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Commutators, Singular Integrals on Lipschitz Curves and Applications

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The topics discussed in this lecture had their origin in the theory of linear partial differential equations. In order to explain how the problem of the so-called commutators and that of the Cauchy integral on Lipschitz curves arose, I will recall and analyze some of the modern methods employed in the theory of linear partial differential equations, and in particular that of the pseudodifferential operators which became widely used in the last decade.

Let us consider the basic idea of the method of pseudo-differential operators. Every linear partial differential operator is a sum of monomial operators

$$a(x)\left(\frac{\partial}{\partial x}\right)^{\alpha}, \tag{1}$$

and the operator $(\partial/\partial x)^{\alpha}$ applied to the function $f(x)$ can be thought of as multiplication of the Fourier transform $\hat{f}(\xi)$ of f by the function $(-i\xi)^{\alpha}$, that is,

$$a(x)(\partial/\partial x)^{\alpha} = a(x)K_{\alpha}, \quad \text{where} \quad (K_{\alpha}f)\hat{} = \hat{f}(\xi)(-i\xi)^{\alpha}.$$

Consequently, a linear differential operator L can be expressed as

$$\begin{gathered} Lf = \sum a_{\alpha}(x)K_{\alpha}f, \ (K_{\alpha}f)\hat{} = \hat{f}(\xi)(-i\xi)^{\alpha}, \quad \text{or} \\ Lf = \frac{1}{(2\pi)^{n}}\int \sum a_{\alpha}(x)(-i\xi)^{\alpha}e^{-ix\cdot\xi}\hat{f}(\xi)\,d\xi. \end{gathered} \tag{2}$$

Now, pseudo-differential operators are obtained by replacing the function

$$\sum a_{\alpha}(x)(-i\xi)^{\alpha} \tag{3}$$

in the preceding expression, which in the case of differential operators is a polynomial in ξ, by more general functions $p(x, \xi)$ in such a way that the resulting class of operators be closed under composition, adjunction, inversion if possible, etc. One should observe here that if the class is to be closed under composition, differential operators contained in it should be freely composable. As is well known, a differential operator can be freely composed with itself only if its coefficients are infinitely differentiable. Thus, classes of pseudo-differential operators which are closed under composition cannot possibly contain differential operators with non-smooth coefficients.

Another method, which preceded chronologically the one above and avoids this obstacle, is that of the singular integral operators. It consists in writing the polynomial in (3) as

$$\frac{1}{(2\pi)^n}\sum a_\alpha(x)(-i\xi)^\alpha = [q(x, \xi)+r(x, \xi)]\varphi(\xi)^m$$

where m is the degree of the polynomial, $\varphi(\xi)$ is a positive infinitely differentiable function such that $\varphi(\xi)=|\xi|$ if $|\xi|\geq 1$, and

$$q(x, \xi) = |\xi|^{-m}\sum_{|\alpha|=m} a_\alpha(x)(-i\xi)^\alpha.$$

Then if

$$Kf = \int q(x, \xi)e^{-ix\cdot\xi}\hat{f}(\xi)\,d\xi+Sf,$$
$$Sf = \int r(x, \xi)e^{-ix\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{4}$$

we have

$$Lf = K\Lambda^m f, \quad \text{where} \quad (\Lambda f)^\wedge = \varphi(\xi)\hat{f}(\xi). \tag{5}$$

