A talk with professor I. M. Gelfand

A student and teacher who followed his own interests and instincts

Recorded by V. S. Retakh and A. B. Sosinsky

ISRAEL MOISEYEVICH GELFAND is one of the greatest living mathematicians. He's the author of around 500 works—books and articles not only on mathematics *per se* but also on mathematical physics, cell biology and neurobiology, and applications in medicine, seismology, and other areas. Gelfand is a member of the Soviet Academy of Sciences, the US National Academy of Sciences, the American Academy of Arts and Sciences, the London Royal Society in England, the French Academy of Science, the Royal Swedish Academy, and many other foreign academies. He has received honorary doctorates from Oxford, Paris, Harvard, and many other universities. He has also received such distinguished prizes as the Kyoto Prize, the Wolf Prize, and the Wigner Medal.

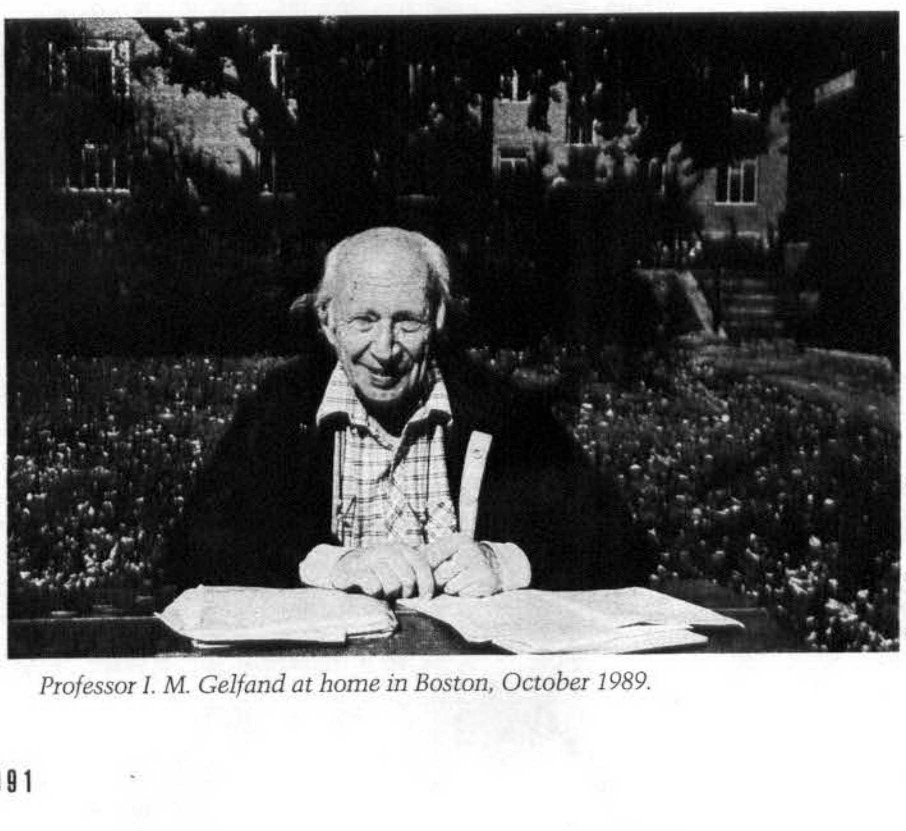
For some 45 years now, first-year students and famous scholars have gathered on Monday evenings at Moscow University for Gelfand's renowned mathematics seminar. Several generations of outstanding mathematicians have been nurtured by this seminar.

Gelfand founded the Mathematics Correspondence School, which has students throughout the Soviet Union, and is the chairman of its governing committee. The main goal of this school is to reach out and help those students who are practically deprived of mathematical literature and contact with scholars. These are generally students who live outside of Moscow, Leningrad, and other big cities where there is access to good books and good mathematicians. Created 25 years ago, this correspondence school was the first such school in the Soviet Union and served as an example for other correspondence schools that followed.

Interviewers from our sister magazine *Kvant* planned this conversation with professor Gelfand in the usual way—that is, by proposing questions that would be of interest to both Gelfand and *Kvant*'s student readers. Gelfand glanced at the list of questions and said they were very interesting but he didn't consider himself competent enough to answer them.

"You see," he said, "I don't think I have the right to impose my opinions on your readers. It would be better if I just tell what I was doing mathematically at their age—13 to 17 years old. I'm not sure I can recall now all the problems I was working on at that time, but the problems I'll talk about I remember very well."

And now—I. M. Gelfand's story.



Professor I. M. Gelfand at home in Boston, October 1989.

