ so that p and q induce isomorphisms $\mathbf{H}^i\mathbf{P}^\cdot \xleftarrow{\cong} \mathbf{H}^i\mathbf{R}^\cdot \xrightarrow{\cong} \mathbf{H}^i\mathbf{Q}^\cdot$ for all i, or equivalently if for all open sets U, p and q induce isomorphisms $\boldsymbol{H}^i(U, \mathbf{P}^\cdot) \leftarrow \boldsymbol{H}^i(U, \mathbf{R}^\cdot) \to \boldsymbol{H}^i(U, \mathbf{Q}^\cdot)$.

If there is a quasi-isomorphism from $\mathbf{P}^{\cdot}$ to $\mathbf{Q}^{\cdot}$, then $\mathbf{P}^{\cdot}$ and $\mathbf{Q}^{\cdot}$ are called *quasi-isomorphic*. This is an equivalence relation. Quasi-isomorphic sheaves are interchangeable for all calculations with cohomological functors.

For any point p in X we choose a local analytic embedding of X near p in C^m and call B_p the intersection of a small open ball in C^m centered at p with X. It follows from stratification theory that the topology of B_p depends only on p. Straight forward geometric arguments show that the complex $\mathbf{IC}^{\cdot}$ satisfies the following four properties:

(0) *Boundedness and constructibility*: $\mathbf{IC}^i = 0$ if $i < 0$ or if i is large enough; and for some stratification $\{S_a\}$ of X, $\mathbf{H}^i\mathbf{IC}^{\cdot}\,|\,S_a$ is locally constant and finitely generated for all i for all a.

(1) *Support*: for all $i > 0$,

$$\dim_C\{x \in X \mid \; H^i(B_x; \mathbf{IC}^{\cdot}) \neq 0\} < n-i.$$

(2) *Cosupport*: For all $i > 0$,

$$\dim_C\{x \in X \mid \; H_c^{2n-i}(B_x; \mathbf{IC}^{\cdot}) \neq 0\} < n-i.$$

(3) *Normalization*: For some stratification $\{S_a\}$ of X, $\mathbf{H}^i\mathbf{IC}^{\cdot}/(X-\Sigma)$ is zero for $i \neq 0$ and is the constant sheaf for $i = 0$.

THEOREM [4]. *The sheaf* $\mathbf{IC}^{\cdot}$ *is uniquely characterized up to quasi-isomorphism by the above conditions* (0), (1), (2), *and* (3).

It is useful to consider also intersection homology with twisted coefficients. The coefficients will be a local system L (i.e. a locally constant sheaf) on $X-\Sigma$. The group $IH_i(X, L)$ may be defined as in § 2 or we may define the sheaf $\mathbf{IC}^i(L)$ directly as follows. The value of $\mathbf{IC}\,(L)$ on an open set $U \subset X$ is those $2n-i$ chains c with closed supports on $U \cap (X-\Sigma)$ with coefficients in L satisfying the allowability condition: the closure in U of the support of c intersects each singular stratum $S_a \subset \Sigma$ in a set of real dimension less than $(2n-i)-c_a$ and the closure in U of the support of the boundary ∂c intersects each singular stratum $S_a \subset \Sigma$ in a set of real dimension less than $(2n-i-1)-c_a$. The sheaf $\mathbf{IC}^{\cdot}(L^{\cdot})$ may be characterized as in the above theorem replacing the normalization condition (3) by

(3′) *Normalization*: $\mathbf{H}^i\mathbf{IC}^{\cdot}(L)|X-\Sigma$ is zero for $i \neq 0$ and is L for $i = 0$.

Conjecturally, if L is a polarized variation of Hodge structure then $H^i(\mathbf{IC}^{\cdot}(L)) = IH^i(X, L)$ has a pure Hodge structure. This is verified for curves [82].

Sheaf theory enables one to give local expressions for some basic properties of intersection homology. For example Poincaré duality of § 2 becomes the statement that $\mathbf{IC}^{\cdot}(Q)$ is quasi-isomorphic to its dual (see [16] or [75]).

§ 9. Perverse sheaves. Intersection homology sheaves are objects in a beautiful Abelian category called the category $P(X)$ of perverse sheaves [12].