The function $q(x, \xi)$ is homogeneous of degree zero in ξ, and, as is readily verified, the operators S and $S(\partial/\partial x)$ are bounded in L^2, or more generally in L^p, $1<p<\infty$, provided the coefficients $a_\alpha(x)$ are bounded. Now the operators K are generalized in the following manner: one replaces $q(x, \xi)$ by a function which is homogeneous of degree zero in ξ and bounded but otherwise arbitrary, and S by any operator with the properties described above. Evidently, $q(x, \xi)$ cannot be assumed to be more regular, as a function of x, than the coefficients of the differential operators we want to be included in the theory. On the other hand, if one considers general differential operators whose coefficients have a certain degree of regularity, it seems reasonable to exclude those whose terms of highest order have coefficients not satisfying at least a Lipschitz condition. This becomes clear if one considers the case of first order operators. If one allows the coefficients not to satisfy a Lipschitz condition there can arise pathologies such as the nonuniqueness of trajectories of the associated vector fields. This suggests restricting the generalization to the operators K in (4) to those for which the function $q(x, \xi)$ is bounded homogeneous of degree zero and regular in ξ and Lipschitzian in x. Then every differential operator whose coefficients are bounded and Lipschitzian for the highest order terms and merely bounded for the remaining ones can be represented as in (5) with such a $q(x, \xi)$. However, in order that this description be useful the operators

K in (4) thus generalized should form an algebra under composition. This is indeed the case, and this algebra becomes an instrument which allows us to manipulate effectively general linear differential operators and obtain for them results on existence, uniqueness, a priori estimates, etc. Even in the case of operators with smooth coefficients this allows us to obtain estimates which depend only on the bounds of the coefficients and the bounds of the first order derivatives of the coefficients of highest order terms. But let us return to the generalized operators K as in (4). The problem of showing that the composition of two such operators is one of the same kind can be reduced without great difficulty to the following problem: let A be, in the case of one variable, the operator multiplication by the bounded Lipschitzian function $a(x)$ and Hf the Hilbert transform of f. As is well known, this transform can be expressed as follows

$$(Hf)(x) = \frac{i}{2} \int_{-\infty}^{+\infty} sg\xi e^{-ik\xi} \hat{f}(\xi)\, d\xi$$

and this makes it clear that A, H and AH are operators of the type of the generalized K, and the simplest of their kind. In order to show that HA is of the same type, since

$$HA = AH + (HA - AH),$$

it would suffice to show that $(HA-AH)D$, $D = d/dx$, is bounded in L^p, $1 < p < \infty$. This was done in 1965 in [4] with the aid of the theory of analytic functions and a result closely related to an old conjecture of Littlewood. If we denote now by $C_a(K)$ the commutator of K and A, that is $AK-KA$, then

$$(AH-HA)D = C_a(H)D = C_a(HD) - HC_a(D)$$

and since the operator $C_a(D)$ is multiplication by $a'(x)$, which is a bounded function if $a(x)$ is Lipschitzian, $HC_a(D)$ is bounded in L^p and the continuity of $C_a(H)D$ is equivalent to that of $C_a(HD)$. Now, it is easy to see that

$$C_a(HD)f = \text{p.v.} \int_{-\infty}^{+\infty} \frac{(-1)}{x-y} \left[\frac{a(x)-a(y)}{x-y}\right] f(y)\, dy. \tag{6}$$

The integral on the right, which in the case $a(x)=x$ reduces to the Hilbert transform, is the one studied in [4] and is the so-called first commutator. Thus, its role in the theory of partial differential equations becomes apparent.

Next let us consider some generalizations of (6) whose interest we will explain later. The first one

$$C_a^m(HD^m)f = \text{p.v.} \int_{-\infty}^{+\infty} \frac{(-1)^m m!}{x-y} \left[\frac{a(x)-a(y)}{x-y}\right]^m f(y)\, dy \tag{7}$$

is the so-called mth commutator. This equality is not evident but also not difficult to prove. Aside from the intrinsic interest of the left-hand side of (7) and the analogy of the right-hand sides of (6) and (7), the integral in (7) is a special case of

$$\text{p.v.} \int_{-\infty}^{+\infty} \frac{1}{x-y} F\left[\frac{a(x)-a(y)}{x-y}\right] f(y)\, dy \tag{8}$$

where F is an analytic function of its argument. Several classical integrals are special cases of (8). Let Γ be the graph of the real valued function $a(x)$, $x \in \boldsymbol{R}$, that is, the range of the function $x+ia(x)$ in the complex plane, and let us regard the function $f(x)$ as a function on Γ. Consider now the Cauchy integral of this function on Γ

$$G(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(y)[1+ia'(y)]}{z-[y+ia(y)]}\, dy.$$