ONE OF GRAHAM GREENE'S NOVELS is called *The Loser Takes All.* My mathematical experience was such a wonderful and happy one, for many years it seemed to be the realization of Greene's title. Why was I so fortunate? Briefly stated: first, I didn't study at a university (or any institution of higher learning, for that matter); second, because of certain difficulties in my family life I found myself in Moscow without parents, and jobless, at sixteen and a half years of age.

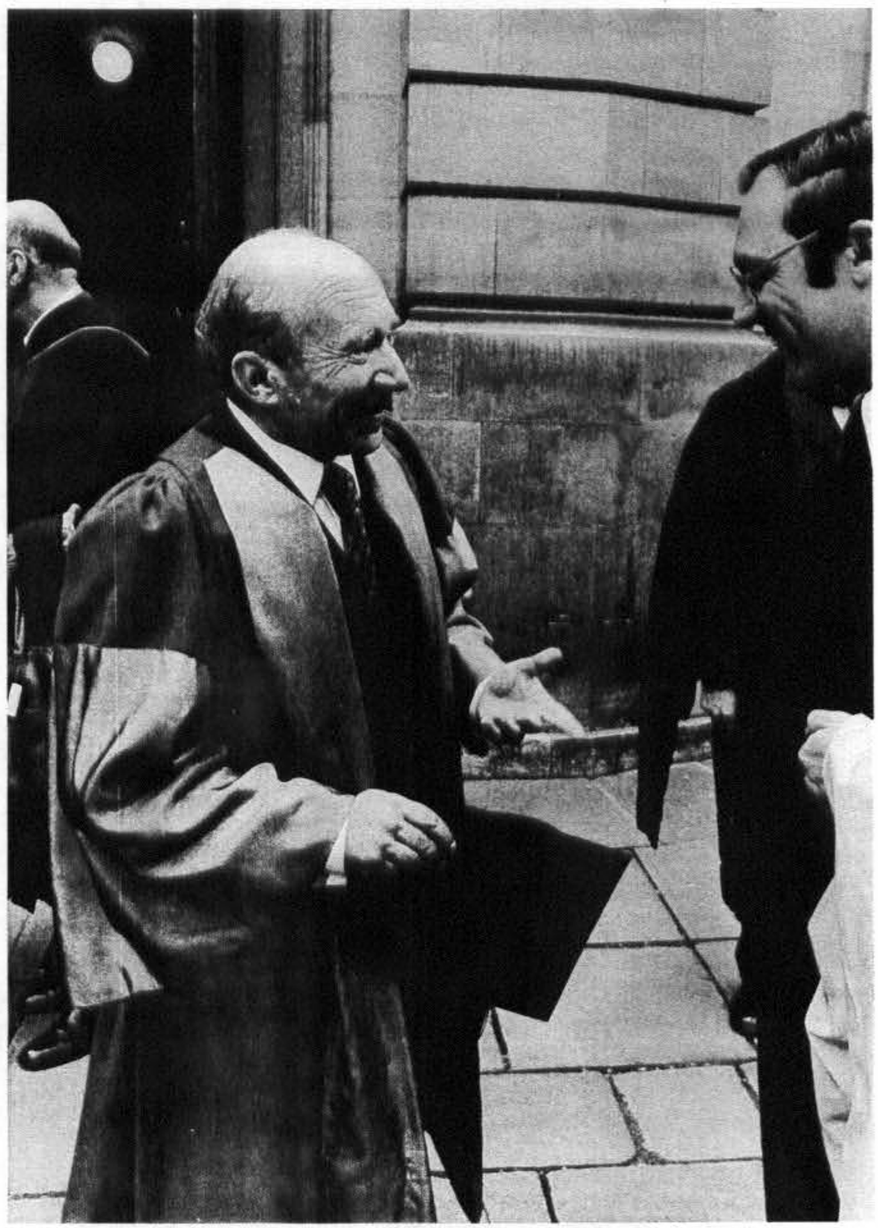
I'll try to illustrate the meaning of the expression "the loser takes all" with the help of another English writer, Somerset Maugham. The hero of the story, a church sexton, suffers a misfortune: during certification of church personnel it comes to light that he's illiterate, and so he's fired. He starts selling cigarettes, then buys a tobacco stand, then several others, and ends up making a brilliant career in commerce. He becomes the richest man in the city. He becomes the city's mayor. Someone comes to interview him—just as you're doing now—and he explains to the journalist that he's illiterate. The stupefied journalist exclaims, "What heights you could have attained if you had been literate!" Without a pause the mayor replies, "I'd have been a sexton."