The category C of complexes of sheaves on X is deficient from the point of view of homological functors because quasi-isomorphisms $\mathbf{P}^{\cdot} \leftarrow \mathbf{R}^{\cdot} \rightarrow \mathbf{Q}^{\cdot}$ may not be morphisms in C, and even if they are (when $\mathbf{P}^{\cdot} \overset{=}{\leftarrow} \mathbf{P}^{\cdot} \rightarrow \mathbf{Q}^{\cdot}$) they may not be invertible in C. This situation may be remedied by introducing formal inverses of morphisms in C which are quasi-isomorphisms [46], [34]. The resulting category $D(X)$ is called the *derived category* of the category of sheaves on X. The isomorphisms in $D(X)$ are exactly the quasi-isomorphisms. But $D(X)$ is not Abelian: instead it has the structure of a "triangulated category" which is quite complicated [77].

The category $P(X)$ of perverse sheaves on X is the full subcategory of $D(X)$ whose objects satisfy the following slight weakening of the conditions characterizing the intersection homology sheaf $\mathbf{IC}^{\cdot}$:

DEFINITION. A complex of sheaves $\mathbf{K}^{\cdot}$ on X is called a *perverse sheaf* if it satisfies the following three properties:

(0) *Boundedness and constructibility*: $\mathbf{K}^{\cdot} = 0$ if $i < 0$ or if i is large enough; and for some stratification $\{S_a\}$ of X, $\mathbf{H}^i\mathbf{K}^{\cdot}|S_a$ is locally constant and finitely generated for all i and for all a.

(1) *Support*: For all i,

$$\dim_C\{x \in X \mid \boldsymbol{H}^i(B_x; \mathbf{IC}^{\cdot}) \neq 0\} \leqslant n-i.$$

(2) *Cosupport*: For all i,

$$\dim_C\{x \in X \mid \boldsymbol{H}_c^{2n-i}(B_x; \mathbf{IC}^{\cdot}) \neq 0\} \leqslant n-i.$$

We now digress to show how intersection homology sheaves provide a rich supply of perverse sheaves.

DEFINITION. An *enriched subvariety* of X is a pair $(V, \boldsymbol{L})$ where V is locally closed, non-singular, equidimensional subvariety and $\boldsymbol{L}$ is a local system of coefficients on V. Two enriched subvarieties $(V, \boldsymbol{L})$ and $(V', \boldsymbol{L}')$ are considered equal if $V \cap V'$ is dense in V and in V', and $\boldsymbol{L}|(V \cap V') = \boldsymbol{L}'|(V \cap V')$. An *irreducible* enriched subvariety $(V, \boldsymbol{L})$ is one where V is an irreducible variety and $\boldsymbol{L}$ is an irreducible local system on V. An enriched subvariety (V, L) gives rise to a complex $\mathbf{IC}^{\cdot}(\overline{V}, L)$ on X called the intersection homology sheaf of (V, L) by extending the complex $\mathbf{IC}^{\cdot}(L)$ on $\overline{V}$ by zero in X.

If $\mathbf{A}^{\cdot}$ is a complex of sheaves and c is an integer, we define $\mathbf{A}^{\cdot}[c]$ to be the same complex with the numbering shifted by c. In other words, the ith sheaf in $\mathbf{A}^{\cdot}[c]$ is $\mathbf{A}^{i+c}$.

Comparison of the definition of a perverse sheaf with the theorem of § 8 shows that if c is the codimension of V in X, $\mathbf{IC}^{\cdot}(\overline{V}, L)[-c]$ is a perverse sheaf on X. For example, if $x \in V$, we have $\boldsymbol{H}^{c_a}(B_x; \mathbf{IC}^{\cdot}(V, L)[-c]) = L_x$ and $\boldsymbol{H}_c^{2n-c_a}(B_x; \mathbf{IC}^{\cdot}(V, L)[-c]) = L_x$.

THEOREM. *The category $P(X)$ of perverse sheaves on X is an Artinian Abelian category whose simple objects are the complexes* $\mathbf{IC}^{\cdot}(\overline{V}, L)[-c]$ *for irreducible enriched subvarieties (V, L) of X.*

The category $P(X)$ is important because of its applications (§ 11, §13 and § 16). It would be interesting to understand its structure more directly (see [60] and [35]).

Applications to analysis

§ 10. L^2 cohomology. The L^2 cohomology of the nonsingular part $X-\Sigma$ of any compact analytic variety X, provided with an appropriate polyhedral metric, was found to be finite-dimensional and to satisfy Poincaré duality [24], [25]. This led to the question, resolved affirmatively, of whether this L^2 cohomology was in fact intersection homology with real coeficients. In this section we address the same question for metrics more naturally associated with the analytic structure of X.