Then the limit of $G(z)$ when z approaches $x+ia(x)$ from above Γ and non-tangentially, if it exists, is given by

$$\frac{1}{2} f(x) + \frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{+\infty} \frac{1}{x-y} \left[1+i\,\frac{a(x)-a(y)}{x-y}\right]^{-1} f(y)[1+ia'(y)]\, dy, \tag{9}$$

and this integral, except for the presence of the factor $[1+ia'(y)]$ which can be incorporated in the function $f(y)$, is of the form (8). Thus, the study of the behaviour at the boundary of analytic functions given by integrals of the Cauchy type reduces to the study of an integral of the type (8).

On the other hand, let us consider the derivative at the point $x+ia(x)$ and in the direction of the vector $(a'(x), -1)$ of the logarithmic potential of the mass distribution $f(y)\, dy$ on Γ. This derivative, if it exists, is given by the following expression

$$\pm \frac{1}{2} f(x) + \frac{1}{2\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{(x-y)a'(x)-a(x)+a(y)}{(x-y)^2+[a(x)-a(y)]^2} f(y)\, dy \tag{10}$$

which is also essentially of the form (8). Similarly, the value on Γ of the potential of a double layer distributed on Γ is given by the transpose of the preceding expression. As is well known, these potentials are used to obtain and represent solutions of boundary value problems for the Laplace equation such as the Dirichlet problem, the Neumann problem, etc. The Neumann problem, for example, reduces to the integral equation obtained by equating (10) to the given function on the boundary Γ. The applicability of this method depends on Γ being such that the resulting integrals have reasonable continuity properties. However, it depends less on the specific type of boundary value problem or equation under consideration than the methods using potential theory which are inapplicable to the Neumann problem, for example. For this reason it is natural to expect that the study of the integral (8) and its generalizations will yield an effective tool for the treatment of boundary value problems for elliptic equations in domains with nonregular boundaries. In fact, some interesting results have already been obtained.

Having justified the interest of the integral (8), let us see what can be said about it. In the first place we observe that if one develops in power series the function F in (8), that integral appears as a series of integrals as in (7). Thus, it would suffice in principle to study these, which are apparently simpler. As was mentioned before, the case $m=1$ of (7) could be treated by means of a technique based on the use of analytic functions. Unfortunately, this method fails utterly if $m \geqslant 2$, and this case resisted all efforts to extend to it the results known for $m=1$ until 1975, when R. Coifman and Y. Meyer [11] settled the case $m=2$ with an entirely different approach. They succeeded by using simultaneously the Fourier and the Mellin transforms and certain real variable methods of M. Cotlar and C. P. Calderón. Soon afterwards they extended their results to all m, and more recently and by using a generalization of the theory of the function g of Littlewood and Paley, they obtained the continuity in L^p, $1<p<\infty$, of the left-hand side of (7) with HD^m replaced by a pseudo-differential operator in $S^m_{1,0}$ in several variables. Unfortunately, the estimates for the norms of the operators (7) obtained by these methods do not allow to sum the series resulting from the power series expansion of the function F in (8). However, last year it was observed, [5], that the technique of analytic functions used in the treatment of (6), strengthened by the results on weighted inequalities between a function, its maximal function and area function of B. Muckenhoupt, R. P. Gundy and R. L. Wheeden, [20], and certain results on conformal mapping, is applicable to the Cauchy integral in (9). It was already known that from results on the Cauchy integral there follow corresponding results on the integral in (8). Specifically, it was shown that the integral in (8) represents a bounded operator in L^p, $1<p<\infty$, provided that $\|a'\|_\infty<\varrho\alpha$, where ϱ is the radius of a disc centered at the origin where F is analytic, and α is an absolute positive constant. By means of the so-called rotation method one can extend this result to functions of several variables and prove that, for example, if $k(x,z)$, $x, z\in \boldsymbol{R}^n$, is bounded, homogeneous of degree $-n$ and even (odd) in z, F is odd (even) and analytic in a disc of radius ϱ centered at the origin, and $a(x)$ is Lipschitzian and such that $\|\nabla a\|_\infty<\varrho\alpha$, then the operator