So in February 1930, at sixteen and a half, I came to Moscow to live with my distant relatives, and I was often unemployed. I tried many temporary jobs, but mostly I went to the Lenin Library and "pulled together" all the knowledge I didn't get in school and in the technical training I didn't finish. At the library I met university students and started going to seminars. At 18 I was already teaching, and at 19 I found myself in graduate school. The rest of my mathematical career proceeded quite normally, taking the usual track for mathematicians.

But it's not this part of my life that I want to talk about. I want to tell your readers about the earlier period. I'd like to do this for two reasons. First, it's my deeply held conviction that mathematical ability in most future professional mathematicians appears precisely at that time—at 13 to 16 years of age. (Of course, there are exceptions—some who develop earlier, some later, at 20 to 30 and even 40—among very strong mathematicians.) Second, this early period formed my style of doing mathematics. The subject of my studies varied, of course, but the artistic form of mathematics that took root at this time became the basis of my taste in choosing problems that continue to attract me right up to the present time. Without an understanding of this motivation, I think it's impossible to make head or tail of the seeming illogic of my ways of working and the choice of themes in my work. In the light of this motivating force, however, they actually come together sequentially and logically.

The first thing I remember happened when I was around 12. I understood then that there are problems in geometry that can't be solved algebraically. I drew up a table of ratios of the length of the chord to the length of the arc in increments of 5 degrees. Only much later did I learn that there are such things as trigonometric (not algebraic!) functions and that, in essence, I was drawing up trigonometric tables.

At about this time I was working through a book of problems in elementary algebra. I had no accompanying textbook, I didn't know the theory, but sometimes I had to



Receiving an honorary doctorate at Oxford University in 1973.

solve some pretty tough problems, using formulas that I didn't know at the time. When I couldn't figure out how to solve a certain problem, I'd look at the answer, and I learned how to reconstruct methods of solving problems from the way they're set up and from the answers given. In particular, I understood then, and remembered for the rest of my life, that you can master a subject by solving problems and that there's nothing wrong with looking at the answer since we always have a hypothesis about the answer while we're working on any problem. Doing research in mathematics is similar to solving problems in which something about the answer is known. This is the difference between working in mathematics and training for university entrance exams (which is necessary as well, of course).

At the age of 12 or 13 I turned my attention to geometry problems in which there was often a right triangle with sides 3, 4, 5 and even with sides 5, 12, 13. I wanted to find all right triangles with integer sides, and I derived a general formula for their sides. That is, I found all Pythagorean triples.[1] (Of course, I didn't know the term at the time.) Unfortunately, I don't remember how I did it.

I worked at mathematics when I was sick and when I was on vacation. Even now I can't help noticing how much strong students manage to do when they stay home because of illness. And so I would keep my own sons home a few extra days after they got better.

In the geometry textbook we used, some theorems were given as problems. I got my hands on a notebook (not an easy thing in those days) and wrote out the statement of a theorem on each page. Over the course of the summer I covered almost all the pages with proofs. That's how I learned to write out my mathematical work.

I'll skip over a stretch here. I'll mention only the book by Davydov on algebra in which you can find clever ways of solving problems about maxima and minima by means of elementary techniques (that is, without using differential calculus). For example: given $a + b$, find the maximum of ab; for a given perimeter, find the rectangle with the maximum area; find the maximum of the product of nonnegative numbers $a_1 a_2 ... a_n$, given their sum $a_1 + a_2 + ... + a_n$; little squares are cut out of a square with a given side and a box is made out of the remainder—what size must the little squares be for the volume of the box to be maximal?

Combinatorics and Newton's binomial formula made a great impression on me, and I thought about them for a long time.

I lived in a small town with only one school. My mathematics teacher was a kind but stern-looking man by the name of Titarenko. He had a huge Cossack moustache. I haven't met a better teacher, although I knew more than he did and he knew it. He liked me a lot and

[1]See "Genealogical Threes" in the Nov./Dec. 1990 issue of *Quantum*.—Ed.

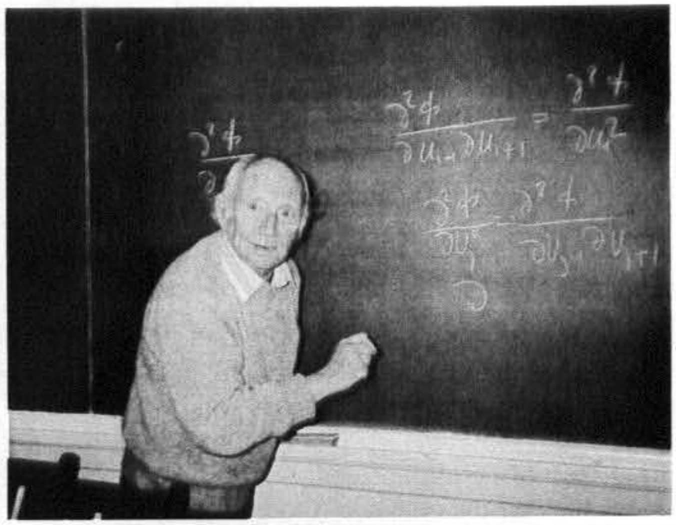


Photo by T. Alekseyevskaya

Lecturing at the Massachusettes Institute of Technology (MIT) in 1989.

encouraged me in every way. Offering encouragement is a teacher's most important job, isn't it?