We define $\Omega^i_{(2)}(X-\Sigma)$ to be the space of smooth i forms ω on $X-\Sigma$ such that

$$\int_{X-\Sigma} \omega \wedge *\omega < \infty, \qquad \int_{X-\Sigma} d\omega \wedge * d\omega < \infty.$$

We define the L^2 cohomology of $X-\Sigma$, $H^i_{(2)}(X-\Sigma)$, to be the ith cohomology of the complex $\Omega^{\cdot}_{(2)}(X-\Sigma)$. Since the operator $*$ depends on the Riemannian metric chosen on $X-\Sigma$, the group $H^i_{(2)}(X-\Sigma)$ also depend on this metric. We are interested in two contexts which differ by the choice of this metric:

Context I. The variety X is embedded analytically in a Kähler manifold M; the metric chosen on $X-\Sigma$ is induced via this embedding from the Kähler metric on M.

Context II. The manifold $X-\Sigma$ with its metric has finite volume and is obtained as the quotient space of a discrete group acting properly on a Hermitian symmetric domain; X is the Baily–Borel compactification of $X-\Sigma$.

The following conjectures may be summarized by the philosophy that analysis on $\Omega_{(2)}(X-\Sigma)$ should be related to analysis on $\tilde{X}$ where $\tilde{X}\to X$ is a small resolution (see § 5).

CONJECTURE. $H^i_{(2)}(X-\Sigma)\cong IH_{2n-i}(X;\boldsymbol{R})$ (see [26] for context I and [83] for context II).

CONJECTURE. Every class in $H^i_{(2)}(X-\Sigma)$ contains a unique harmonic (closed and coclosed) representative. (This is well-known in context II because the metric is complete.) A Hodge theory can be obtained by splitting the L^2 harmonic forms with values in $\boldsymbol{C}$ into their (p,q) pieces and it satisfies the properties of the conjecture of § 4.

CONJECTURE. The index of the $L^2\bar{\partial}$ complex is the arithmetic genus of $\tilde{X}$ where $\pi\colon\tilde{X}\to X$ is any resolution of singularities of X. More generally, the $K_*(X)$ element obtained from this complex by the analytic procedure of [8] coincides with the π pushforward of the K orientation of $\tilde{X}$ [10].

Work on these conjectures has been difficult for lack of adequate analytic methods to study $\Omega^{\boldsymbol{\cdot}}_{(2)}(X-\Sigma)$ near a singularity of X and for lack of adequate information on the metric structure of X near a singularity. Nevertheless, partial results are known. The first holds in context I for metrically conical singularities [25] and for some surfaces [49] and in context II for the case where Σ is a manifold [15] and for some rank 2 cases [84], [23]. The third holds in context I for curves [47].

§ 11. Relation to algebraic analysis. Holonomic $\mathfrak{D}$-modules are an important and beautiful subject in their own right, but here we treat them only in relation to our topological constructions. Suppose M is a non-singular analytic variety. Let $\mathfrak{D}$ be the sheaf of linear differential operators with analytic coefficients. Then $\mathfrak{D}$, as a sheaf of noncommutative rings on M, is filtered by the order. The associated graded $\mathrm{Gr}\,\mathfrak{D}$ is commutative; sections of it over $U\subset M$ may be interpreted as functions on T^*U. Suppose $\mathfrak{M}$ is a coherent sheaf of $\mathfrak{D}$-modules. $\mathfrak{M}$ is called *holonomic r.s.* (for regular singularities) if it has a filtration as a module over filtered $\mathfrak{D}$ so that $\mathrm{Gr}\,\mathfrak{M}$ has support in T^*M which is reduced and of pure dimension m. (This definition provides no intuitive feeling for holonomic r.s. $\mathfrak{D}$-modules; for this we refer the reader to [66], [13].) We denote by $\mathrm{ch}(M)$ (for characteristic variety) the algebraic cycle in $T^*(M)$ determined by $Gr\,\mathfrak{M}$. Any $\mathfrak{D}$-module $\mathfrak{M}$ determines a DeRham complex $\mathrm{DR}\,\mathfrak{M} = \mathfrak{M}\otimes\Omega^0 \xrightarrow{d} \mathfrak{M}\otimes\Omega^1 \xrightarrow{d}$ in the derived category $D(M)$ (see § 8).

THEOREM ([63], [50], [18]). *The category of holonomic r.s. $\mathfrak{D}$-modules on M is equivalent to the category $P(M)$ of perverse sheaves on M. The equivalence is given by the DeRham functor* DR.

Therefore the irreducible holonomic r.s. $\mathfrak{D}$-modules correspond to intersection homology sheaves of subvarieties of M. If $\mathrm{DR}\,\mathfrak{L} = \mathbf{IC}^{\cdot}(X)$ where $X\subset M$, then $\mathrm{ch}\,\mathfrak{L} = \mathrm{ch}(X)$ as defined in § 6.