$$\text{p.v.} \int_{R^n} k(x, x-y) F\left[\frac{a(x)-a(y)}{|x-y|}\right] f(y)\, dy \tag{11}$$

is well defined and continuous in L^p, $1<p<\infty$. Later on we shall outline the proof of the results of R. Coifman and Y. Meyer as well as those on this last integral.

Before proceeding to describe some of the applications of the foregoing results, I would like to mention still another result on commutators due to R. Coifman, R. Rochberg and G. Weiss, [17], which is of a different character. Let $k(x)$, $x\in \boldsymbol{R}^n$, be homogeneous of degree $-n$, of mean value zero on $|x|=1$, and sufficiently regular in $x\neq 0$, and K the operator convolution with k, then if $a(x)$ is of bounded mean oscillation the operator

$$C_a^m(K)f = \text{p.v.} \int (a(x)-a(y))^m k(x-y) f(y)\, dy$$

is bounded in L^p, $1<p<\infty$.

Now let us turn to applications. Let Γ be a simple rectifiable arc in the complex plane. Then the function

$$G(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\, dw,$$

where $f(w)$ is a function on Γ which is integrable with respect to arc length, has a limit almost everywhere in Γ as z approaches nontangentially a point of Γ. In the case of several variables one has similar results about double layer potentials and derivatives of single layer potentials of functions defined on graphs of functions which are of bounded variation in the sense of Tonelli. This gives an affirmative answer to old problems about the existence of such limits.

Another application is the following result due to D. E. Marshall (personal communication) which confirms an old conjecture of A. Denjoy (C. R. Acad Sci. Paris **149** (1909), 258—260): the analytic capacity $\gamma(E)$ of a compact subset E of a rectifiable arc in the complex plane is zero if and only if its one-dimensional Hausdorff measure vanishes.

Finally, I will mention some applications to the theory of partial differential equations which motivated the study of our subject. In the first place, on the basis of the preceding results it is possible to construct algebras of singular integral operators [6] which allow to extend automatically to equations with bounded coefficients and terms of highest order with bounded Lipschitzian coefficients the results on the uniqueness of the Cauchy problem and the existence and uniqueness of solutions of totally hyperbolic systems obtained in [6] and [7]. On the other hand, results such as the ones obtained by E. Fabes, M. Jodeit and N. M. Riviere for the Laplace equation, [9], with the method of the integral equations on the boundary described earlier, are surely also obtainable for much more general elliptic systems. Let us see what these results are. Let Ω be a domain in R^n with boundary $\partial\Omega$ of class C^1. Let N_y be the interior normal unit vector at the point y of $\partial\Omega$, and Λ_y a cone with vertex at y, with fixed height and aperture, and except for its vertex, entirely contained in Ω. Then, in the case of the Dirichlet problem, one has the following: if $g(y)$ is a function in $L^p(\partial\Omega)$, $1<p<\infty$, there is a unique function $u(x)$, harmonic in Ω, and such that

$$u^*(y) = \sup\{|u(x)|\,|x\in\Lambda_y\}\in L^p(\partial\Omega),\ \lim_{x\to y}\{u(x)|x\in\Lambda_y\} = g(y) \quad \text{p.p.}$$