There was a definite lack of mathematical books. I saw ads for books on higher mathematics and figured higher mathematics must be pretty interesting. My parents couldn't order these books—they didn't have the money. But once again I was lucky. At the age of 15 I was taken to Odessa to have my appendix taken out. I told my parents I wouldn't go to the hospital until they bought me a book on higher mathematics. My parents agreed and bought me the textbook on higher mathematics written by Belyayev in Ukrainian for use in technical institutes. But they only had enough money for the first part, which was about differential calculus and analytical geometry in the plane.

I was lucky that I didn't start with a full-fledged university course. This was a very elementary book. You can judge the level of Belyayev's book by its introduction—in particular, it says there are three kinds of functions: analytical, as defined by formulas; empirical, as defined by tables; and correlational. I didn't find out about correlational functions until many years later, from a student who was studying probability theory.

On the third day after the operation I picked up the book and read it, alternating it with novels by Émile Zola, for nine days. (In those days you'd stay in the hospital for twelve days after an appendectomy.) That was enough time for me to finish Belyayev's book.

I took away two remarkable ideas from this book. First, any geometric problem in the plane and in space can be written as formulas. (I had suspected this earlier.) I also learned about the existence of some remarkable figures—the ellipse, for example.

The second idea turned my world view upside down. This idea is the fact that there's a formula for calculating the sine: $\sin x = x - x^3/3! + x^5/5! - \dots$. Before this I thought there are two types of mathematics, algebraic and geometric, and that geometric mathematics is basically "transcendental" relative to algebraic mathematics—that is, in geometry there are some notions that can't be expressed by formulas. Consider, for example, the formula for circumference—it contains the "geometric" number π; or, say, the sine—it's defined in a completely geometric way.

When I discovered that the sine can be expressed algebraically as a series, a barrier came tumbling down, and mathematics became one. To this day I see the various branches of mathematics, together with mathematical physics, as a unified whole.

Of course, I became convinced that problems of the extreme are solved automatically (that is, by means of an exact algorithm). Although they lose their charm, you have in your hands a powerful tool (calculus) for solving them.

Studying differential calculus I learned that there is also integral calculus, which has to do with areas and volumes. But what it consisted of, I had no idea—I didn't have the second volume of Belyayev's textbook!

Now's a good time to mention another problem I recall. The next autumn we studied the volumes of solids of revolution at school. A classmate of mine, D. P. Milman, who later became a famous mathematician, brought the following problem to my attention: find the volume of a body formed by the rotation of a circle about its tangent. To solve it I divided the circle into strips. Then I calculated the differences of the volumes of the corresponding cylinders obtained by rotation. Finally, I found the sum of these differences. This brought me face to face with the need to find the sum

$$\cos \varphi + \cos 2\varphi + \cos 3\varphi + \ldots + \cos n\varphi. \qquad (1)$$

The rest, as usual, was a mixture of inventiveness and stupidity. I passed over an elementary solution based on standard trigonometry, using instead the formula

$$e^{i\varphi} = \cos \varphi + i \sin \varphi.$$

(This formula is called Euler's formula, but I didn't know that.) I got this formula from the power series for sin x, cos x, and e^x, which had made a deep impression on me. It remained for me to find the sum of the geometric progression $e^{i\varphi} + e^{2i\varphi} + \ldots$ and, from that, to derive the sum (1), which I did.

This problem led to my habit of thinking about a problem even after I'd solved it. And I came up with something else: I moved the circle away from the line and understood that rotation produces a body that looks like the rubber cushion my friend's hemorrhoidal grandfather used to sit on. Knowing the radius r of a circle and the distance d from its center to the line, I used the method described above to determine the volume of the solid of revolution, $2\pi^2r^2d$. I was stunned by the simplicity of this formula. I rewrote it in the form $\pi r^2 \cdot 2\pi d$ and understood that if we cut the rubber cushion and pull it into a cylinder whose side equals the length of the trajectory formed by the center of the circle, then the volume of the cylinder would be the same. A similar fact is true for the area of a surface, and I understood that it was not by chance. What will happen if we rotate some other figure instead of a circle—for example, a triangle?