There is a filtration of the $\mathfrak{D}$-module $\mathfrak{L}$ such that $\mathrm{DR}\,\mathfrak{L} = \mathbf{IC}^{\cdot}(X)$ which conjecturally gives rise to the Hodge filtration suggested in § 4 on $IH_i(X)$ [18]. The Fourier transformation, taking the category of radially homogeneous holonomic r.s. $\mathfrak{D}$-modules on $\boldsymbol{C}^n$ to itself, leads to a similar operation on $P(\boldsymbol{C}^n)$ with interesting applications [22], [19].

Applications to algebraic geometry

§ 12. The decomposition theorem. The decomposition theorem says that the pushforward of an intersection homology sheaf by a proper algebraic map is a direct sum of intersection homology sheaves. It contains as special cases the deepest homological properties of algebraic maps that we know. It was conjectured in [36] and proved in [12].

THEOREM. (1) *Let $f\colon X\to Y$ be a proper projective map of complex algebraic varieties. Then there exist a unique set of irreducible enriched subvarieties $\{(V_a, \boldsymbol{L}_a)\}$ in Y (V_a smooth in Y; $\boldsymbol{L}_a$ a local system of $\boldsymbol{Q}$ vector spaces on V_a) and polynomials $\{\varphi^a = \varphi_0^a+\varphi_1^a t+\ldots\}$ such that there is a quasi-isomorphism*

$$f_*\mathbf{IC}^{\cdot}(X) \approx \bigoplus_{a,i} \mathbf{IC}^{\cdot}(\overline{V}_a, \boldsymbol{L}_a)[-i]\otimes\boldsymbol{Q}^{\varphi_i^a}. \qquad (**)$$

(2) *The coefficients of φ^a are palindromic around $k_a = \dim X-\dim V_a$ (i.e. $t^{k_a}\varphi^a(t^{-1}) = \varphi^a(t)$) and the odd and even degree terms are separately unimodal (i.e. if $i\leqslant k_a$ then $\varphi_{i-2}^a\leqslant\varphi_i^a$).*

Applying hypercohomology to the above quasi-isomorphism, we get

$$P^X(t) = \sum_a \varphi^a(t)P^a(t),$$

where P^X is the Poincaré polynomial for $IH^*(X)$ and P^a is the Poincaré polynomial for $IH^*(\overline{V}_a, L_a)$. If X is compact, the $\overline{V}_a$ will also be compact so P^X and P^a have the character of the Poincaré polynomial of a smooth projective variety. Part (2) of the theorem asserts that the φ^a also have this character (palindromicity $\approx$ Poincaré duality and unimodality $\approx$ hard Lefschetz). So the theorem asserts that $IH^*(X)$ is a sum of terms, each of which is like the intersection homology of the product of a fictitious fiber variety (with Poincaré polynomial φ^a), and an enriched subvariety (V_a, L_a) of Y. (Conjecturally, L_a should be a polarized variation of Hodge structure.)

The quasi-isomorphism in the decomposition formula (**) can be made canonical by the following procedure [28] given a factorization of f as $X \hookrightarrow Y \times P^m \to Y$. The hyperplane class $[H]$ in $H^2(P^m)$ induces a map $\eta\colon f_* \mathbf{IC}^\cdot(X) \to f_* \mathbf{IC}^\cdot(X)$ [2] essentially by transversally intersecting cycles with $[H]$. If we denote by $\mathbf{A}^\cdot(i)$ the sum of the right-hand side of (**) over a so (**) reads $f_* \mathbf{IC}^\cdot(X) \approx \bigoplus_i \mathbf{A}^\cdot(i)$, then η decomposes into pieces $\eta_{ij}\colon \mathbf{A}^\cdot(i) \to A^\cdot(i-j)$ [2]. Set $\eta_j = \sum_i \eta_{ij}$. Then there is a unique quasi-isomorphism in (**) satisfying $(\mathrm{ad}\,\eta_0)^{i-1}\eta_i = 0$ for all i (where $\mathrm{ad}\,\eta_0\xi$ is $\eta_0\xi - \xi\eta_0$).

If $X \to Y$ is a resolution of singularities of Y, then one of the enriched subvarieties in the decomposition will be (Y, Q) and the corresponding φ^a will be 1. Thus $IH^*(Y)$ sits canonically in the cohomology of X given a factorization of the resolution as in the last paragraph. Conjecturally, this embedding determines the Hodge structure on $IH^*(Y)$ from that on X.