This result was obtained for the first time by B. E. J. Dahlberg with different methods which show that if $p\geq 2$ the same holds even if $\partial\Omega$ is merely Lipschitzian. If in addition $g(y)\in L_1^p(\partial\Omega)$, then $|\nabla u|^*(y)$, whose definition is similar to that of $u^*(y)$, also belongs to $L^p(\partial\Omega)$. On the other hand, in the case of the Neumann problem, one has that if $g(y) \in L^p(\partial\Omega)$ there exists a harmonic function $u(x)$, which is unique up to an additive constant, such that $|\nabla u|^*(y)$ is in $L^p(\partial\Omega)$ and

$$\lim_{x \to y} \{\nabla u(x) \cdot N_y \,|\, x \in \Lambda_y\} = g(y) \quad \text{p.p.}$$

These results are also valid if $\partial\Omega$ is merely Lipschitzian, provided that the local oscillation of N_y does not exceed a constant which depends on p but not on Ω, or only on certain global properties of Ω.

An interesting consequence of the preceding results is the following. Given p, $1<p<\infty$, there is a positive ε such that if the local oscillation of the normal N_y to the boundary $\partial\Omega$ of a Lipschitzian domain Ω is less than ε, then every harmonic measure on $\partial\Omega$ is absolutely continuous with respect to surface area and has a density in $L^p(\partial\Omega)$.

The method of R. Coifman and Y. Meyer. We shall now outline the elegant way in which these authors treat the problem of the commutators by its reduction to the continuity of certain multilinear operators. We shall confine ourselves to the bilinear case where the ideas and techniques they employ are already apparent.

THEOREM 1. *Let* $\hat{\varphi}^{(1)}(\xi)$ *and* $\hat{\varphi}^{(2)}(\xi)$ *be two infinitely differentiable functions with compact support in* $\boldsymbol{R}^n$ *such that at least one of them vanishes in a neighborhood of the origin. Let* $\varphi_t(x)=t^{-n}\varphi(x/t)$, $t>0$, *where* $\varphi(x)$ *is the inverse Fourier transform of* $\hat{\varphi}(\xi)$. *Let*

$$g = \int_0^\infty (f_1 * \varphi_t^{(1)})(f_2 * \varphi_t^{(2)})\, m(t) \frac{dt}{t}$$

where $m(t)$ *is a bounded function. Then*

$$\|g_2\| \leqslant c\|f_1\|_2 \|f_2\|_\infty \|m\|_\infty$$

where c *depends only on the functions* $\varphi^{(j)}(x)$.

In order to show this let us assume first that both functions $\hat{\varphi}^{(j)}(\xi)$ vanish near the origin, that is, they both have support in $0<a\leqslant|\xi|\leqslant b$. It is easy to see that if $\hat{\eta}(\xi)$ has compact support and equals 1 in $\xi\leqslant 2b$, and $\hat{\eta}(\xi)=\hat{\eta}(-\xi)$, then

$$g = \int_0^\infty \eta_t * [(f_1 * \varphi_t^{(1)})(f_2 * \varphi_t^{(2)})]\, m(t) \frac{dt}{t}$$

and

$$\int hg\, dx = \int\int_0^\infty (h * \eta_t)(f_1 * \varphi_t^{(1)})(f_2 * \varphi_t^{(2)})\, m(t) \frac{dt}{t}\, dx. \tag{12}$$

If we assume now that f_2 is bounded then $|(f_2 * \varphi_t^{(2)})|^2 (dx\, dt/t)$ is a Carleson measure, that is, if Q is a cube in $\boldsymbol{R}^n \times \{t \geqslant 0\}$ with base B in $t=0$, and denote this measure by μ, then $\mu(Q) \leqslant c\|f_2\|_\infty^2 |B|$. Hence, as is well known,

$$\int |(h * \eta_t)|^2\, d\mu \leqslant c\|h\|_2^2 \|f_2\|_\infty^2 .$$

On the other hand, by taking Fourier transforms one verifies readily that

$$\int_0^\infty \int |(f_1 * \varphi_t^{(1)})|^2 \frac{dx\,dt}{t} \leq c\|f_1\|_2^2$$

and using these inequalities in estimating (12) the desired result follows.