In this case the volume of the solid of revolution coincides with the volume of a prism whose base is a triangle and whose height equals the length of the trajectory formed by the common intersection of the medians of the triangle. From a physics book I knew that this point is the triangle's center of mass. Seeing what happens when a section is rotated, I understood that the center of a circle is its center of mass as well.

I found a general definition of the center of mass in some textbook on the strength of materials—I have no idea where I got a hold of it. Not only did I immediately start rotating various figures, I'd move them along various curves and calculate the volumes of the bodies obtained and their surface areas. The rigor of the thinking was important here. I was very proud that I could find the center of mass of a half circumference (half circle) and of a half disk (half of the interior of a circle) given the volume of a ball and the area of its surface.

And I was lucky yet again. An extraordinarily well-educated man (in my opinion at the time) came to our town. He had graduated from the Odessa Pedagogical Institute in physics and math. Among the books he brought with him were Kagan's *Theory of Determinants* and Hvolson's *Course in Physics*. Kagan's book was useful and detailed. It even contained a chapter on determinants of infinite order.

I should also mention the biology textbook by Filippenko, the well-known biologist from the school of the famous geneticist N. K. Koltsov. This was a fine book, and it naturally influenced my work in biology some 15 or 20 years later.

But to get back to mathematics. I was still interested in problems of areas and volumes. I began with a calculation of the area under the segment between two points of a parabola. This problem reduces to a calculation of the sum $1^2 + 2^2 + \ldots + n^2$, which I did easily.

Then I wanted to find the area under the curve $y = x^p$, where $p = 2, 3, 4, \ldots$; that is, to find the sum $S_0 = 1^p + 2^p + \ldots + n^p$ for every positive integer p.

By analogy with the formula

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

I decided that S_0 is a polynomial in n of degree $p + 1$. I didn't notice that to find the area under the curve it's sufficient to know only the first coefficient of the polynomial S_0, so I started searching for the entire polynomial. This turned out to be quite interesting. First of all, I generalized the problem: instead of x^p I considered $f(x)$ and started looking for the sum

$$S_0 = f(1) + f(2) + \ldots + f(n).$$

Let $F(x)$ be a function such that $F'(x) = f(x)$. From Taylor's formula we get

$$F(2)-F(1)=f(1)+\frac{f'(1)}{2!}+\frac{f''(1)}{3!}+\dots,$$

$$F(3)-F(2)=f(2)+\frac{f'(2)}{2!}+\frac{f''(2)}{3!}+\dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$F(n+1)-F(n)=f(n)+\frac{f'(n)}{2!}+\frac{f''(n)}{3!}+\dots.$$

I added these equalities and got

$$F(n+1)-F(1)=S_0+\frac{S_1}{2!}+\frac{S_2}{3!}+\dots,$$

where S_0 is the sum that interested me and

$$S_1=f'(1)+f'(2)+\dots+f'(n),$$
$$S_2=f''(1)+f''(2)+\dots+f''(n),\ \dots.$$

Then I wrote the following system:

$$F(n+1)-F(1)=S_0+\frac{S_1}{2!}+\frac{S_2}{3!}+\frac{S_3}{4!}+\dots,$$

$$f(n+1)-f(1)=S_1+\frac{S_2}{2!}+\frac{S_3}{3!}+\dots,$$

$$f'(n+1)-f'(1)=S_2+\frac{S_3}{2!}+\dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

This is an infinite system with an infinite number of unknown variables $S_0, S_1, S_2, \dots$. As I mentioned earlier, Kagan's book touched on determinants of infinite order, so I was able to use Cramer's rule to find S_0:

$$S_0=\frac{\begin{vmatrix} F(n+1)-F(1) & 1/2! & 1/3! & 1/4!\dots \\ f(n+1)-f(1) & 1 & 1/2! & 1/3!\dots \\ f'(n+1)-f'(1) & 0 & 1 & 1/2!\dots \\ \dots & \dots & \dots & \dots \end{vmatrix}}{1}$$

I expanded the determinant in the numerator of this "fraction" in the elements of the first column and the corresponding minors and got

$$S_0=B_0(F(n+1)-F(1))+B_1(f(n+1)-f(1))$$
$$+B_2(f'(n+1)-f'(1))+\dots, \qquad (2)$$

where $B_0=1$, B_1, B_2, ... are numerical determinants of infinite order. The expression I got is called the *Euler–Maclaurin formula*, but of course I didn't know that. To calculate this expression I needed to know the coefficients $B_0, B_1, B_2, \dots$.