The decomposition theorem contains, for example, the invariant cycle theorem and the degeneration of the spectral sequence for f in case f is a topological fibration. See [40] for a discussion of some of its consequences. It is also one of the most powerful techniques for calculating intersection homology (see [5]). At the moment, the only proof of it goes through characteristic p algebraic geometry. (As a consequence, it is unproved for proper analytic maps.) It would be very interesting to find an analytic proof, either using $\mathfrak{D}$-modules or using L^2 techniques.

§ 13. Specialization. Suppose $\pi\colon X \to C$ is a map of an algebraic variety to a smooth algebraic curve, o is a point in C, and $c \in C$ is a nearby general point (i.e. π is a topological fibration near c). There is a continuous map called ψ from X_c, the fiber over c, to X_o the fiber over o, which roughly collapses points near a given stratum S in X_o to S [41]. There is also a map $\mu\colon X_c \to X_c$ with $\psi \circ \mu = \psi$ called the *monodromy* which represents the result of tracing paths over c as it moves once around o. Therefore there is a complex of sheaves $\psi_* \mathbf{IC}^\cdot(X_c)$ on X_0 called the *nearby cycles* of $\mathbf{IC}^\cdot$ with an action of μ_* on it. Although there is some choice in the specification of ψ and μ, the complex $\psi_* \mathbf{IC}^\cdot(X_c)$ and the action of μ_* are independent up to quasi-isomorphism of the choice. (Moreover, they can be defined purely algebraically [69].)

PROPOSITION ([41], [12]). *The complex $\psi_* \mathbf{IC}^\cdot$ is a perverse sheaf.*

Therefore we dispose of the Abelian category structure of $P(X_0)$ to analyze the monodromy μ_*. There is a factorization $\mu_* = F \cdot (1+N)$, where F has finite order and N is nilpotent. Then there is a unique increasing filtration W^i on $\psi_* \mathbf{IC}^\cdot$ such that N sends W^i to W^{i-2} and N^i takes $\mathrm{Gr}^i \mathbf{IC}^\cdot$ isomorphically to $\mathrm{Gr}^{-i} \mathbf{IC}^\cdot$, where Gr^i is the associated graded to the filtration W.

THEOREM ([33]). *The graded pieces $\mathrm{Gr}^i \psi_* \mathbf{IC}^\cdot(X_c)$ are semi-simple in $P(X_0)$. In other words, they are direct sums of intersection homology sheaves of irreducible enriched subvarieties of X_0.*

The study of $\psi_* \mathbf{IC}^\cdot$ can be generalized to the case where X_0 is replaced by an arbitrary hypersurface [78].

Applications to group theory

§ 14. Weyl group representations. Here and in the succeeding sections, we will treat only the case of $\mathrm{GL}(k, \boldsymbol{C})$. Lie theorists can imagine the generalization to an arbitrary reductive complex algebraic group with the aid of occasional parenthetical remarks.

Consider the variety $\mathfrak{N}$ of all $k \times k$ complex matrices all of whose eigenvalues are zero. It is singular, and it has a stratification $\{S_a\}$ indexed by partitions a of the number k. The stratum S_a consists of matrices whose sizes of Jordan blocks are given by a. Let c_a be the codimension of S_a in $\mathfrak{N}$. The variety $\mathfrak{N}$ also has a resolution $\pi\colon \tilde{\mathfrak{N}} \to \mathfrak{N}$ constructed as follows: Let $\mathfrak{B}$ denote the manifold of complete flags $0 \subset F_1 \subset F_2 \subset \ldots \subset F_{k-1} \subset \boldsymbol{C}^k$ in $\boldsymbol{C}^k$. Then $\tilde{\mathfrak{N}}$ is the subvariety of $\mathfrak{N} \times \mathfrak{B}$ consisting of pairs $(x, \{F_j\})$ so that $xF_j \subset F_{j-1}$. The map π is the projection to the first factor. The decomposition formula (§ 12) for $\mathbf{A}^\cdot = \pi_* \mathbf{IC}^\cdot(\tilde{\mathfrak{N}})$ may be computed as follows:

PROPOSITION ([17]).

$$\mathbf{A}^\cdot = \bigoplus_a \mathbf{IC}^\cdot(\bar{S}_a)[-c_a] \otimes \boldsymbol{Q}^{\varphi^a}.$$

So $\mathbf{A}^{\cdot}$ is a semisimple perverse sheaf on $\mathfrak{N}$. (For other reductive groups, enriched strata will be necessary.)