In case one of the functions $\hat{\varphi}^{(i)}(\xi)$, call it simply $\hat{\varphi}(\xi)$, does not vanish near the origin while the other vanishes in $|\xi| \leq a$, one decomposes $\hat{\varphi} = \hat{\Phi} + \hat{\varrho}$, where $\hat{\Phi}$ vanishes in $|\xi| \leq a/8$ and $\hat{\varrho}$ has support in $|\xi| \leq a/4$. The contribution of $\hat{\Phi}$ is treated exactly as is the preceding case. The contribution of $\hat{\varrho}$ is treated essentially the same way, with the difference that now one chooses $\hat{\eta}(\xi)$ so that it equals 1 in $a/4 \leq |\xi| \leq 2b$ and vanishes near the origin, and then (12) can again be estimated as above.

Remark. A closer examination of the preceding argument shows that if $\hat{\varphi}^{(j)}(\xi)$ is replaced by $e^{iu_j\cdot\xi}\hat{\varphi}^{(j)}(\xi)$, then

$$\|g\|_2 \leq c(1+|u_1|)^{n/2}(1+|u_2|)^{n/2}\|f_1\|_2\|f_2\|_\infty\|m\|_\infty$$

where again c depends only on the function $\varphi^{(j)}(x)$.

Theorem 2. *Let $p(x, \xi)$, $x=(x_1, x_2)\in R^n\times R^n$, $\xi=(\xi_1, \xi_2)\in R^n\times R^n$, be a symbol of class $S^0_{1,0}$ and $p(x, D)$ the corresponding pseudo-differential operator. Then if $f(x)=f_1(x_1)f_2(x_2)$ and g is the restriction to the diagonal $x_1=x_2$ of $p(x, D)f$ we have*

$$\|g\|_2 \leq c\|f_1\|_\infty\|f_2\|_2.$$

For the sake of simplicity we shall only consider the case where p is independent of x and vanishes near the origin. We shall show that g is a convergent integral of operators like those in the preceding theorem. First we take two infinitely differentiable functions with compact support $\hat{\varphi}(\xi)$ and $\hat{\psi}(\xi)$, $\xi\in R^n$, one of which vanishes near the origin, and such that

$$r(\xi_1, \xi_2) = \int_0^\infty [\hat{\varphi}(t\xi_1)^2\hat{\psi}(t\xi_2)^2+\hat{\psi}(t\xi_1)^2\hat{\varphi}(t\xi_2)^2]\frac{dt}{t} \neq 0, \ \ |\xi_1|+|\xi_2| > 0.$$

Then if

$$p(\xi_1, \xi_2) = q(\xi_1, \xi_2)r(\xi_1, \xi_2)$$

and

$$p_t(\xi_1, \xi_2) = q(\xi_1/t, \xi_2/t)[\hat{\varphi}(\xi_1)\hat{\psi}(\xi_2)+i\hat{\psi}(\xi_1)\hat{\varphi}(\xi_2)],$$

$$p_t(\xi_1, \xi_2) = \int e^{i(\xi_1\cdot u_1)+i(\xi_2\cdot u_2)}m(t, u_1, u_2)\,du_1\,du_2,$$

we have $|m(t, u_1, u_2)| \leq c_k(1+|u_1|+|u_2|)^{-k}$, $\forall k$. Now, if $x\in R^n$, then

$$g(x) = \frac{1}{(2\pi)^n}\int p(\xi_1, \xi_2)e^{-ix\cdot(\xi_1+\xi_2)}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\xi_1\,d\xi_2$$

and replacing p by rq and using the preceding identities we obtain

$$g = \frac{1}{(2\pi)^n} \int_0^\infty \int e^{i(\xi_1 t \cdot u_1 + \xi_2 t \cdot u_2 - x \cdot \xi_1 - x \cdot \xi_2)} (\hat{\varphi}(t\xi_1)\hat{\psi}(t\xi_2) - i\hat{\psi}(t\xi_1)\hat{\varphi}(t\xi_2))$$