To do this, I used arguments that would now be called "functorials." Taking advantage of the fact that the coefficients B_0, B_1, ... don't depend on f, I picked a function f such that the left part of the system formed a geometric progression (which I knew how to sum). The function $f(x) = e^{\alpha x}$ suits this purpose. Inserting it into formula (2) (I'll leave the intermediate steps for you to work out!), I got

$$B_0+\alpha B_1+\alpha B_2+\dots=\frac{\alpha}{(e^{\alpha}-1)}.$$

That is, I got the power series for the numbers I was after. (These numbers B_0, B_1, B_2, ... are called *Bernoulli numbers*, and the polynomial S_0 for $f(x) = x^p$ is called *Bernoulli's polynomial*.)

I remember two other problems from this period. The first arose out of the problem in our book of algebra problems: express $x_1^2 + x_2^2$ and $x_1^3 + x_2^3$ via the coefficients of a quadratic equation with the roots x_1 and x_2. A natural generalization of this problem leads to another: express the sum $x_1^2 + \dots + x_n^2$ and the sum $x_1^3 + \dots + x_n^3$ via the coefficients of the equation $x^n + a_1x^{n-1} + \dots + a_n = 0$, where $x_1, \dots, x_n$ are roots of this equation. At this point Bezout's theorem helped me, which I knew from Davydov's book. I went further and posed a more general problem for myself: express the sum of kth degrees of the roots of an algebraic equation of nth degree via the coefficients of this equation. I managed to solve this problem (the solution is known as Newton's formula).

The second problem I solved at that time arose when I discovered that the number cos ix is real because

$$\cos ix=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots.$$

I pondered this unexpected fact and came up with the following general theorem: *every even real-valued function takes real values on the imaginary axis.*

To prove this I had to refine the notion of a "function." I thought about what to call a function and arrived at this definition: a function is the sum of a convergent power series. After this, the proof of the theorem is almost self-evident.

This problem was probably the last one I thought about before I came to Moscow. I solved it in the summer of 1929. The next six months were very difficult for my family and me. Mathematics was far from my mind.

The next period of my studies in Moscow was no longer "pure experimentation." In Moscow I was exposed to many completely different influences, and my development no longer drifted on its own course. At this time, as I mentioned earlier, I studied independently in the Lenin Library and lived on occasional earnings from odd jobs. For a while I actually worked behind a check-out desk at the library. I met mathematics students from the university. One of them told me that expressions of the form $f(n+1) - f(n)$, which greatly interested me, were part of a whole science called the theory of finite differences. He told me I had to read Nörlund's book *Differenzenkalkül* on this topic. It was in German, but I mastered it with the help of a dictionary.

I started going to university seminars, and there I found myself under intense psychological stress. I discovered that my style of doing mathematics wasn't good for anything. New breezes were blowing in mathematics—new demands for rigorous proofs, great interest in the theory of functions of a real variable. (Today this level of rigor and this particular theory are considered old-fashioned and obsolete, but at the time . . .)

Then I realized it's very important that a function doesn't have to be continuous, that a continuous function doesn't have to be differentiable, that a differentiable function doesn't have to be twice differentiable, and so on; that even if a function has derivatives of all orders, the Taylor series for this function isn't necessarily convergent, and that even if it is, its sum doesn't necessarily coincide with the value of the function! If this coincidence takes place, the function is called analytic, and this class of functions (so the devotees of the real-variable function theory maintained) is so narrow that it lies outside the bounds of mainstream mathematics. And these were the only functions I'd been looking at!

Under the pressure of this point of view, I read the "modern, rigorous" textbook on analysis by Vallee Poussin. It's similar to the texts currently used at Moscow University by students of mathematics and mechanics, but better. So I sympathize with those first-year students who are allowed to experience the beauties of mathematical analysis only after a year's probation, a sort of trial by the fire of its "rigorous foundation."

But even here I was lucky. I began reading I. I. Privalov's remarkable book on the theory of functions of a complex variable. While reading this book I understood why, for the function $f(x) = 1/(1 + x^2)$, the Taylor series is divergent at $x = 1$ even though its graph is continuous. (As a matter of fact, the corresponding complex function has a **singularity** for $x = i$). After the first 100 pages I felt a fresh wind. I discovered that if a complex function has a first derivative, it has derivatives of all orders, and then the Taylor series converges at the value of this function in some domain. Everything fell into place, and harmony was restored.