The symmetric group on k letters, denoted W (the Weyl group of $\mathrm{GL}(k)$), acts on $\mathbf{A}^{\cdot}$ by automorphisms in $P(\mathfrak{N})$, the category of perverse objects on $\mathfrak{N}$. This action, constructed first in [72], has several descriptions; we follow [71]. We map $\mathfrak{G}$, the space of all $k\times k$ complex matrices, to C^k by the coefficients of the characteristic polynomial. Then $\mathfrak{N}$ is the fiber over 0, and if $c\in C^k$ is a nearby non-singular point, it turns out that $\mathbf{A}^{\cdot}$ is $\psi_*\mathbf{IC}^{\cdot}(\mathfrak{G}_c)$. Just as in § 13, the fundamental group of the complement of the discriminant in C^k acts on $\mathbf{A}^{\cdot}$ by monodromy. This fundamental group is the braid group on k strands; its action factors through the quotient map to W.

Proposition ([17]). *The action of W on $\mathbf{A}^{\cdot}$ induces an isomorphism from the group ring of W to the endomorphism ring of $\mathbf{A}^{\cdot}$ in $P(\mathfrak{N})$.*

It follows that isotypical components $\mathbf{IC}^{\cdot}(\bar{S}_a)[-c_a)\otimes Q^{v^a}$ of $\mathbf{A}^{\cdot}$ correspond bijectively to irreducible representations of W. This correspondence between partitions a and irreducible W representations P_a is that of Young. Applying hypercohomology, we obtain as another corollary the formula that the multiplicity of p_a in the standard action of W on $H^i(\mathfrak{B})$ is $\dim IH^{i-c_a}(\bar{S}_a)$ ([17], [54]).

§ 15. Representations of Hecke algebras. We sketch the contents of the papers [51], [52], whose historical importance in stimulating the recent development of the material of this report cannot be overemphasized. The form of our presentation relies on these later developments.

The variety $\mathfrak{B}\times\mathfrak{B}$, where $\mathfrak{B}$ is the flag manifold as in § 14, is stratified by orbits S of the diagonal action of $\mathrm{GL}(k, C)$. The orbits are parametrized by elements a of the symmetric group W. A pair of flags $\{F_j\}$, $\{F'_j\}\in\mathfrak{B}\times\mathfrak{B}$ corresponds to $a\in W$ if there is a basis $e_1, e_2, \ldots, e_k$ of C^k so that F_j is the span of $e_1, e_2, \ldots, e_j$ and F'_j is the span of $e_{a(1)}, e_{a(2)}, \ldots, e_{a(j)}$.

The Hecke algebra $\mathfrak{H}$ is the set of formal linear combinations of intersection homology sheaves $\sum_{a,i} m_{a,i}\mathbf{IC}^{\cdot}(\bar{S}_a)[i]$ with integral multiplicities $m_{a,i}$. Addition is addition of multiplicities; multiplication is obtained by regarding the intersection homology sheaves as "sheaf valued correspondences" on $\mathfrak{B}\times\mathfrak{B}$. That is $\mathbf{A}^{\cdot}\cdot\mathbf{B}^{\cdot} = p_{13*}(p_{12}^*\mathbf{A}^{\cdot}\otimes p_{23}^*\mathbf{B}^{\cdot})$ where p_{13}, p_{12}, and p_{23} map $\mathfrak{B}\times\mathfrak{B}\times\mathfrak{B}$ to $\mathfrak{B}\times\mathfrak{B}$ by forgeting the factor not named. The Hecke algebra is an algebra over $\boldsymbol{Z}[t, t^{-1}]$ where t sends $\mathbf{IC}^{\cdot}(\bar{S}_a)[i]$ to $\mathbf{IC}^{\cdot}(\bar{S}_a)[i-1]$. (For general reductive Lie groups, $\mathfrak{H}$ depends only on the Weyl group W, even though $\mathfrak{B} \times \mathfrak{B}$ and the singularities of the strata are not determined by W alone.)

There is a positive cone $\mathfrak{K} \subset \mathfrak{H}$ consisting of actual complexes of sheaves on $\mathfrak{B} \times \mathfrak{B}$, i.e. $\mathfrak{K}$ consists of formal linear combinations where all multiplicities $m_{a,i}$ are non-negative. Clearly $\mathfrak{K}$ is closed under multiplication in $\mathfrak{H}$. This defines a partial ordering $\leqslant_L$ of the elements of W by $a \leqslant_L a'$ if $\mathfrak{K} \cdot \mathbf{IC}^{\cdot}(\bar{S}_a)$ contains an element where $\mathbf{IC}^{\cdot}(\bar{S}_{a'})[i]$ occurs with positive multiplicity. A *left cell* in W is a $\leqslant_L$ equivalence class of elements of W. Any left cell $C \subset W$ gives rise to a representation $R(L)$ of the Hecke algebra $\mathfrak{H}$ with C as a $\boldsymbol{Z}[t, t^{-1}]$ basis as follows $R(C)$ is a subquotient of the regular representation of $\mathfrak{H}$; say $R(C) = \mathfrak{S}/\mathfrak{T}$. Here $\mathfrak{S} \subset \mathfrak{H}$ is generated over $\boldsymbol{Z}[t, t^{-1}]$ by $\mathbf{IC}^{\cdot}(\bar{S}_a)$ where $\{C\} \leqslant_L a$ and $\mathfrak{T}$ is generated by $\mathbf{IC}^{\cdot}(\bar{S}_a)$, where $\{C\} <_L a$.