$$\cdot \hat{f}_1(\xi_1)\hat{f}_2(\xi_2) m(t, u_1, u_2)\, du_1\, du_2\, d\xi_1\, d\xi_2 \frac{dt}{t}$$

$$= \int_0^\infty \int ((\varphi_t^{u_1} * f_1)(\psi_t^{u_2} * f_2) - i(\psi_t^{u_1} * f_1)(\varphi_t^{u_2} * f_2)) m(t, u_1, u_2) \frac{dt}{t}$$

where φ^u is the inverse Fourier transform of $e^{iu\cdot\xi}\hat{\varphi}(\xi)$. Now an application of the preceding theorem yields the desired result.

THEOREM 3. *Let $p(x, D)$ be a pseudo-differential operator of type $S^0_{1,0}$ and $a(x)$ a function with bounded derivatives. Let C_a denote the commutator of multiplication by $a(x)$ and*

$$g = [C_a p(x, D)]\, \partial f/\partial x_1;$$

then

$$\|g\|_2 \leq c\|f\|_2 \|\nabla a\|_\infty .$$

We shall assume again that p is independent of x. Then, as is readily verified

$$g = \frac{1}{(2\pi)^n} \int \xi_1 [p(\xi) - p(\xi+\eta)] e^{-ix\cdot(\xi+\eta)} \hat{f}(\xi)\hat{a}(\eta)\, d\xi\, d\eta.$$

Now we decompose

$$\xi_1[p(\xi) - p(\xi+\eta)] = \sum_1^n q_j(\xi, \eta)\eta_j + p(\xi) r_j(\xi, \eta)\eta_j - p(\xi+\eta) s_j(\xi, \eta)\eta_j$$

where the q_j are functions in the class $S^0_{1,0}$ multiplied by homogeneous functions of degree zero, and the r_j and s_j are homogeneous of degree zero and infinitely differentiable away from the origin. The preceding theorem applies to these functions thought of as symbols. If we denote now by $B(u)$ (f_1, f_2) the bilinear operator of the preceding theorem associated with the symbol $u(\xi, \eta)$. Then the following identities are readily verified

$$B(q_j \eta_j)(f, a) = B(q_j)(f, \partial a/\partial x_j),$$

$$B(p(\xi) r_j(\xi, \eta)\eta_j)(f, a) = B(r_j)(p(D)f, \partial a/\partial x_j),$$

$$B(p(\xi+\eta) s_j(\xi, \eta)\eta_j)(f, a) = p(D) B(s_j)(f, \partial a/\partial x_j)$$

and an application of the preceding theorem yields the desired result.

The method of the Cauchy integral. We will see now how the study of the integral (11) reduces to that of (8), this to that of (9), and in turn, this one to that of an integral similar to the one which appears in (6). Unfortunately the study of the latter is too complicated to allow a brief description and we are compelled to refer the reader to the literature in this respect (see [4] and [5]).

Let us write the integral (8) as

$$g(s) = \text{p.v.} \int_{-\infty}^{+\infty} t^{-1} F\left[\frac{a(s)-a(s-t)}{t}\right] f(s-t)\, dt$$

and let us assume that for a given function F and a positive number M we have $\|g\|_p \leq c\|f\|_p$ whenever the function $a(t)$ satisfies the condition $|a(s)-a(t)| \leq M|s-t|$. We also write the integral in (11) as

$$g(x) = \text{p.v.} \int_{R^n} k(x, y) F\left[\frac{a(x)-a(x-y)}{|y|}\right] f(x-y)\, dy,$$

and, if v denotes a unit vector and t a real variable, we define

$$g(x, v) = \frac{1}{2} k(x, v)\, \text{p.v.} \int_{-\infty}^{+\infty} t^{-1} F\left[\frac{a(x)-a(x-t)}{t}\right] f(x-t)\, dt.$$