I raced through **Hurwitz** and Courant's book on the theory of functions of a complex variable. I was mostly

CONTINUED ON PAGE 26

A note on international prizes in mathematics

There is no Nobel Prize in mathematics. One story has it that the woman Alfred Nobel loved left him for the famous Swedish mathematician M. G. Mittag-Leffler. So when Nobel decided what prizes should be awarded by his foundation, he established one in physics, one in economics, one in literature, as well as other areas, but somehow either forgot or wasn't especially enthusiastic about establishing a Nobel Prize in mathematics.

Whatever the real reason, this injustice was corrected by Riccardo Wolf, another wealthy industrialist. Being of Jewish origin, he emigrated from Germany after Hitler came to power and settled in Latin America, making a brilliant career in the steel industry. Even though he was a capitalist, he was a friend of Fidel Castro, who even sent him as the Cuban ambassador to Israel. When Cuba broke off diplomatic relations with Israel after the Yom Kippur war in 1973, he decided to stay in Israel. Wolf, who was already in his eighties at the time, founded the Wolf Foundation. The foundation, which is based in Israel, each year awards several Wolf Prizes for achievements in different areas of science and the arts, and among them is the Wolf Prize in mathematics! Wolf Prizes were awarded for the first time in 1978, and the first Wolf Prize in mathematics went to I. M. Gelfand of the Soviet Union and C. L. Siegel of Germany. In 1978 the Soviet Union rarely allowed its citizens to travel to Israel and back, so Gelfand wasn't allowed to go to Jerusalem to receive the prize. It was only ten years later, in May 1988, when *perestroika* was gathering steam, that Gelfand was able to come to Jerusalem and attend the yearly presentation ceremony in the Knesset (the Israeli Parliament).

The Kyoto Prize, awarded by the Inamori Foundation (established by the well-known Japanese industrialist Kazuo Inamori in 1984), is different. There are no annual prizes in specific predetermined fields like physics, chemistry, economics, or mathematics. Instead, there are three broad categories: advanced technology, basic sciences, and creative art and moral sciences. Each year a specific field is selected from each of the three categories, and a laureate is then chosen from that field. For instance, the 1986 Kyoto Prize in basic sciences was awarded in biology, in 1987—in Earth sciences and astrophysics, and so on. You see that to get a Kyoto Prize for a mathematician is much more difficult since it's not awarded every year. I. M. Gelfand received the Kyoto Prize in 1989 when the field chosen in basic sciences was mathematics.

Another award that should be mentioned here is the Fields Medal. At the 1924 International Congress in Toronto, a resolution was adopted that two gold medals should be awarded at each international mathematical congress, held every four years. Professor J. D. Fields, a Canadian mathematician who was secretary of the 1924 congress, later donated funds establishing the medals, which were named in his honor. Fields wished that the awards be open to the entire world and recognize both existing work and the promise of future development, so the medals are restricted to mathematicians not over the age of forty. In 1966 the number of medals that could be awarded at each international congress was increased to four in light of the great expansion of mathematical research in the world.

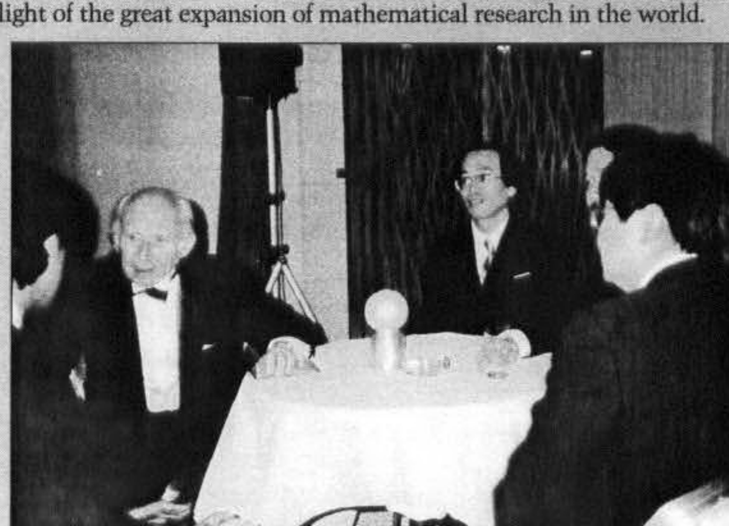


Photo by T. Alekseyevskaya

After the Kyoto Prize award ceremony, Professor Gelfand talks with Japanese mathematicians (November 1989).

CONTINUED FROM PAGE 25

impressed by the chapters on elliptic functions written by Gourvits. And once again fashion made a fool of me—this branch of mathematics was considered obsolete. The theory of elliptic functions was looked down on as "barely extended trigonometry." Many years would pass before this area once again became a focal point of mathematicians' attention.