THEOREM. *The representations $R(C)$ are irreducible. All irreducible representations of $\mathfrak{H}$ occur as $R(C)$ for some left cell C.*

(This theorem does not generalize easily to other reductive groups.) The main interest in this theorem is that the combinatorics of it can be explicitly spelled out. The left cells C arise from partitions of k by the Robinson–Schensted algorithm. The representations $R(C)$ may be written with respect to the basis $\{\mathbf{IC}^{\cdot}(S_a), a \in L\}$ using inductively computable combinatorial objects called W-graphs. If t is specialized to 1, then $\mathfrak{H}$ becomes the group ring $\boldsymbol{Z}[W]$ of W and the theorem gives the irreducible representations of the symmetric group W with a basis with particularly agreeable properties: for example, all elements are represented by matrices with integral entries.

The methods of this section can be used to compute both the local and global intersection homology groups of the Schubert varieties $\bar{S}_a$. A fascinating phenomenon is that these groups are all zero in odd degrees. The same holds for the computations for toric varieties [5], nilpotent varieties [17], and K_C orbits in $\mathfrak{B}$ [57]. One would like a general explanation for this phenomenon.

§ 16. Lie algebra representations. In this section, we discuss the serendipitous entry of intersection homology into the study of infinite-dimensional representations of Lie algebras.

Recall that the $k \times k$ complex matrices under the bracket operation, denoted $\mathfrak{G}$, is the Lie algebra of the group of invertible $k \times k$ matrices under composition, denoted G. The sub-Lie algebra $\mathfrak{n}^+$ of upper triangular matrices that are zero on the diagonal is the Lie algebra of the subgroup

N^+ of upper triangular matrices that are one on the diagonal. A $(\mathfrak{G}, N^+)$ representation V is a possibly infinite-dimensional representation of $\mathfrak{G}$ such that the action of $\mathfrak{n}^+$ exponentiates to representation of N^+ and, for all vectors $v \in V$, N^+v is contained in a finite-dimensional subspace on which N acts algebraically. In order to focus on the part of the theory where the transition from algebra to topology is fully understood, we will consider only $\bar{\mathcal{O}}_1$ the category of $(\mathfrak{G}, N^+)$ representations V satisfying the following condition:

The space V has a finite decomposition into summands V_i on each of which the center Z of $U(\mathfrak{G})$, acts as it does on some irreducible finite-dimensional representation ϱ_i of $\mathfrak{G}$.

Recall that $U(\mathfrak{G})$ is the universal enveloping algebra of $\mathfrak{G}$. It is the associative algebra containing $\mathfrak{G}$ such that the embedding induces an equivalence between the representation theory of $\mathfrak{G}$ and that of $U(\mathfrak{G})$. Every element of its center acts on a finite-dimensional irreducible representation ϱ by a scalar.

The passage to topology proceeds in four steps:

Step I. The category $\bar{\mathcal{O}}$ is the direct sum of the categories $\bar{\mathcal{O}}_\varrho$ for ϱ a finite-dimensional irreducible representation of $\mathfrak{G}$, defined by replacing $*$ by the condition that Z acts on all of V as it does on ϱ. So it suffices to understand $\bar{\mathcal{O}}_\varrho$.

Step II. The categories $\bar{\mathcal{O}}_\varrho$ are all equivalent to each other by an algebraic process called coherent continuation [14]. So it suffices to understand $\bar{\mathcal{O}}_1$ where 1 is the trivial one-dimensional representation.