Given our assumptions on the parities of k and F, integration with respect to v on the unit shpere Σ yields

$$\int_{\Sigma} g(x, v)\, d\sigma = g(x).$$

Now, on account of our assumptions on the one-dimensional case and boundedness of $k(x, y)$ we have

$$\int_{-\infty}^{+\infty} |g(x+tv, v)|^p\, dt \leq c \int_{-\infty}^{+\infty} |f(x+tv)|^p\, dt$$

and integrating with respect to x on a hyperplane perpendicular to v we obtain $\|g(x, v)\|_p^p \leq c\|f\|_p^p$. Integrating $g(x, v)$ with respect to v on the unit sphere Σ and using Minkowski's integral inequality we find that $\|g\|_p \leq c\|f\|_p$. Next let us see how the integral (8) reduces to that in (9). Let L denote the operator represented by the integral (8) and

$$A_z f = \text{p.v.} \int_{-\infty}^{+\infty} [s-t-z^{-1}(a(s)-a(t))]^{-1} f(s)\, ds.$$

Then, if $F(z)$ is analytic in $|z| \leq \varrho = \sup |(a(s)-a(t))/(s-t)|$, we have

$$(Lf)(t) = \frac{1}{2\pi i} \int_{|z|=\varrho} F(z)(A_z f)(t) \frac{dz}{z}.$$

Now, the integral defining A_z is not of the form (9) because there the function $a(x)$ is real, while in general this is not the case in A_z. However, introducing the new variables $\bar{t} = t - ua(t)$, $\bar{s} = s - ua(s)$, where $z^{-1} = u + iv$, A_z takes the form of the integral (9), except for the absence of the factor $(1 + ia'(s))$ in the integrand, which is irrelevant. This makes it possible to estimate the operator L by means of integrals of the form (9).

Finally, we will show how (9) can be estimated by means of integrals similar to that in (6). For this purpose let $z(\lambda)=t+i\lambda a(t)$, $w(\lambda)=s+i\lambda a(s)$, and consider the operator

$$A(\lambda)f = \text{p.v.} \int_{-\infty}^{+\infty} \frac{f(s)\, ds}{z(\lambda)-w(\lambda)}.$$

This operator has essentially the form (9) for each real value of λ, and for $\lambda=0$ it reduces to the Hilbert transform. Differentiating with respect to λ one obtains the operator

$$B(\lambda)f = \frac{d}{d\lambda} A(\lambda)f = \text{p.v.} \int_{-\infty}^{+\infty} \frac{-i}{z(\lambda)-w(\lambda)} \left[\frac{a(t)-a(s)}{z(\lambda)-w(\lambda)}\right] f(s)\, ds,$$

whose analogy with (6) is apparent. As was said before, the method used in the study (6) can be applied to $B(\lambda)$ and in this manner one obtains

$$d\|A(\lambda)\|/d\lambda \leq \|B(\lambda)\| \leq c[1+\|A(\lambda)\|]^2,$$

where the norms denote operator norms in L^2 and c denotes a constant depending on $\|a'\|_\infty$. From this differential inequality and the fact that $A(0)$ is the Hilbert transform and consequently $\|A(0)\|=\pi$, there follows that

$$\|A(1)\| \leq 2\pi(1-\|a'\|_\infty \alpha^{-1})^{-1}-\pi, \quad \alpha > 0,$$

where α is an absolute constant. This result can be extended to L^p, $1<p<\infty$, by means of well known techniques.

Problems. There still are some basic unresolved problems in the subject we have been discussing. Consider the integral in (9). Are the results obtained so far about it also valid without restrictions on the norm $\|a'\|_\infty$? It is natural to expect an affirmative answer to this question. However, since the integral depends on a in a non-linear fashion, a negative answer cannot be ruled out. More generally we may ask which are the function spaces in which the operator given by the integral (8) is continuous with the only condition that the quotient $(a(x)-a(y))/(x-y)$ remain in a compact subset of the domain of analyticity of F. A clarification of this question would be very important in the study of boundary value problems for elliptic equations in general Lipschitzian domains. The methods employed so far seem to be insufficient for the treatment of these problems.

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