I gained a lot from the university seminars. Meeting with mathematicians of every stripe, I was able to compare my romantic, antiquated (that is, unfashionable) views of mathematics with what was actually happening then. I studied with many remarkable mathematicians and continue to try to learn this way.

A little later I read—studied in great depth, really—a remarkable book by Courant and Hilbert called *Methods of Mathematical Physics*. I understood then the need to read basic works. Here it's important not to regret the time spent thinking about the very foundations of a theory. The work of Herman Weyl (1925) on the representations of classical groups belongs to that category. But, unfortunately, we didn't have access to even older fundamental works by Cayley, Schur, and other authors of the "pre-Hilbert period."

I learned a lot from L. G. Shnirelman, M. A. Lavrentiev, L. A. Lusternick, I. G. Petrovsky, A. I. Plesner, and even more from Andrey Nikolayevich Kolmogorov.[2] In particular, I learned from him that a true mathematician nowadays must be a philosopher of nature.

But my story has turned into the standard scientific biography. This genre is usually very misleading. A true scientific biography is simply a collection of the scientist's works. One's own impressions about one's works are no more significant than the impressions of any other reader. And so it's time I ended my tale. ◙

[2]For more on A. N. Kolmogorov, see the Innovators department in the Jan. 1990 issue of *Quantum*.—*Ed.*

CONTINUED FROM PAGE 19

point midway between the cylinders. If the distance between the cylinders is $2d$, the stick has a weight w and length l, and the coefficient of friction between each cylinder and the stick is μ, describe how the stick moves.

Please send your solutions to *Quantum*, 1742 Connecticut Avenue NW, Washington, DC 20009. The best solutions will be acknowledged in *Quantum* and their creators will receive free subscriptions for one year.

Click, click, click

We were disappointed that we received no correct solutions to this contest problem. We are confident that our readers could have solved part A. Don't get discouraged. If you can answer part A but not part B or C, send us a note anyway. Your solutions will help us judge what you might like to see.

In the Contest Problem involving Newton's toy you were asked to find the mass of a middle ball so that the velocity of the small ball will be greatest in a three-ball collision. Applying the laws of conservation of energy and momentum to the first collision, we have $m_1v_1 = m_1v_1' + m_2v_2'$, $(1/2)m_1v_1^2 = (1/2)m_1v_1'^2 + (1/2)m_2v_2'^2$.

Solving for v_1' in the first equation and substituting in the second equation, we arrive at $0 = -2m_1v_1v_2' + m_2v_2'^2 + m_1v_2'^2$. Solving for v_2', we find that $v_2' = 0$ and $v_2' = 2m_1v_1/(m_1 + m_2)$. We ignore the solution $v_2' = 0$ since this corresponds to the case of no collision. Since the second collision is similar to the first, we can write down the relevant equation immediately: $v_3'' = 2m_2v_2'/(m_2 + m_3)$. Combining the last two equations, we get

$$v_3'' = \frac{4m_1m_2v_1}{(m_1+m_2)(m_2+m_3)}. \quad (1)$$

To find when the value of m_2 for which v_3'' will be a maximum, we can take the derivative of v_3'' with respect to m_2 and set it equal to zero. The solution is that the mass m_2 should be the geometric mean of the other masses. Specifically,

$$m_2 = \sqrt{m_1m_3}. \quad (2)$$

For those of you who aren't knowledgeable about calculus, we suggest that you take arbitrary values for m_1 and m_3 (that is, $m_1 = 1$ and $m_3 = 100$) and plot a graph of v_3 versus m_2 for different values of m_2. You'll find that the graph reaches a peak where $m_2 = 10$, as predicted by equation (2).

Part B of the problem is an extension of this solution to a collision of five balls. In this case, the masses of the balls follow the relation $m_1/m_2 = m_2/m_3 = m_3/m_4 = m_4/m_5$. Part C of the problem asks about the middle mass given a coefficient of restitution e. You may be surprised to find out that the ratio of masses is the same, independent of e, and is therefore the same solution as in part A.

Burt Lowry, our colleague from Whitman High School in Bethesda, Maryland, was quick to point out that other collision possibilities exist mathematically in the Newton toy that obey energy and momentum conservation. These never occur because the masses are independent. One ball always hits a second ball. The incoming ball never "sees" a ball of twice the mass, but rather sees a single-mass ball. This probably explains the importance of always leaving a small space between the balls when you build one of these toys. ◙

Does your library have Quantum?

If not, talk to your librarian!

Quantum is a resource that belongs in every high school and college library. **"Highly recommended."—*Library Journal***

See page 40 for subscription information.

Share the QUANTUM experience!