Step III [21], [11]. The category $\bar{\mathcal{O}}_1$ is equivalent to the category of holonomic r.s. $\mathfrak{D}$-modules on the flag manifold $\mathfrak{B}$ constructible with respect to the stratification $\{S_a\}$ of $\mathfrak{B}$ by N^+ orbits. (Saying that a holonomic r.s. $\mathfrak{D}$-module $\mathfrak{M}$ is constructible with respect to $\{S_a\}$ means that its characteristic variety $\mathrm{ch}(\mathfrak{M})$ is contained in the union of the conormal bundles $C(S_a)$. The strata S_a are Schubert cells: they are restrictions to a slice point $\times \mathfrak{B}$ of the strata of $\mathfrak{B} \times \mathfrak{B}$ of § 15. So the S_a are again naturally indexed by the symmetric group W.) The equivalence is given by associating to a $\mathfrak{D}$-module $\mathfrak{M}$ the vector space of its global sections. This is a representation of $\mathfrak{G}$ because elements of $\mathfrak{G}$ give vector fields on $\mathfrak{B}$ which are global sections of $\mathfrak{D}$.

Step IV. As in § 9 the category of holonomic r.s. $\mathfrak{D}$-modules on $\mathfrak{B}$, constructible with respect to the strata S_a is equivalent to the category of perverse sheaves on $\mathfrak{B}$ constructible with respect to S_a. (A perverse sheaf $\mathbf{A}^\bullet \in P(\mathfrak{B})$ is constructible with respect to $\{S_a\}$ if all of its homology sheaves $\mathbf{H}^i(\mathbf{A}^\bullet)$ are locally constant on the S_a.)

With the transition completed, we see that irreducible representations in $\bar{\mathcal{O}}$ correspond to intersection homology sheaves $\mathbf{IC}^\bullet(\bar{S}_\alpha)[-c_\alpha]$. Any purely category-theoretic question about $\bar{\mathcal{O}}$ can be answered topologically using $P(\mathfrak{B})$. As an example (the historically motivating one), the Kazhdan–Lusztig conjectures [51], were proved this way [11], [21]. A Verma module in $\bar{\mathcal{O}}$ is a representation that is $U(\mathfrak{n}^-)$ free where $\mathfrak{n}^-$ is the lower triangular matrices zero on the diagonal. It corresponds in $P(\mathfrak{B})$ to a sheaf whose stalk Euler characteristic is 1 on one S_β and 0 on all the other strata. If L is the irreducible module in $\bar{\mathcal{O}}_p$ corresponding to $\mathbf{IC}^\bullet(\bar{S}_\alpha)[-c_\alpha]$, it has a resolution $\ldots V_2 \to V_1 \to V_0 \to L$ where the V_i are direct sums of Verma modules. The Kazhdan–Lusztig conjectures state that $\sum(-1)^i m_i$ where m_i is the multiplicity of the Verma module in $\bar{\mathcal{O}}_p$ corresponding to S, is the Euler characteristic of the stalk homology of $\mathbf{IC}^\bullet(\bar{S}_\alpha)[-c_\alpha]$ at a point in S_β. Given all of the above facts, the proof is an exercise.

There is an extension of the above theory to $(\mathfrak{G}, K)$ modules whenever K has only finitely many orbits on $\mathfrak{B}$ [11], [79]. This applies to Harish-Chandra modules. There are also two other similar but unproved conjectured relations between algebra and topology: one for representations of p-adic groups [81] and one for modular representations [55].

§ 17. Other subjects. Only some subjects relating both to topology and to complex algebraic geometry have been treated. This leaves out much interesting work on intersection homology in the two fields separately.

In topology, there is an L class in $H_*(X)$ which relates to the intersection homology signature [3], [24]. There is a singular cobordism theory for a class of spaces called rational Witt spaces which are the most general on which rational Poincaré duality holds. The cobordism groups are the Witt ring of Q in dimension $4n$ and 0 otherwise [70]. There is an integral version of this [65] and an application of these ideas to the proof of the *Hauptvermutung* [64]. There is a theory of intersection homology operations [38], and a theory of obstructions to immersion [44].

In algebraic geometry, there is a construction of $\mathbf{IC}^\bullet(X)$ in the l-adic topology for varieties in characteristic p. If X is complete, Frobenius F acts on $IH^i(X)$ with eigenvalues of absolute $p^{i/2}$, so statements about the number of points defined over $F(p^k)$ analogous to those of the Weil conjectures are obtained [27], [12]. For an algebraic group G defined over $F(p^k)$, there is a collection of intersection homology sheaves on G such that interesting characters of G are evaluated at $p \in G$ by taking the alternating trace of a particular automorphism of the sheaves in the stalk over p [56].

We close with a general remark on the enterprise of studying the global topology of singular spaces. Usually a given concept in the topology of manifolds such as homology or a characteristic class has several plausible extensions to singular spaces. The art is to find the most useful one. When this is done, it often happens that one finds new results about the non-singular case. For example, results like those of [12] and [13] would not have been anticipated five years ago, even when all spaces involved are smooth.

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